Existence of unpreserved extreme points in the disc algebra $\mathbb{A}$

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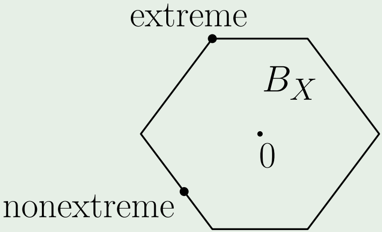
Homenaje a Richard Aron en su 70 cumpleaños

Extreme points

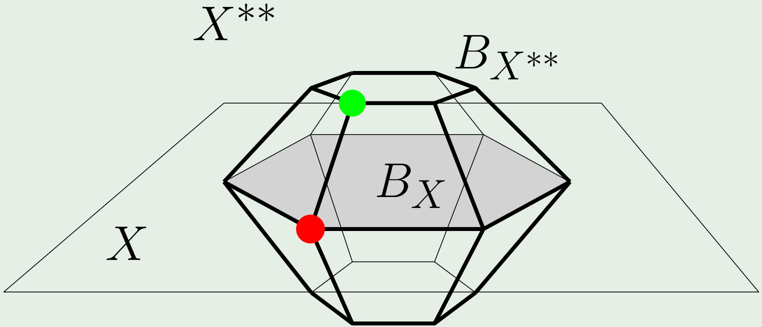
Extreme Points

X Banach. B_X closed unit ball, S_X unit sphere
$x \in S_X$ is extreme if

$$x = \frac{y+z}{2} \text{ and } y, z \in B_X \Rightarrow y = z$$



Looking at the bidual



Reason for the green point: compactness and Krein–Milman.

Looking at the bidual

Reflexive

If X is reflexive $\Rightarrow B_X = B_{X^{**}}$, so Ext$(B_X)$ =Ext$(B_{X^{**}})$.

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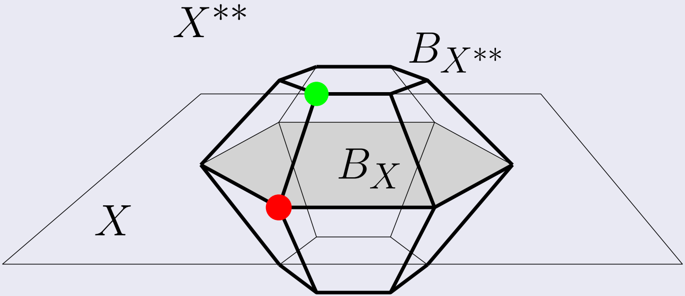
If X is reflexive $\Rightarrow B_X = B_{X^{**}}$, so $\mathrm{Ext}(B_X) = \mathrm{Ext}(B_{X^{**}})$.

NonReflexive

If X nonreflexive then $B_{X^{**}}$ has extreme points not in X.

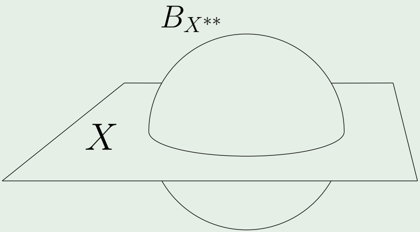


Preserved extreme points

Preserved Extreme

$x \in S_X$ is **preserved extreme**

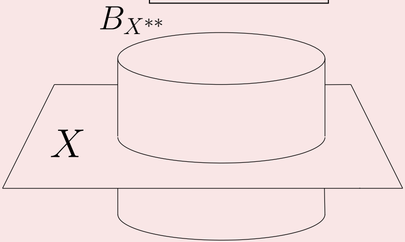
whenever extreme of $B_{X^{**}}$



all $x \in S_X$ preserved

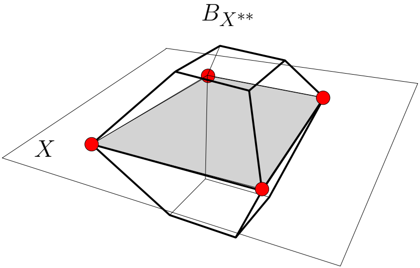
Unpreserved Extreme

Otherwise, **unpreserved**

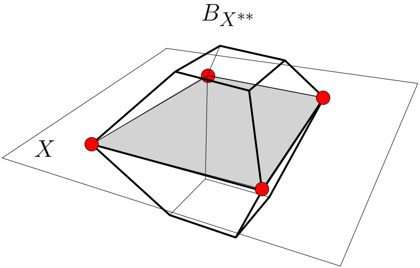


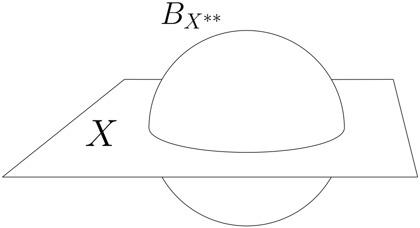
all $x \in S_X$ unpreserved

All extreme points are preserved in $C(K)$,



All extreme points are preserved in $C(K)$, L^p, $1 \leq p \leq \infty$





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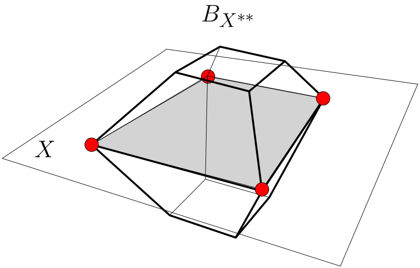

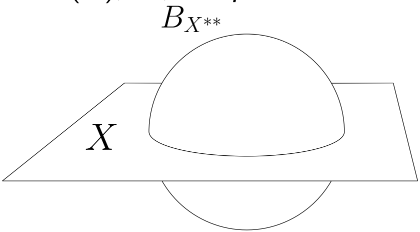

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Question (Phelps'61)

Does there exist any unpreserved extreme point?

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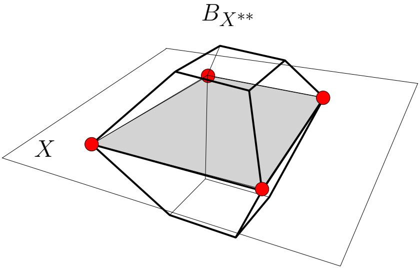

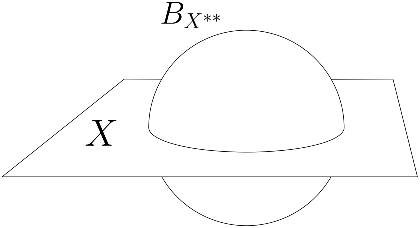

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Question (Phelps'61)

Does there exist any unpreserved extreme point?

Answer (Katznelson'61)

Disk algebra $\mathbb{A}$.

Some distinguished points of the unit sphere

MLUR	β-Extreme	$\Rightarrow$	ω-β-Extreme	$\Rightarrow$	Extreme	R
	$\Uparrow$					
	Denting	$\Rightarrow$	Continuity		$\Uparrow$	
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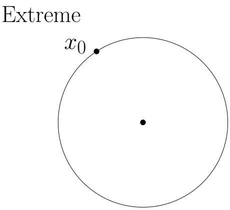

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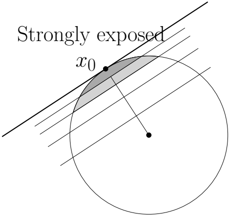

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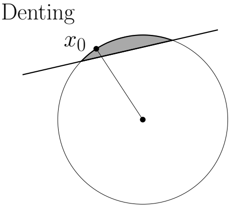

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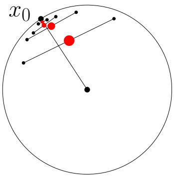
(w-) Strongly extreme



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Extreme points in $\mathbb{A}$

$$\mathbb{A} = (\{f \in C(\overline{\mathbb{D}}, \mathbb{C}) \colon f\restriction_{\mathbb{D}} \in H(\mathbb{D})\}, \|\cdot\|_\infty) \subset (C(\mathbb{T}, \mathbb{C}), \|\cdot\|_\infty)$$

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Lemma (Phelps-61)

$f \in S_{\mathbb{A}}$ is extreme point of $B_{\mathbb{A}}$ iff $g \in \mathbb{A}$ is null whenever

$$|f(z)| + |g(z)| \leq 1 \text{ for all } z \in \mathbb{T}$$

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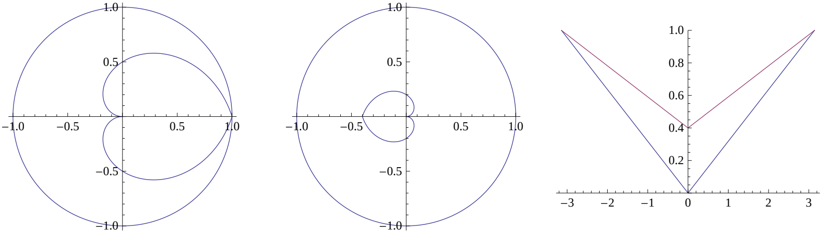

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Theorem (Hoffman-62, pp. 138–139)

$f \in S_{\mathbb{A}}$ is extreme point of $B_{\mathbb{A}}$ iff

$$\int_{-\pi}^{\pi} \log(1 - |f(e^{i\theta})|)\, d\theta = -\infty.$$

Exposed and preserved points in $\mathbb{A}$

Theorem (Phelps 65)

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Theorem (GMZ 14)

$f \in S_{\mathbb{A}}$. Then, following conditions are equivalent

1. *f is β-extreme point.*
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Inner function

$f \in S_{\mathbb{A}}$ is an inner function whenever $f(z) \in \mathbb{T}$ for all $z \in \mathbb{T}$.

The basic tool in $\mathbb{A}$

Question

Given a continuous function on $\mathbb{T}$, can it be regarded as an element of $\mathbb{A}$?

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Theorem (Rudin, Theorem 17.16 and Hoffman, pag. 79)

f positive real-valued in $L^1(\mathbb{T})$ such that $\log(f) \in L^1(\mathbb{T})$. Then, the following function belongs to H^1,

$$\mathcal{G}(f)(z) := \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log(f(e^{i\theta}))\,d\theta\right), \quad \textit{for } z \in \mathbb{D},$$

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Moreover, if f is piecewise continuously differentiable in $\mathbb{T}$, then

- *$h(z) := \lim_{r\to 1^-} \mathcal{G}(f)(r\,z)$ exists and is uniform on $\mathbb{T}$.*
- *$|h(z)| = f(z)$ and $h \in C(\mathbb{T},\mathbb{C})$. (So, $\mathcal{G}(f) \in \mathbb{A}$)*

Unpreserved and exposed

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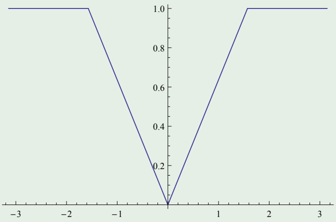

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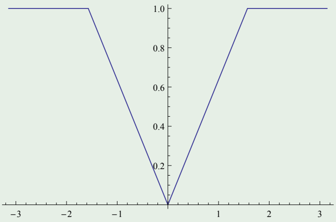

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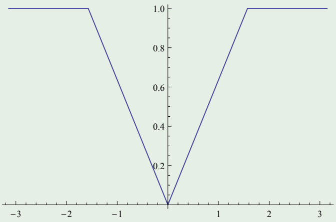

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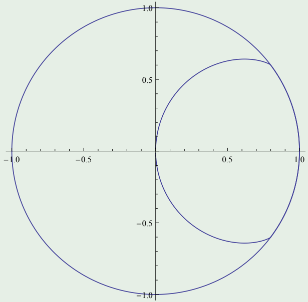

Exposed and extreme, not preserved (is not inner function)

The examples

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$$\omega(t) = 1 - \exp(1 - (\pi/t))$$

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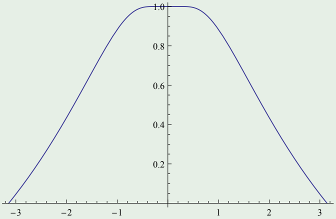

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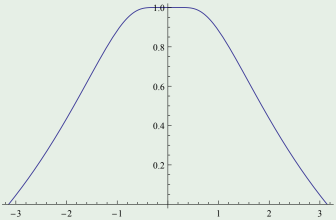

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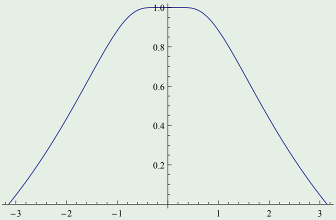

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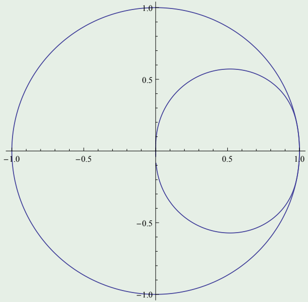

Not exposed and extreme, not preserved (is not inner function)

Referencias

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W. Rudin
Real and complex Analysis

A.J. Guirao, V. Montesinos, V. Zizler
On preserved and unpreserved extreme points
Descriptive Topology and Functional Analysis (Springer)

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Thanks and Congratulations
to Richard!