Chapter 7
The Cognitive Demand of a Gifted Student's Answers to Geometric Pattern Problems

Analysis of Key Moments in a Pre-algebra Teaching Sequence

Angel Gutierrez, Clara Benedicto, Adela Jaime and Eva Arbona

Abstract Mathematically gifted students require specific teaching methodologies to foster their interest in mathematics and their engagement in solving problems. Geometric pattern problems are an interesting context in which to introduce algebra to those students. We present the case of a nine-year-old student engaged in a teaching unit based on geometric pattern problems that was aimed at helping him start learning algebra, equations, and algebra word problems. To analyze and assess the cognitive effort the student made to solve the problems, we used a particularization to this context of the cognitive demand model. We analyzed answers typical of the different kinds of problems posed throughout the teaching unit, showing the student's learning trajectory and related characteristics of mathematical giftedness.

Keywords Geometric pattern problems · Levels of cognitive demand
Linear equations · Mathematical giftedness · Pre-algebra · Primary school

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7.1 Introduction

Mathematically gifted students (hereafter referred to as “gifted students”) tend to show unusual paths of reasoning and methods of solving problems. Authors such as Freiman (2006), Greenes (1981), Krutetskii (1976), and Miller (1990) have suggested a number of characteristics of gifted students related to aspects of their mathematical or social activities. Some characteristics are quite general, such as good memory or the enthusiasm for mathematics, while others are more specific, such as the abilities to identify patterns and relationships among different elements, generalize and transfer mathematical ideas or knowledge from one context to another, or invert mental procedures of mathematical reasoning. These abilities are especially useful in particular contexts, such as the one we are dealing with in this chapter: the use of geometric pattern problems to introduce gifted students to algebraic language and equations. Other traits of gifted students are that they need much less time than average students to solve problems (Budak 2012) and require challenging problems to maintain their interest during the mathematics classes (Kennard 2001). This raises, for teachers and researchers, the issue of finding criteria to determine problems’ appropriateness for certain specific gifted students.

Research also shows that gifted students understand and learn mathematical concepts quite quickly, so it may be useful to teach them some advanced topics that open a door for them into new kinds of challenging problems (Cai and Knuth 2011; Diezmann and Watters 2002; Kennard 2001). For gifted students in upper primary grades, one such topic is elementary algebra, since it provides them with new tools to solve problems when arithmetic is not sufficient. Different researchers have designed curricular variations and teaching units to introduce elementary algebra to ordinary groups, including gifted students (Gavin et al. 2009).

A successful methodology to initiate students into elementary algebra is posing *geometric pattern problems* (Cai and Knuth 2011; Rivera 2013). What we are here calling geometric patterns have also been called visual patterns, pictorial patterns, growing patterns, or just patterns by other authors. Typical geometric pattern problems include questions asking students for the value of the term in a position of the sequence (direct questions). They may also pose questions where the value of a term in the sequence is given and students are asked to calculate the position of that term in the sequence (inverse questions).

The literature has reported many teaching experiments involving direct questions given to students of different ages, from kindergarten (Papic et al. 2011) and early primary (Radford 2011; Rivera 2010; Warren 2005) to lower secondary (Warren et al. 2016). Research has shown a variety of strategies used by students (Rivera 2010) and different focuses of attention to analyze this context (Cai and Knuth 2011), with some research focusing on gifted students (Amit and Neria 2008; Benedicto et al. 2015; Fritzlar and Karpinski-Siebold 2012).

Rivera and Becker (2005) identified two methods to analyze these patterns: numerical and figural. García-Reche et al. (2015) described several strategies to calculate the value of a term of the sequence, labeled: counting, recursive, functional, and proportional. Radford (2006) described several types of generalization, ranging from naïve trial and error to the sophisticated symbolic algebraic generalization.

There are very few publications about inverse questions in geometric pattern problems. Rivera (2013) reported an experiment where Grade 2 students solved pattern problems represented by means of manipulatives and drawings. The students found the inverse tasks very difficult, with very few students solving them meaningfully. According to Rivera, a source of difficulty was the language, since children confused data and result values. Warren (2005) conducted an experiment where Grade 4 students solved several geometric pattern problems including *reversing the thinking* questions. This author reported that this type of questions was very difficult for most children, although she included neither examples nor descriptions of students' answering strategies in the paper.

The context of geometric pattern problems seems especially interesting in promoting access to pre-algebraic concepts in mixed-ability classrooms, since all students, regardless of their mathematical ability and previous knowledge, may explore geometric patterns and obtain some answers (Smith et al. 2007). This context has been shown to be particularly useful for gifted students, since they may advance faster and further than average students. However, there have only been a few publications reporting gifted students' behavior when solving these problems. Amit and Neria (2008) confirmed that generalization via pattern problems is an appropriate gateway to developing the algebraic skills of gifted students. Fritzlar and Karpinski-Siebold (2012) explored the algebraic abilities of primary school students aged 9–10 of varying performance levels, including gifted students. As expected, the more able students got the better results, although none were able to adequately answer questions about the nth term of sequences. Benedicto et al. (2015) asked gifted children in Grades 5, 6 (primary school), and 7 (secondary school) to calculate the values of the 5th, 20th, 100th and nth terms in a triangular numbers pattern. They found that some students in the sample were not able to obtain any kind of expression for the nth term, others verbalized a recursive expression (the nth term is obtained by adding the numbers from 1 to n), and the most talented were able to write an algebraic expression to calculate the nth term (the value of the nth term is $n + (n + 1)/2$). These results prove that some questions commonly asked in geometric pattern problems may be appropriate for some students but not for others, even among gifted students, so teachers need to evaluate the suitability of questions to students. Therefore, again, a reliable criterion to decide on which questions are appropriate for specific gifted students is needed.

Giving a meaning to equations and learning to solve linear equations is one of the possible objectives of posing students geometric pattern problems. When they start learning to solve linear equations, students are often asked to solve simple equations such as $3x + 4 = 19$. However, in the context of geometric pattern problems, students may also be faced with solving complex equations such as $2(x + 1) + 2(x + 2) = 86$. Filloy et al. (2008) suggested the existence of a didactical obstacle in between simple and complex linear equations that explains the difficulties students find in solving complex linear equations even when they solve simple equations easily.

There is increasing agreement among mathematics education researchers and teachers that a successful methodology to promote meaningful learning in all students, particularly in gifted students, is to pose them challenging tasks that promote high-level thinking (Silver and Mesa 2011). An issue in this context is to have at our disposal a theoretical tool to discriminate tasks that promote high-level thinking from those that do not promote it. The *cognitive demand* model may evaluate the intellectual effort required when students solve mathematics problems, so it helps decide on which problems are more appropriate to promote high-level thinking in different kinds of students. To assess the power of tasks to help develop students' mathematical thinking, Stein et al. (1996) analyzed a diversity of types of tasks, varying from ones requiring only recall from memory to others requiring complex and original use of mathematical knowledge. To allow teachers to select tasks with an appropriate level of challenge or demand for their pupils, Smith and Stein (1998) stated a set of criteria to classify mathematical tasks or problems into four *levels of cognitive demand* corresponding to different grades of cognitive effort required to solve them.

Most researchers determine the level of cognitive demand of a problem by analyzing the statement of the problem (Boston and Smith 2009; Wijaya et al. 2015), but this procedure does not acknowledge that a problem may be solved correctly in several ways that require from students different degrees of cognitive effort. Instead, we have adopted an original approach that uses the levels of cognitive demand to also make an analysis of students' answers to those problems. In this way, we can better understand their processes of reasoning and decide on the appropriateness of tasks (Benedicto et al. 2015). This way of using the cognitive demand model has proved in our research experiments to be a framework that reliably identifies problems appropriate to students with diverse mathematical capabilities, in particular to gifted students. It has also allowed us to analyze individual students' answers to different problems, providing information about the students' learning trajectories.

To use the levels of cognitive demand to analyze students' answers to geometric pattern problems, we have rephrased the generic characteristics of the levels to make them appropriate to the particularities of the geometric pattern problems posed and the answers to their questions.

The issues that emerged in this introduction are related to researchers' interest in knowing how gifted students solve problems and progress in learning more abstract and complex strategies and how to determine which questions are appropriate to promoting gifted students' high-level thinking while learning and understanding mathematics (pre-algebra in particular). The objective of this chapter is to offer some answers to these questions in the context of geometric pattern problems and pre-algebra, to gain knowledge about gifted students' behavior, and to evaluate the cognitive effort required to solve the problems posed to them. This information could help teachers prepare sets of problems tailored to the particular needs and expertise of their gifted pupils. The specific objectives of the research we present here are:

(i) To identify and analyze the solution strategies gifted students use to solve geometric pattern problems and the evolution of these strategies throughout the course of a teaching unit.
(ii) To analyze the relationships between types of geometric pattern problems and the cognitive demand required by gifted students' solution methods.
(iii) To analyze the relationships between the complexity of the generalizations made by gifted students and the cognitive demand required by their methods to answer the inverse relationship tasks.

To provide information on these objectives, we carried out a teaching experiment aimed at guiding a nine-year-old gifted student to the ultimate learning objective of being able to solve verbal algebraic problems based on linear equations. The intermediate objectives were to help the student: understand and learn the process of mathematical generalization, contextualized in geometric patterns; start managing the basic components of algebraic reasoning in order to learn to translate verbal descriptions of the general terms of sequences into algebraic expressions; and learn and understand a meaning of linear equations and the procedures for solving them.

7.2 Theoretical Framework

The description and analysis of the teaching experiment presented in the next sections is based on three elements that integrate our theoretical framework: the geometric pattern problems, which are the environment where the teaching experiment took place and the student's algebraic thinking arose; the cognitive demand model, which is the analytic tool used to interpret and categorize the different instances of cognitive effort made by the student participating in our experiment when solving the problems we posed him; and the characteristics of mathematical giftedness, since we have observed a gifted student's behavior to identify traits of giftedness present in his mathematical activity that explain the student's success in learning algebraic language and solving linear equations and algebraic word problems.

7.2.1 The Solution of Geometric Pattern Problems

Radford (2000) differentiated algebraic thinking from algebra; the former refers to the use, possibly intuitive, of basic algebraic concepts such as unknown, variable, and generalization (i.e., expressions of the general term of a sequence) without employing the algebraic symbolic system of signs, and the latter refers to the explicit use of the analytic ways of representing and managing the above mentioned concepts in contexts such as solution of equations. According to Radford (2010), algebraic thinking may adopt different forms depending on the kinds of tasks posed. Our research is situated in the context of algebraic thinking, so it is necessary to characterize the particularities of our teaching experiment.

Geometric pattern problems typically show, as data, a pictorial representation of the first terms of an increasing sequence of natural numbers (see some examples below), although some authors may present as data non-consecutive terms. The problems pose students *direct questions* (Amit and Neria 2008) about some terms of the sequence, usually asking them: to calculate the values V_n of *immediate*, *near*, and *far* terms (Stacey 1989); to *verbalize a general rule* valid for calculating any specific term; and to *write an algebraic expression* for such a rule [i.e., to write an algebraic expression in mathematical terms for the function $V_n = f(n)$]. Students may also be asked *inverse questions*, consisting of calculating specific cases of the *inverse relationship* (Rivera 2013), that is, to get the place n of a term given its value V_n (i.e., mathematically speaking, to solve the equation $f(n) = V_n$).

The pictorial representation of the sequence provides students with objects carrying numerical information and graphically exhibiting the algebraic relationship between the terms of the sequence, that students can identify and induce in different ways (Amit and Neria 2008; Rivera and Becker 2005). Rivera and Becker (2005) differentiated between *figural* and *numerical* procedures of using the information provided by the geometric patterns, depending on whether students use the graphical representation of the terms to answer the questions or only pay attention to the numerical values of those terms (i.e., the number of elements in the pictures of the terms), respectively.

According to García-Reche et al. (2015), students use several strategies to get the numeric answers to the direct questions:

Counting: Students reproduce the graphical pattern by drawing the requested term and count the number of elements in the new drawing to get the numeric answer. Students do not use any mathematical property of the sequence to get the answer.
Recursive: Students identify the graphical or numerical pattern of growth in the sequence, relating each term to the preceding one(s). They then calculate the terms one by one until they get the requested term. This strategy is helpful in calculating immediate and near terms, but it is too time consuming when calculating far terms and does not work to get the general term of the sequence.
Functional: Students identify a mathematical expression that allows them calculate the value of any specific term of the sequence. This expression can also be used as the general term of the sequence.
Proportional: Students use the ratio between the values of two specific terms to calculate the value of any other term of the sequence. For instance, if V_3 is 5 times V_1, then V_9 is 15 times V_1. This strategy usually produces wrong answers.

7.2.2 The Cognitive Demand Model

The cognitive demand model identifies four levels of cognitive effort required from students to solve mathematical problems based on the complexity of the reasoning the students used to produce the answers. The basic characteristics of the tasks typically associated with each level are (Smith and Stein 1998):

- *Memorization*: tasks asking students to reproduce facts, rules, formulas or definitions previously learned or information explicitly presented in the statement of the task.
- *Procedures without connections* to concepts or meaning: tasks focused on getting correct answers but not on connecting to the underlying contents, requiring students to perform in a routine manner an algorithmic process already learned.
- *Procedures with connections* to concepts and meaning: tasks focused on discovering the underlying contents and gaining mathematical understanding of them, requiring students to perform an algorithmic process that is not routine, since it presents some ambiguity on how to carry it out.
- *Doing mathematics*: tasks requiring complex and non-algorithmic thinking from students. Students have to understand the underlying mathematical contents and explore their relationships.

A more detailed and operational description of the levels of cognitive demand can be found in Smith and Stein (1998). As in the cognitive demand model, in this text the terms *algorithm* and *procedure* are equivalent, and they should be understood in a broad way, including the well-known algorithms for arithmetic calculations, solving equations, getting the derivative of a polynomial function, etc., and also any procedure to get a result by means of a purposeful sequence of steps, such as calculating the angles of a convex polygon with a protractor or drawing the element of a geometric pattern following the terms given.

7.2.3 Particularization of the Cognitive Demand Model to the Geometric Pattern Problems

The characterization of the levels of cognitive demand by Smith and Stein (1998) is good to give a general idea of their meaning and to show the main differences between them, but it is not sufficiently precise to evaluate geometric pattern problems or students' answers to these problems. To make operational and helpful use of the levels of cognitive demand to analyze specific students' answers, we needed to particularize the general descriptions of the levels to the specific context of the geometric pattern problems. To do this, we matched up the core characteristics of each level of cognitive demand to the specific characteristics of the questions posed in these problems and rephrased the characteristics to include aspects of geometric pattern problems. We present below a synthetic analysis of each type of task included in the geometric pattern problems used in our teaching unit and then a table with the detailed characteristics of each level of cognitive demand in this context. A thorough description and validation of the process of transformation of the initial characteristics of the levels into the specific characteristics can be read in Benedicto et al. (2017).

The ordinary procedures of calculating immediate and near terms of geometric patterns require only continuing the numeric or geometric structure shown by the terms given in the statement of the problem, either by drawing the requested term and counting its elements or by recursively determining its value (for instance, by adding 3 again and again). To do this, students do not need to be aware of the algebraic relationship underlying the pattern (i.e., the general term of the sequence) and only need to make a limited cognitive effort to do such calculations, which corresponds to the procedures without connections level of cognitive demand.

Far terms cannot be calculated without finding an algebraic relationship underlying the sequence and using it. To solve these tasks, typical students analyze previous information (terms in the statement of the problem and immediate and near terms already calculated) to connect relevant data from them and get a relationship or rule that can be used to calculate the value of any other specific term. This is not a routine procedure, since it requires using underlying algebraic relationships and has to be carefully applied in different ways to different patterns. Therefore, the cognitive effort necessary to calculate far terms corresponds to the procedure with a connections level of cognitive demand.

To verbally or algebraically express a general rule for calculation of terms, students need to find an expression for the algebraic relationship underlying the sequence (i.e., its general term) and be able to abstract that relationship in order to express it without the support of specific terms. There is not an algorithmic guide that can help primary school students to solve this kind of task, so they need complex non-algorithmic thinking to analyze the task, extract useful information, and make appropriate use of it. All this activity requires originality and a considerable cognitive effort from students, which corresponds to the doing mathematics level of cognitive demand.

To calculate inverse relationships, typical students' answers are based on the rules of generalization obtained in direct questions. The rules may have a simple algebraic structure (for instance, $V_n = 3n + 2$) or a complex one (for instance, $V_n = 3n + 2(n + 1) + 1$), so the levels of cognitive demand required to solve the inverse tasks vary depending on the complexity of the generalization made. When the rule of generalization has a simple structure, students simply need to make arithmetic calculations in the appropriate order, determined by an easy algorithm, so students have to make limited cognitive effort, which corresponds to the procedures without connections level.

When the rule of generalization has a complex structure, students need to explicitly state and solve an equation to get the answer. When these tasks are posed before students know how to solve equations, they can only find the solution by making a (sometimes carefully organized) trial-and-error checking of possible values. This process of solution is quite straightforward and it does not require understanding of the algebraic structure of the sequence, so it only requires limited cognitive effort, which corresponds to the procedures without connections level.

When students have learned to solve equations, they may state and solve an appropriate equation derived from the rule of generalization. Stating a correct equation cannot be made without understanding the algebraic relationship underlying the pattern, so it requires a quite high cognitive effort, which corresponds to the procedures with connections level.

Table 7.1 presents the characterization of the levels of cognitive demand particularized to geometric pattern problems (Benedicto et al. 2017) that we have used to analyze the student's outcomes presented in next sections. Table 7.1 does not include the level of memorization because it is not used in the analysis made in this chapter.

Table 7.1 Characterization of the cognitive demand of answers to geometric pattern problems

Levels of Cogn. Dem.	Categories	Characteristics of the task
Procedures without connections	Process of solution	• Is algorithmic. The procedure consists of following the pattern shown in the statement to calculate (either recursively or functionally) immediate or near terms, either graphically by drawing the terms and counting their elements, or arithmetically by calculating the number of elements of the terms. However, the students do not understand the underlying algebraic structure of the sequence. The calculation of the inverse relationship is based on a learned sequence of basic arithmetic operations or on checking possible answers by trial and error
	Objective	• Focus students' attention on producing a correct answer (the number of elements in an immediate or near term), but not on developing understanding of the algebraic structure of the sequence
	Cognitive effort	• Solving it correctly requires limited cognitive effort. Little ambiguity exists about what has to be done and how to do it, because the statement clearly shows how to continue the sequence
	Implicit content	• There is implicit connection between the underlying structure of the sequence and the procedure used. However, students do not need to be aware of such connection since they may answer the question by drawing terms and counting their items
	Explanations	• Requires explanations that focus only on describing the procedure used. It is not necessary to identify the relationship between the answer and the term
	Representation of solution	• A geometric representation is used to get the number of elements and an arithmetic one to write the result. Students use the representations without establishing connections between them or with the algebraic structure of the sequence
Procedures with connections	Process of solution	• The data or the answers to previous tasks suggest general functional procedures that are connected to the underlying algebraic structure. The students understand the algebraic structure of the sequence and they use it to solve the task, but

(continued)

Table 7.1 (continued)

Levels of Cogn. Dem.	Categories	Characteristics of the task
		they are not able to obtain a general algebraic expression. The calculation of the inverse relationship is based on solving the equation of the general procedure previously obtained
	Objective	• It directs students' attention to the use of general procedures aiming to deepen their understanding of the underlying algebraic structure of the sequence
	Cognitive effort	• Solving it correctly requires rather considerable cognitive effort. Students may use a general procedure, but they need to have some understanding of the algebraic structure of the pattern
	Implicit content	• To solve the task, students need to explicitly consider the algebraic relationship between any term and its value underlying the correct procedures of solution
	Explanations	• Requires explanations referring to the general algebraic relationship between the terms and their values, based on using specific cases (particular terms of the sequence)
	Representation of solution	• The solution connects several representations. Geometric, arithmetic, and algebraic representations may be used, and students use those which help them to make an abstract reasoning
Doing mathematics	Process of solution	• Requires complex and non-algorithmic thinking. The statement does not suggest any way to get the general term of the sequence. Students have to understand and analyze the algebraic structure of the sequence to get a general algebraic expression that lets them obtain any term of the sequence
	Objective	• Requires students to analyze the solutions to previous tasks and possible limitations to get an algebraic expression of the general term of the sequence
	Cognitive effort	• Requires considerable cognitive effort, since it is necessary to use abstract reasoning to determine how to algebraically represent the general term
	Implicit content	• Requires that students access relevant knowledge and previous experiences (immediate, near, and far terms) and make appropriate use of them in working through the task to get an algebraic expression of the general term
	Explanations	• Explanations consist of the proof of the algebraic expression of the general term
	Representation of solution	• The solution is based on an algebraic representation, which may be connected to geometric and/or arithmetic representations

7.2.4 *Characteristics of Mathematically Gifted Students*

As mentioned in Sect. 7.1, researchers have determined quite many of the characteristics of gifted students' behavior. A few of these characteristics are pertinent to the differential particularities of geometric pattern problems (Greenes 1981; Krutetskii 1976; Miller 1990):

- An unusual quickness in learning, understanding, and applying mathematical ideas: Gifted students understand and learn very quickly. They usually need only a few explanations; they even grasp new ideas before the teacher has finished explaining them.
- Ability to see mathematical patterns and relationships, sometimes in original ways: Gifted students have a high capability for identifying regularities and complex structures in patterns, extracting them from empirical contexts, and characterizing them in general terms.
- Ability to generalize and transfer mathematical ideas to another context: Gifted students are able to detect general relationships when observing specific cases and are able to extract the relationships they have identified in specific contexts and formulate them in general terms.
- Ability to invert mental procedures of mathematical reasoning: Gifted students are able to manage unidirectional relationships in ways that allow them to see whether they can be inverted and, when possible, the direct or the inverse relationship can be used. They are then able to create new procedures by inverting the steps in known procedures.
- Flexibility to change: Gifted students are able to quickly move from one problem-solving strategy to another if they believe that the new one will be more useful or easier.
- Development of efficient strategies and abbreviation of problem-solving processes: Because gifted students tend to see better procedures of solving a type of problem, they are able to more efficiently solve other problems of the same type.

We present in this chapter a research experiment where a nine-year-old gifted student solved a teaching sequence based on geometric pattern problems aimed to teach him to generalize and to solve verbal algebraic problems based on linear equations. The analysis of the student's behavior presented in next sections will show that he had the above mentioned traits of mathematical giftedness and he put them to work when solving the different kinds of problems.

7.3 The Research Methodology

This research is based on an experimental case study, analyzed qualitatively to provide answers to the specific objectives stated in the first section. We present and analyze data from a teaching experiment with a gifted student who solved a sequence of problems aimed at guiding him in understanding and learning the basic algebraic concepts necessary to solve verbal problems based on linear equations. Most problems in the sequence were geometric pattern problems.

7.3.1 Sample and Experimental Setting

The subject participating in this study was Juan (a pseudonym), a 9 year old who had been identified as gifted student after having been administered the standard identification procedure used by the educational authority. Furthermore, Juan had proved to have a very high mathematical talent during his participation in several mathematics workshops conducted by the authors over several years. In the Spanish educational system, 9-year-old children are typically in primary school Grade 3, but Juan had been accelerated one grade. We invited Juan to participate in our study because he had showed a high interest for mathematical problem solving and he was willing to learn more complex mathematics.

The teaching experiment was an out-of-school workshop having the format of clinical interviews, which were conducted by the fourth author. It started in August during the summer holidays after Juan had finished Grade 4 and ended in December of the same year, when Juan was studying Grade 5. It was not possible for the researcher-teacher and the student to meet each other, so the sessions were conducted by means of videoconferences using Skype, which were audio- and video-recorded using screen-capture software.

As the first action to start working on a problem, the teacher posted a document for Juan with the statement of the problem. Juan could draw or write in a notebook, but he had to answer and give explanations verbally, except when he had to write algebraic expressions. The teacher asked Juan to explain his answers whenever he did not do it spontaneously. When Juan started writing algebraic expressions, he used a word processor and shared his screen with the teacher so she could read what Juan was writing. The information written in the notebook was not relevant for our analysis because it mostly consisted of calculations that Juan described verbally to the teacher when necessary.

7.3.2 The Teaching Unit

The experimental teaching unit consisted of a sequence of problems divided into three parts. To easily identify the problems in each part of the teaching unit in this text, we have labeled them as Problems 1.*n*, 2.*n*, and 3.*n*. Some problems were taken from the literature and the others were created by the authors to fit specific requirements for methods of solution or structure of the general relationships of their sequences.

The first part had the objectives of (i) teaching Juan to identify and verbally express generalizations of the relationships underlying the sequences represented by the patterns and handle those generalizations, and (ii) helping Juan start implicitly using variables and unknowns while solving the problems. Over nine sessions, Juan solved 20 geometric pattern problems ordered according to their difficulty. All problems had the same structure and included the same tasks (Fig. 7.1):

Marc and his friend want to make sets of houses with sticks as follows:

a) How many sticks will they need to make 6 houses? How did you know?

b) How many sticks will they need to make 11 houses? How did you know?

c) Can you tell me a way to calculate how many sticks will they need to make 44 houses? How did you know?

d) If they have 51 sticks, how many houses can they make? Explain how you got the answer.

Fig. 7.1 A typical geometric pattern problem (1.8) from the first part of the teaching unit

Three direct questions (a–c) asking for calculation of the values of an immediate, a near, and a far term in the sequence, and an inverse question (d) asking for the term in the sequence having a given value.

From the very beginning, Juan correctly solved most problems, showing a high ability to generalize from the data presented in the problems, which is one of the previously mentioned characteristics of mathematical giftedness.

The second part of the teaching unit aimed at (i) introducing Juan to the use of algebraic symbols (letters, equal sign, parentheses, etc.) to algebraically represent his generalizations and (ii) introducing Juan to the solution of linear equations contextualized by an applet representing a balance. Over three sessions, Juan solved six geometric pattern problems different from the ones in the first part. All problems had the same structure and included the same tasks (Fig. 7.2): A direct question (a) asking for calculation of a near term of the sequence, a question (b) asking a written algebraic representation of the calculations made in question a, and an inverse question (c) aimed to be solved using an equation. The aim of these problems was no longer to teach Juan to get generalizations, so we removed the

At the school we have learned how to build wooden cabinets having as many shelves as you like. We used pieces of wood to make the cabinets in this way:

a) How many pieces of wood do we need to make a cabinet with 13 shelves? How did you know?

b) Write down the formula you used in the previous question.

c) If we have 98 pieces of wood, how many shelves can the cabinet have? How did you know?

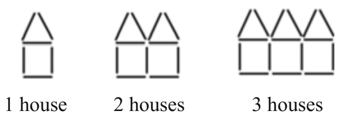
Fig. 7.2 A typical geometric pattern problem (2.5) from the second part of the teaching unit unnecessary questions and included new questions focusing on the algebraic representation of the general relationship.

To answer question b, Juan learned to write algebraic expressions representing the calculations made in question a. Next, to answer question c, Juan learned to write an equation by using his answer to question b and the data in question c. In the first problem of this part, Juan was introduced to the meaning of equation as equilibrium using an applet (NLVM 2016) showing a balance that allows representation and solution of linear equations by adding pieces to or removing them from the balance beams. When there is not equilibrium, the balance swings down. Juan quickly learned to write algebraic expressions and solve linear equations with the help of the balance model, showing one of the traits of mathematical giftedness.

The third part of the teaching unit aimed at (i) teaching Juan to transform algebraic expressions and simplify complex linear equations, (ii) teaching Juan solve algebraic word problems and gain practice in solving linear equations, and (iii) showing Juan the usefulness of equations in solving a diversity of mathematical problems. Over two sessions, Juan solved seven geometric pattern problems, including shortened versions of six problems from the first part of the teaching unit in which Juan was not able to correctly solve the inverse question or he solved it by trial and error. He also solved four linear equations to gain practice and six algebraic word problems. All geometric pattern problems had the same structure and included the same tasks (Fig. 7.3): A question (a) asking for a written algebraic representation of the general term of the pattern, a question (b) asking about the possibility of shortening the algebraic expression produced in a, and an inverse question (c) asking for a transformation of the expression in b into an equation and its solution.

In objective (iii) of this part of the teaching unit, the geometric pattern problems used were the ones from the first part of the teaching unit whose inverse questions Juan had found very difficult to solve and on which he had had to use trial and error (Table 7.3). This showed Juan that having learned to state and solve equations allowed him to solve these problems easily.

Marc and his friend want to make sets of houses with sticks as follows:



a) Write down an algebraic formula to calculate the number of sticks necessary to make any number of houses.

b) Do you believe that it is possible to get a simpler formula? If so, write it down.

c) If they have 96 sticks, how many houses can they make? Explain how you got the answer.

Fig. 7.3 A typical geometric pattern problem (3.5) from the third part of the teaching unit, related to problem 1.8 (Fig. 7.1)

7.3.3 Source and Analysis of Data

In this chapter we analyze Juan's answers to the geometric pattern problems he solved in the three parts of the teaching unit, but we do not take into consideration the word problems or the linear equations he solved in the third part. We make a multi-faceted analysis of those answers based on the three theoretical constructs described in the second section of the chapter. The main analysis is based on matching the answers to the characteristics of the levels of cognitive demand in Table 7.1, in order to get information about the cognitive effort required to solve the problems and look for a learning trajectory throughout the course of the teaching unit. However, we have also analyzed Juan's answers in relation to his procedures of using the graphic data of the problems (numerical and figural; Rivera and Becker 2005), the procedures he used to calculate specific terms of the sequence (counting, recursive, functional, and proportional; García-Reche et al. 2015), and the appearance of traits of mathematical giftedness. This multi-faceted analysis provides a more complete picture of this student's behavior and helps in understanding why he made more or less cognitive effort in the solution of different problems.

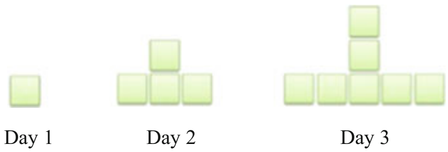
7.4 Analysis of the Cognitive Demand of Student's Solutions

In this section, we present examples of the various types of strategies Juan used to solve the geometric pattern problems and, based on Table 7.1, analyze the levels of cognitive demand necessary to produce those answers. Due to the differences between strategies of solution of direct and inverse questions, we present them in two different sub-sections.

7.4.1 Analysis of the Cognitive Demand of Solutions to Direct Questions

Two relevant aspects of the strategies of solution of direct questions in geometric pattern problems are the ways of using the graphical information and the procedures used to calculate the values of specific terms and obtain the general relationships needed to calculate any term. We are considering Rivera and Becker's (2005) figural and numerical procedures of use of the pictorial information. Juan used one or the other depending on the complexity of the geometric pattern of each problem. We are also considering the counting, recursive, functional, and proportional types of calculations described by García-Reche et al. (2015). Examples 1–4 present a diversity of answers corresponding to the different types, showing that Juan made calculations of the recursive and functional types, but he never used the counting and proportional types.

My mother has bought a strange plant. It grows during the night and, when we get up in the morning, we see new leaves:



a) How many leaves will the plant have on day 5? How did you know?

b) How many leaves will the plant have on day 13? How did you know?

c) How many leaves will the plant have on day 65? How did you know?

Fig. 7.4 Problem 1.7

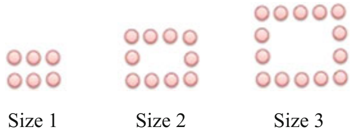
Example 1: *Recursive numerical counting* from the pattern and a cognitive demand in the *procedures without connections level*. In a few problems, Juan identified the constant difference between the total number of objects in two consecutive given terms and recursively used this difference to get the answers for the immediate or even the near terms. In problem 1.7 (Fig. 7.4), in order to calculate the number of leaves the plant has after 5 and 13 days, Juan made recursive calculations:

Juan: On the fifth day it has 13.
Teacher: Very good. How do you know?
Juan: I was adding 3.
Teacher: And what about day 13?
Juan: 38 [the correct answer is 37, but he made a mistake in the calculations].
Teacher: It is 37. How did you calculate this?
Juan: Plus 3, plus 3, plus 3.

The geometric pattern clearly suggests that each day the plant has three new leaves, so the geometric pattern induced Juan to use a recursive strategy to solve the immediate and near terms. However, when he had to answer question c, Juan moved to a functional strategy. The recursive strategy is algorithmic, since it consists of adding 3 a number of times and requires a very low cognitive effort since following it does not require awareness of the algebraic relationship underlying the sequence. The objective of this solution method was producing a correct result but not gaining understanding of the structure of the sequence. Juan's answers to a and b were typical of the procedures without connections level.

Example 2: *Functional numerical counting* from the pattern and cognitive demand in the *procedures without connections level*. In some problems, Juan counted the total number of objects in each given term and worked with the numeric sequence without paying attention to the graphical information of the pattern. In problem 1.5, in order to calculate the number of tiles around a pool of size 5 (Fig. 7.5), Juan made a guess about a general functional relationship:

We want to build a swimming pool with tiles around it. We want to know how many tiles will be needed for pools with different sizes:



a) How many tiles do we need for the pool of size 5? How did you know?

Fig. 7.5 Problem 1.5

Juan: We need 22 tiles.
Teacher: Why?
Juan: I discovered that it [the number of tiles] is 4 times the size of the pool plus 2.
Teacher: How did you discover this?
Juan: Looking at the examples. The first pool has 6 [tiles], and $4 \times 1 + 2 = 6$. The second has 10, and $4 \times 2 + 2 = 10$. And the third has 14, and $4 \times 3 + 2 = 14$.

Juan either was not able to find an adequate decomposition of the geometric pattern or directly got the numeric values of the given terms and used trial and error to look for an arithmetic way to relate the value of each term to its position. Despite the result he obtained, consisting of finding a general formula, he did not show an awareness of the underlying algebraic structure of the sequence, since he described the procedure used to solve the task but was not able to give a reason for it. The strategy of the solution chosen may have required rather cognitive effort to produce the general relationship, but it did not require understanding the structure of the sequence. The objective of this solution was focused only on producing a correct result. This solution then corresponds to the procedures without connections level.

Example 3: *Functional figural decomposition* of the pattern with a cognitive demand in the *procedures with connections level*. Most geometric patterns show procedures to split the figures into parts that can be considered like independent patterns, making it easy to find a general procedure to calculate the values of the terms in the sequence. This is Juan's explanation of his method of calculating the number of chairs around the tables in question a of problem 1.12 (Fig. 7.6):

Juan: I think this is correct. I take 3, or whatever number, times 3 and add 2.
Teacher: Fine. How did you get it [this procedure]?
Juan: For one table, [I added] the numbers [of chairs] above and below, 1 and 2, and $1 \times 3 = 3$. For two tables, 3 above and 3 below, $2 \times 3 = 6$ and $3 + 3 = 6$. And we still have to add those two [chairs on the sides]. For three tables, $5 + 4 = 9$ and $3 \times 3 = 9$.

My parents are organizing a family party. There are many people in my family, and my parents do not know how many will come. They have to decide how many tables and chairs they will need:

1 table 2 tables 3 tables

a) How many chairs will there be around 6 tables? How did you know?

d) If there will be 50 guests, how many tables will they need? How did you know?

Fig. 7.6 Problem 1.12

To answer question a, Juan divided the chairs into three sets (top, bottom, and sides), and this helped him elaborate a general procedure to calculate the number of chairs for any number of tables. This answer offers a functional relationship to calculate the number of chairs based on the algebraic structure of the sequence represented by the geometric pattern. The geometric pattern can be easily split, which helps in understanding the algebraic structure of the sequence. This allowed Juan to understand the algebraic structure of the pattern and produce a generalized functional relationship. The strategy of the solution chosen required rather considerable cognitive effort to produce the general relationship. Therefore, this answer is in the procedures with connections level. This transcript is also an example of the functional strategy of solution of geometric pattern problems that Juan used in most problems.

Example 4: *Functional figural decomposition* of the pattern and cognitive demand in the *doing mathematics level*. Question b in the problems in the second and third parts of the teaching unit (Fig. 7.2) asked explicitly for an algebraic expression of the general rule of calculation of terms that should have been derived from question a. This new question meant an increase in Juan's cognitive effort while solving the problems, particularly for those problems whose pattern was more complex. We present Juan's answer to questions a and b of problem 2.5 (Fig. 7.2). After having read question a, Juan spent about 2:50 min thinking about it. Then he said:

Juan: I have a way. In the [cabinet] 1, $1 \times 4 + 2$ [pieces of wood]. In the 2, $2 \times 4 + 2$. In the 3, $3 \times 4 + 2$.
Teacher: Where do the 4 and the 2 come from?
Juan: The 2 is because … in the first … there are two extra pieces on the top of the cabinets … like the roof. And the 4 is because they [the shelves] are 4 and 4 [each shelf].
Juan: [He wrote] $13 \times 4 + 2 = 54$.

Next, to answer question b, Juan started writing:

Juan: $N =$ [then he deleted the text and wrote again] $S =$
Juan: Will you understand if I write shelves?

Teacher: Yes.
Juan: If I write S instead of N?
Teacher: Yes. You can use any letter you like. S for shelves is fine.
Juan: [He wrote] $S \times 4 + 2$.

To answer question a, Juan divided the cabinets into shelves (made of four pieces) and the two extra pieces on the top. This was a quite complex graphical structure that did not suggest a procedure of decomposition to get the general term. The abstract explanation he offered for his answer to question a correctly justified the meaning of the coefficients in the algebraic expression. To answer question b, Juan had to use the information produced in question a and transform it into a functional algebraic expression using the initial letter of *shelf* to give meaning to the algebraic expression. The cognitive effort required to produce the answers was high, since Juan had to decompose the graphical pattern and obtain an algebraic expression for such decomposition, following a non-algorithmic complex way. This solution was in the doing mathematics level. The teaching method of suggesting that students write the initial letter of the unknown instead of its whole name proved to be successful in helping Juan make the transition from verbal expressions to algebraic expressions and understand the use of letters in algebraic expressions. This is clearly seen when he changed the letter N to S because the unknown were shelves.

The examples of solutions presented here demonstrate that Juan was able to interpret correctly most patterns presented to him, showing an ability to interpret them and convert them into meaningful arithmetic or algebraic information after a process of generalization from the particular cases worked out. This behavior is quite different from what is observed in average students in Grade 5, therefore showing traits of mathematical giftedness related to identification of patterns and generalization of mathematical ideas.

7.4.2 Analysis of the Cognitive Demand of Answers to Inverse Questions

As explained in Sect. 7.2.3, strategies for calculating inverse relationships are conditioned by students' awareness of algebra (or lack of it) and ability to solve equations (or lack of it). When the teaching experiment started, Juan had not received any previous instruction on algebra, so he neither knew how to write algebraic expressions nor solve equations. He also had not studied square roots. Examples 5–7 demonstrate that, during the first part of the teaching experiment, he showed three strategies to calculate inverse relationships (trial and error, wrong inversion, and correct inversion) that we consider typical of students without knowledge of algebra. Examples 8 and 9 then show the great change that occurred in Juan's ways of approaching and solving the problems when he started to learn algebraic concepts, use its system of signs, and solve equations.

Example 5: *Trial and error direct calculations*, with a cognitive demand in the *procedures without connections level*. When the general procedures Juan had found for the direct questions were of the linear type $y = ax + b(x \pm c) \pm d$ or the quadratic types $y = x^2$ or $y = (x \pm a)(x \pm b)$, Juan was unable to find a procedure to invert such complex expressions and he resorted to trial and error. We present below Juan's answer to the inverse question in the problem of the walls (problem 1.11, Fig. 7.7), where the generalization he got in the direct questions was the functional relationship $V_n = (n + 1) \times 2 + n$.

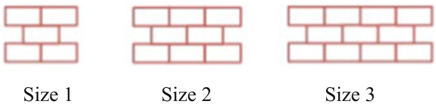
Juan: I had a procedure, but it doesn't work. … I did 38 divided by 2 minus 2.
Juan: [after 1:35 min of making calculations] I believe it is 13.
Teacher: Why?
Juan: I have tried numbers [for the size of the wall].
Teacher: What did you do while you were trying numbers?
Juan: I did $13 + 1 = 14$ … No, sorry, it is 12. Because $12 + 1 = 13$, $13 \times 2 = 26$ and $26 + 12 = 38$.

Trial and error is an algorithmic process that does not connect to the algebraic structure of the pattern and is only aimed at getting the correct answer. Juan did not understand the algebraic structure of the generalization he had obtained in previous questions of the problem, so he could not use it. He made a limited cognitive effort to get the answer, since he only had to make direct arithmetic calculations by checking different values for n until he found the correct one. He used an arithmetic representation that did not help him to connect to the algebraic properties of the sequence. He only correctly answered the question when the teacher helped him. Therefore, this strategy required a cognitive effort in the procedures without connections level.

Even though this trial and error strategy was far from our objectives of learning, Juan showed an ability to look for a different strategy of solution when he was not able to use a more correct one. Furthermore, nobody taught him this trial and error strategy; rather, it was Juan who developed it. Therefore, Juan demonstrated traits of mathematical giftedness related to flexibility in changing his focus and ability to develop efficient strategies to solve problems.

Example 6: *Correct inversion* of the order of operations, with a cognitive demand in the *procedures without connections level*. When the direct calculations were of types $y = ax$ or $y = ax \pm b$, most of the time Juan correctly applied the inverse

A group of masons has to build walls of different sizes, as shown below:



d) If they have used 38 bricks, what size is wall they have built? How did you know?

Fig. 7.7 Problem 1.11

arithmetic operations to get the position of the term, such as in question d of problem 1.12 (see Example 3, Fig. 7.6). To answer the direct questions, Juan used the general relationship $V_n = 3n + 2$. His answer to the inverse question d was:

Juan: It is 16 [tables]. … I did 50 minus 2, which is 48, and then divided by 3.

Inverting arithmetic operations is not just a matter of memory: it is a simple algorithm that can be applied in a straightforward way, requiring very limited cognitive effort because the student only needed to make basic arithmetic calculations. The aim of this procedure is to get a correct solution: the explanation was just a description of the calculations made, and it is not necessary to be aware of the algebraic structure of the sequence to get the correct answer. Therefore, this solution required a cognitive effort in the procedures without connections level.

Example 7: *Wrong inversion* of the order of operations, with a cognitive demand in the *procedures without connections level*. Although Juan correctly solved most problems like the previous example, sometimes he was not aware of the relevance of the order of calculations. In the problem of the friezes (problem 1.10, Fig. 7.8), the generalization he got in the direct questions was the functional relationship $V_n = 2n + 1$.

Juan was first asked to calculate the number of triangles made with 20 sticks (question d). After his wrong answer, the teacher helped him by making him aware of the need to consider the order of calculations.

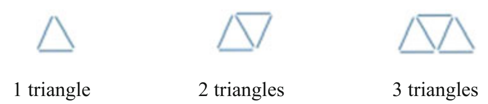
Juan: I did 20 divided by 2 minus 1.
Teacher: Well, … look, before [in questions a to c] you first multiplied by 2 and then added 1. Now, what do you have to do first, subtraction or division?
Juan: Subtraction.

Juan was then asked to calculate the number of triangles made with 31 sticks (question e).

Juan: It is 15. … I subtracted 1 from 31 and got 30, and 30 divided by 2 is 15.

Juan did not understand the algebraic structure of the pattern, which induced him to decide on an incorrect order for the calculations. He made a limited cognitive effort to give a solution because he did not try to analyze the way he had made the calculations in the direct questions. Furthermore, the teacher helped Juan to

John wants to make friezes with sticks to decorate his bedroom. The friezes are made of triangles, as follows:



d) If John has 20 sticks, how many triangles can he make? How did you know?

e) If John has 31 sticks, how many triangles can he make? How did you know?

Fig. 7.8 Problem 1.10

understand the order in which the inverse calculations had to be made, so he repeated the teacher's instructions (first subtraction, then division). Besides, as he did not know how to represent the arithmetic expression algebraically, he had to use the arithmetic representation of the data. So, this solution required a cognitive effort in the procedures without connections level.

At the beginning of the second part of the teaching experiment, Juan learned to state and solve linear equations. From that moment, he was able to answer any inverse question by writing and solving an equation, as shown in Example 8.

Example 8: *Statement and solution* of an equation, with a cognitive demand in the *procedures with connections level*. We present now Juan's answer to the inverse question (c) in problem 2.5 of the second part (Fig. 7.2), which follows the answer we presented in Example 4, where Juan wrote $S \times 4 + 2$ to answer question b. When the teacher asked him to answer question c, he immediately gave the correct answer (24 shelves) by doing the inverse calculations. However, the teacher asked him to solve the question by using algebra.

In previous sessions, Juan had solved problems 2.1–2.4. When solving problem 2.1, the teacher introduced him to the use of the virtual balance (NLVM 2016) to represent and solve equations. The teacher guided Juan to understand the objective of solving equations by maintaining the balance in equilibrium while reducing the number of pieces in order to isolate the unknown. He practiced by solving a few equations ($ax + b = c$) with the virtual balance.

After having solved problem 2.1 with the virtual balance, Juan did not need to use it anymore; instead, he preferred to write a simulation of the virtual balance using the word processor (Fig. 7.9):

Teacher: First, you have to write the balance, like the equation. OK?
Juan: Do I write S times 4 plus 2?
Teacher: S times 4 plus 2 is a side of the balance. Now we know the [number of] pieces of wood. In question a, we knew the number of shelves but not the number of pieces. Now we do not know the number of shelves, so we write S instead of 13. So, what do we need now?
Juan: The balance?
Teacher: We already have a part. We need the other part, OK? … What should we write in the other part [of the balance]?
Juan: X.
Teacher: Look at question a. We had the first part of the balance [she meant $S \times 4 + 2$] and, as we knew how many shelves we had, we used 13 instead of S. Right? And we calculated the number of pieces we needed. Do we now know how many pieces of wood we have?
Juan: Yes.
Teacher: So we write it down, OK?
Juan: [wrote] $S \times 4 + 2 = 98$.

Juan now continued representing the equation on the word processor screen as if it were in the balance (Fig. 7.9a) and solving it by reproducing the compensations

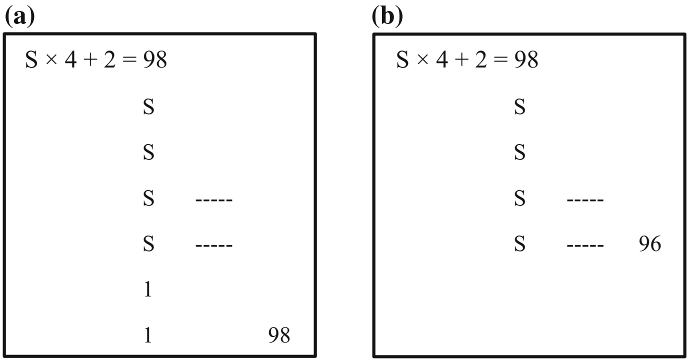


Fig. 7.9 Steps in the solution of problem 2.5, in the second part of the teaching unit

that he would make in the balance (Fig. 7.9b). Finally, he calculated 96 ÷ 4 = 24 shelves.

Juan's first answer to this question was based on arithmetical inversion, like in Example 6, because that procedure was easy for him. However, unlike in Examples 5–7 above, Juan had now learned to state and solve linear equations, so, when asked by the teacher, he was now able to correctly connect the pattern and the generalization he had obtained to the algebraic expression representing it and state (with some help by the teacher) and solve the equation.

Once Juan learned the procedure to write equations for the geometric pattern problems, this procedure became algorithmic for him, since he learned to combine the answers to questions a and b and the data in question c in a meaningful way related to the algebraic structure of the sequence represented, and from this he was able to write the corresponding equation. This algorithmic procedure cannot be followed automatically, but it is necessary to understand the specific algebraic structure of the sequence to decide on the way to write the equation. We see that, in this problem, the solution required Juan to put forth a quite high cognitive effort in order to decide which parts of the information available were useful and how to combine them. Juan used algebraic and diagrammatic (a balance-like diagram) representations to state and solve the equation. Therefore, this solution required a cognitive effort in the procedures with connections level.

Example 9: *Algebraic solution* of a problem that Juan *could not solve arithmetically in the first part of the teaching unit*, with a cognitive demand of the new solution in the *procedures with connections level*. To close this section, we present an example of the solution of a geometric pattern problem from the third part of the teaching unit. As mentioned in the description of the teaching unit, these problems were aimed at showing Juan the power that equations have in solving the problems that he found too difficult during the first sessions.

Problem 3.6 was an advanced version of problem 1.11 (Example 5, Fig. 7.7). As shown in Fig. 7.3, the new version of the problem explicitly asked for an algebraic expression of the general term of the sequence, then it asked for a simplification of that expression if possible, and, finally, it asked an inverse question, to be answered by solving an equation. The algebraic expression Juan wrote in question a was $(S + 1) \times 2 + S$, the same expression he used in problem 1.11. After question b, Juan simplified this formula to $3S + 2$. Question c asked for the size of the wall having 50 bricks, so Juan wrote, without any help:

Juan: $3S + 2 = 50$
$3S = 48$
$S = 48 \div 3$
$S =$ size 16

This answer required a cognitive effort in the procedures with connections level, for the same reasons stated for Example 8. The cognitive effort of a long answer depends on its most complex parts. In this case, for the problems in the third part of the teaching unit, Juan was able to solve question c (state and solve equations) more efficiently, since he had gained practice in operating with parentheses, simplifying the equations, and solving them. However, he had to make the same mental reasoning and the same cognitive effort as in previous problems in order to understand the algebraic structure of the graphical pattern and adequately combine the data to translate them into an equation.

The examples of solutions presented in this section demonstrate that Juan was able to interpret relationships and invert them arithmetically or algebraically. This behavior corresponds to the ability of gifted students related to identification and inversion of mental procedures.

7.5 Discussion on the Analysis of the Student's Answers

In the previous section we have presented examples of the different types of answers to the geometric pattern problems solved by Juan, and we have analyzed them from the viewpoints of the cognitive demand model and algebraic thinking. To discuss this student's behavior and his learning trajectory in the teaching experiment, we present information about the types of answers and the cognitive demand required to answer the direct questions (Table 7.2) and the inversion questions (Table 7.3) in all the problems solved in the first part of the teaching unit. After that, we also analyze the answers to the geometric pattern problems solved in the second and third parts of the teaching unit.

The first striking result from Table 7.2 is that, since the very first problem, Juan was able to solve correctly all direct questions without significant help (although he sometimes made errors in mental arithmetical calculations) by formulating and using appropriate verbalizations of the general term of the sequences. Only in problem 1.17, he was not able to calculate by himself the near and far terms, and the teacher helped him to get a generalization so he could try to answer the inverse question.

Table 7.2 Cognitive demand of the answers to the direct questions in the first part of the teaching unit

Problems	**a) Immediate terms**	**b) Near terms**	**c) Far terms**	**Cognitive demand***	
1.1	Numer. - Recurs.	Figural - Funct.	Figural - Funct.	P. wout c.	P. with c.
1.2	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.3	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.4	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.5	Numer. - Funct.	Numer. - Funct.	Numer. - Funct.	P. wout c.	
1.6	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.7	Numer. - Recurs.	Numer. - Recurs.	Numer. - Funct.	P. wout c.	
1.8	Figural - Recurs.	Figural - Funct.	Figural - Funct.	P. wout c.	P. with c.
1.9	Numer. - Recurs.	Figural - Funct.	Figural - Funct.	P. wout c.	P. with c.
1.10	Numer. - Funct.	Numer. - Funct.	Numer. - Funct.	P. wout c.	
1.11	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.12	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.13	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.14	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.15	Numer. - Funct.	Numer. - Funct.	Numer. - Funct.	P. wout c.	
1.16	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.17	Numer. - Recurs.	No answer	No answer	P. wout c.	- -
1.18	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.19	Numer. - Funct.	Numer. - Funct.	Numer. - Funct.	P. wout c.	
1.20	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	

* Levels of cognitive demand: *P. wout c.* = procedures without connections; *P. with c.* = procedures with connections.

Furthermore, Juan made calculations of recursive and functional types, but he never used the counting and proportional types. He only used valid mathematical procedures to solve the problems, trait related to the unusual quickness in learning, understanding, and applying mathematical concepts typical of gifted students.

The facts that Juan had never had prior contact with mathematical generalization or algebra and that he had never solved geometric pattern problems before this teaching experiment clearly point to the presence of some traits characteristic of mathematically gifted students. Some of these are the ability to recognize mathematical patterns and structures, the ability to generalize mathematical ideas, and the ability to invert mental processes. It is clear from Table 7.2 that these abilities were very well developed in Juan.

Table 7.2 also shows that the student did use functional procedures to solve most of the 20 geometric pattern problems. He used the recursive strategy to find the immediate term in five problems and to find the near term in one of those problems. This is a clear sign of high mathematical talent, since this flexibility in changing the basic strategy of solution to a more efficient one is not found in average Grade 5 students. However, it is most interesting to note that he used this method of solution even to calculate the immediate terms, for which it is not necessary. He soon noticed that the last questions in the problems required a generalization, so when he started solving a problem, he worked to find a general rule and he applied it to calculate all the terms, exhibiting his ability to develop efficient and shorter problem-solving strategies.

Table 7.3 presents the types of solutions to the inversion questions in the first part of the teaching unit. As showed by Examples 5–7, all the solutions are in the procedure without connections level, since both arithmetic inversion and arithmetic

Table 7.3 Types of answers to the inversion questions in the first part of the teaching unit (two types in the same problem mean two attempts of solution)

Problems	d) Inverse questions	
1.1	Correct inversion	
1.2	Trial and error	
1.3	Wrong inversion	
1.4	Correct inversion	
1.5	Correct inversion	
1.6	Wrong inversion	Trial and error
1.7	Trial and error	
1.8	Trial and error	
1.9	Trial and error	
1.10	Wrong inversion	Correct inversion
1.11	Wrong inversion	Trial and error
1.12	Correct inversion	
1.13	Trial and error	
1.14	Trial and error	
1.15	Trial and error	
1.16	Trial and error	
1.17	- -	
1.18	Trial and error	
1.19	Trial and error	
1.20	Trial and error	

trial and error procedures were applied without needing understanding of the algebraic structure underlying the geometric patterns representing the sequence.

Table 7.3 shows other traits of giftedness, as the abilities to invert mathematical processes and develop efficient strategies. In this case, the efficient use of such abilities by Juan was limited by his ignorance of algebraic language and linear equations. However, Juan managed to find an alternative procedure to solve the inversion questions, since he consistently used arithmetical inversion, and, when this strategy was not useful due to the complexity of the general expression he had found (Table 7.4), he resorted to a careful trial and error. The very few publications analyzing students' answers to inverse questions in geometric pattern problems have not yet described a new behavior that we found in our experiment: a student systematically using, from the very beginning, an *organized trial and error* procedure to solve inversion questions. He tried a possible solution, and, if it was too big or small, he tried a smaller or bigger number, continuing this process until he found the correct solution.

Cai and Knuth (2011) described several learning trajectories of students solving geometric pattern problems, but none of them fit Juan's behavior. Tables 7.2 and 7.3 show that Juan was quite consistent throughout the course of the teaching unit in his methods of solving problems with shared characteristics (for instance, problems with the same grade of algebraic complexity), which is another contribution of this research to the knowledge about gifted students.

The problems in the second part of the teaching unit (Fig. 7.2) were focused on introducing our student to the basic concepts of algebra, to algebraic symbolization, and to the statement and solution of linear equations. On the direct questions of problems 2.1–2.3, Juan needed some help from the teacher to correctly write the algebraic expressions—in particular the use of parentheses—but he no longer needed help in the three last problems. To represent the unknown, he even wrote the initial letter of the objects presented in the pattern (days, shelves, etc.). In the direct questions, as shown by Example 4, Juan worked at the doing mathematics level, since he was discovering new algebraic ideas. On the inverse questions, all answers consisted of stating and solving an equation based on the data in the question, so the cognitive demand in all the student's answers was in the procedures with connections level.

The problems in the third part of the teaching unit (Fig. 7.3) were all solved by stating an algebraic expression for the generalized relationship in the sequence and

Table 7.4 Solutions to the inverse questions in the first and third parts of the teaching unit

Problems	Generalization	Inverse questions	Problems	Inverse questions
1.1	$y = ax$	Correct inversion	3.1	Equation
1.2	$y = ax + b$	Trial and error	3.2	Equation
1.6	$y = ax + b(x + c)$	Wrong inv.—Tr. & err.	3.3	Equation
1.7	$y = ax + b(x + c)$	Trial and error	3.4	Equation
1.8	$y = ax + b(x + c)$	Trial and error	3.5	Equation
1.11	$y = ax + b(x + c)$	Wrong inv.—Tr. & err.	3.6	Equation
1.16	$y = ax - b(x - c) + d$	Trial and error	3.7	Equation

then stating an appropriate equation and solving it, so the student worked at the procedures with connections level. Table 7.4 presents a comparison of the strategies of solution of the inverse question in the problems posed both in the first and third parts of the experiment. After the student had all the necessary knowledge and abilities to understand and use the algebraic approach to the geometric pattern problems, he used them confidently to solve those problems without difficulty.

In line with gifted students' quickness in learning and understanding mathematical concepts, Juan only needed the help of the virtual balance in very few problems, since he began to solve question c by imagining the balance and writing the transformations of equations paralleling the manipulations made in the balance. This internalization of the balance and the way he represented it on the word processor screen (Fig. 7.9) is a consequence of the giftedness trait of development of efficient strategies.

7.6 Conclusions

We have presented the case of a nine-year-old student in primary school Grade 5 who worked on an experimental pre-algebra teaching unit. As an answer to the first research question, we have shown that, as the experiment advanced, the student also progressed in his learning of the different concepts, structures, and procedures necessary to meaningfully learn basic algebra and linear equations and appropriately modified his strategies to solve the problems. This experiment shows that mathematically gifted students are much faster than average students in understanding and learning mathematical contents, but they also need to be taught mathematics.

We have used the cognitive demand model to evaluate a student's cognitive behavior in the different questions in the problems and also in the consecutive parts of the teaching unit. Related to the second research question, we have shown that, in trying to solve the problems during the first and second parts of the teaching unit, the student made all necessary cognitive effort, as much as was possible due to his limited knowledge of algebra. The model has proved to be useful to differentiate the cognitive effort required from the student by the different types of questions posed.

As for the third research question, the student exhibited ability to adapt his solving strategies and his cognitive effort to the mathematical complexity of the generalizations he had obtained and the algebraic tools he had learned at each moment. By the end of the teaching unit, the students showed the ability to work confidently on solving linear equations and algebraic word problems. This behavior is also typical of mathematical giftedness.

This research shows only the case of one student, so we do not suggest that this behavior may be generalized; however, it has some similarities to—and also differences from—other cases reported in the mathematics education literature. This research experiment was done in a laboratory context, but we believe that the teaching unit may be modified to adapt it to the context of ordinary schools, where it could be useful for teachers of the upper primary grades and lower secondary grades when starting to teach pre-algebra to all their pupils, but with the possibility of paying special attention to the gifted students.

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