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ANALYSIS OF VISUALIZATION AS AN INDICATOR OF MATHEMATICAL GIFTEDNESS

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We present results from a research project we have conducted with primary school students who participated in an olympiad. We focus on the use of visualization as a possible characteristic of mathematically gifted students. We designed several types of problems requiring the use of visualization, to be solved in the olympiad, and observed which problems were useful for such identification and also which components of visualization, evidenced in students' solutions, were good discriminating the students with best scores, who can be considered potential mathematically gifted students. We have identified and defined seven descriptors associated with the problem-solving process, and identified the descriptors used by students in each problem. Here we show examples of the different types of problems and evidence of the descriptors of visualization evidenced in students' answers.

INTRODUCTION

There is broad agreement among researchers in mathematics education that mathematically gifted students (MG students hereafter) develop faster and use more effectively than their peers certain central abilities or skills in mathematics, such as generalization, abstraction, transfer, deductive reasoning, creativity, or flexibility (Krutetskii, 1976; Leikin, 2021; Miller, 1990). However, when analyzing the relationship between MG and visualization in mathematics, there are discrepant research results. Clements (1999) points out that "many studies have shown that children with specific spatial abilities are more mathematically competent. However, other research indicates that students who process mathematical information by verbal-logical means outperform students who process information visually" (p. 72). This diversity of findings is still present in more recent literature. It is therefore relevant for mathematics education to study this apparent contradiction and provide information to help decide whether mathematical visualization is a trait of MG students.

We present part of a research in which we have analyzed the presence of mathematical generalization, visualization, and flexibility in the solutions of students who participated in the mathematical olympiad for primary school of Costa Rica (OLCOMEP). We will focus on the ability of visualization and the problems we posed in the olympiad that required its use. One objective of our research is to identify which of these problems were more suitable for discriminating students with higher scores and, therefore, potential MG students. Another research objective, related to the previous one, is to identify which visualization features evidenced in students' solutions showed adequate power to discriminate the students with higher scores. The objective of this paper is to identify the descriptors of visualization evidenced in students' solutions and inform on their power to discriminate potential MG students. To this end, we present the different types of visualization problems and an analysis of the descriptors of visualization that some examples of students' solutions used to solve them.

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 207-214). ICMI.

REVIEW OF LITERATURE

Several authors (Clements & Batista, 1992; Giaquinto, 2007; Gutiérrez, 1996) point out the relevance of diagrams, figures, real objects, etc., for both mathematics teaching and mathematical research. Visualization is "being recognized as a key component of reasoning (deeply engaging with the conceptual and not the merely perceptual), problem solving, and even proving" (Arcavi, 2003, p. 235) in school contexts and, on the other hand, "research mathematicians can think through and communicate their proofs to one another in a quite casual way, ... often indicating their intentions with diagrams" (Giaquinto, 2007, p. 81). From the cognitive point of view, Krutetskii (1976) identifies analytic and geometric (visual) types of mathematical thinking and Cain (2019) recalls that some great mathematicians showed preference for analytic thinking (e.g., Hardy or Russell) while others used geometric thinking in their research (e.g., Hadamard).

To evaluate the use of visualization by mathematics students, both psychometric tests and mathematics problem sets are used. Several researchers conclude that it is more effective and reliable to analyze solutions to problems than answers to tests (Ramírez & Flores, 2017).

Interpretations of visualization from mathematics education

Several authors have proposed different conceptualizations of visualization and its use in mathematics: Gutiérrez (1996) states that mathematical visualization is composed of three main elements: mental images (created and manipulated to solve problems or any other mathematical activity; Presmeg, 1986), processes (of converting available information into mental images and interpreting these to obtain new information; Bishop, 1983) and abilities (the skills necessary to carry out the processes and create or manipulate images; Del Grande, 1990). Clements (1999) considers that spatial thinking, or spatial sense, is composed of spatial orientation, spatial visualization, and imagery, along with skills such as creating and transforming mental images and representing them graphically. More recently, Lowrie et al. (2020) consider spatial reasoning as integrated by spatial visualization, mental rotation, spatial orientation, and spatial structuring. Miragliotta and Baccaglini-Frank (2017) call spatial reasoning "the set of cognitive processes by which mental representation for spatial objects, relationships, and transformations are constructed and manipulated" (p. 3953) and characterize it by various skills, in which the use of memory plays an important role. These interpretations of visualization are different but have as common characteristics the creation and transformation of mental images and the need to develop skills to properly use visualization.

Visualization and mathematical giftedness

There is a disparity of conclusions in the research on the use of visualization by MG students. Krutetskii (1976) recognizes the importance of visualization in mathematics by including, among "the component mathematical abilities that arise from the basic characteristics of mathematical thought ... the ability for spatial concepts" (pp. 87-88). But, when he analyzes the presence of such components in MG students' solutions, he concludes that some of them "are not obligatory in the structure of mathematical giftedness" (p. 351), among which he mentions "an ability for spatial concepts" and "an ability to visualize abstract mathematical relationships and dependencies" (p. 351). Presmeg (1986), based on Krutetskii's findings, identifies several reasons for MG students' low use of visualization, internal (natural predisposition and need to use analytical reasoning) and external (textbooks, teaching methodologies, and examination procedures). Van Garderen et al. (2014), after observing students with learning disabilities (LD) and high ability (HA), obtain similar conclusions: "a higher percentage of HA students than students with LD did not consistently utilize a diagram as a strategy to solve a problem. ..... [and] a higher percentage of students identified as HA than students with LD had difficulty using a diagram to reason with as they solved the problem" (p. 147).

On the other hand, Ramírez (2012), analyzing the strategies of secondary school students, concludes that "mathematically talented students have shown significantly higher intelligence and visual ability in tests than students in the control group" (p. 336). Other research shows that, when solving math problems in which visualization facilitates the solution, (potentially) MG students use it more consistently and effectively than average students (Escrivá et al., 2017; Ramírez & Flores, 2017). Therefore, we agree with Webb, Lubinski and Benbow (2007) that observing visualization ability, through problem solving, should be included in the processes of identifying MG students, in order to identify a type of MG students that is sometimes uncovered.

THEORETICAL FRAMEWORK

Based on Gutiérrez (1996), we define the *capacity of visualization* as the capacity to reason using spatial or visual elements, both physical (photos, diagrams, solid objects, etc.) and mental (mental images), to solve problems, perform proofs, or understand mathematical concepts and properties. Furthermore, based on this author and Del Grande (1990), we define *abilities of visualization* as skills that must be acquired and improved by students to effectively execute the necessary visualization processes with the appropriate mental images when solving problems. For our research, we have taken the definitions proposed by these authors for visual identification, mental rotation, conservation of perception, recognition of positions-in-space, recognition of spatial relationships, and visual discrimination. From them, we have defined a set of operational *specific descriptors* of the capacity of visualization that allowed us to observe evidence of the capacity and skills of visualization in students' solutions to the problems in our experiment:

- DV1. *Visual identification*: to identify a figure that is part of a complex context by isolating it from the context.
- DV2. *Mental rotation*: to produce or transform a mental image by rotating it in space. It can be based on a dynamic image or on a pair of pictorial images.
- DV3.1. *Conservation of perception of partially hidden figures*: to identify a regularity of a partially hidden figure that allows to imagine how the figure would look like if it could be fully seen.
- DV3.2. *Conservation of perception of completely hidden figures*: to identify a regularity of a structure that allows to assume that it includes hidden figures, and how the figures would look like if they could be fully seen.
- DV4.1. *Recognition of positions-in-space with respect to oneself as observer*: to recognize the positions of objects located in space in relation to oneself as observer and as the reference center.
- DV4.2. *Recognition of positions-in-space with respect to a reference object*: to recognize the positions of objects located in space in relation to another fixed object acting as a reference center.
- DV5.1. *Recognition of spatial relationships between objects*: to recognize the positions of objects located in space in relation to each other.

- DV5.2. *Recognition of spatial relationships between elements of objects*: to recognize the positions of elements of one or more objects located in space in relation to each other.
- DV6. *Visual discrimination*: to compare objects, identifying their visual similarities and differences.

As we have shown in the previous section, some authors argue that being able to choose and effectively use the most appropriate visualization skills to solve a problem is a skill that differentiates MG students.

METHODOLOGY

We gathered the solutions by a sample of 300 students in grades 2 and 4 to 6 (7 to 12 years old) who participated in the phases 2, 3, and 4 (final) of the OLCOMEP olympiad and analyzed the ways in which the students used their capacity of visualization, by identifying which descriptors were evidenced in their solutions. We created 13 problems of visualization, posed in the successive phases of the olympiad of the mentioned grades, and classified them into three types: A) given a complex structure made of simple objects, analyze its different side-views to get some information; B) given a complex structure made of simple objects, identify the quantity or types of objects that form it; C) given a structure, mentally disassemble or rearrange its elements, to fit the instructions stated in the problem. Several problems are presented in Figures 1, 3, and 5 and in Mora and Gutiérrez (2021).

We have made a mixed analysis of the data provided by the students' solutions. The qualitative analysis has allowed us to classify the solutions according to the descriptors evidenced by each one, as shown below. The quantitative analysis has been oriented to identify the problems and descriptors that have shown the highest power of discrimination of students with better visualization skills (Mora et al., 2023).

RESULTS

In this section we analyze several problems, as examples of the types described above, and present a synthesis of the responses to each problem, analyzed in terms of the visualization descriptors evidenced by the students in each problem.

Type A problems: analyze side-views of a complex structure made of simple elements

Luis builds a city with Lego, that has buildings of three colors and the buildings of the same color have the same height. He places them in a grid of 3x3 buildings, so that in the same row and column there are no buildings of the same height. Ana and Juan are two habitants of the city that are placed as shown in the figure.

a) If Juan looks towards the city, how many buildings can he see? Justify your answer.
b) If Anna looks towards the city, how many buildings can she see? Justify your answer.
c) In what position, different from Juan's, could Ana stand to see the same number of buildings that Juan sees?

Figure 1: Problem of phase 3, grade 4

Figure 2 shows a student's solution to the problem in Figure 1. To solve it, students needed to use *visual identification* (DV1), to correctly interpret the information, to identify the polygonal shapes that form the visible faces of the buildings, the complete buildings (with their walls and roofs), and the complete structure of the city. All students scoring high used DV1 to solve the problem, while only 20% of the students scoring low did it. Therefore, this problem allowed to discriminate very well the use of DV1 by the best visualizers, who are thus potential MG students.

To answer questions a) and b), students had to use the *conservation of perception* to, considering the regularity of the structure, identify the buildings that, from John or Anne's position, are *partially hidden* (DV3.1) and those that are *completely hidden* (DV3.2). In this problem, DV3.1 was observed in 75% of the students and DV3.2 in only 62,5% of them, so using DV3.2 was more difficult for students than DV3.1. In the solutions to this problem, all students scoring high used both descriptors to solve the questions, but only 20% of students scoring low did the same. Then both descriptors may well differentiate students with high visualization ability and thus potential MG students.

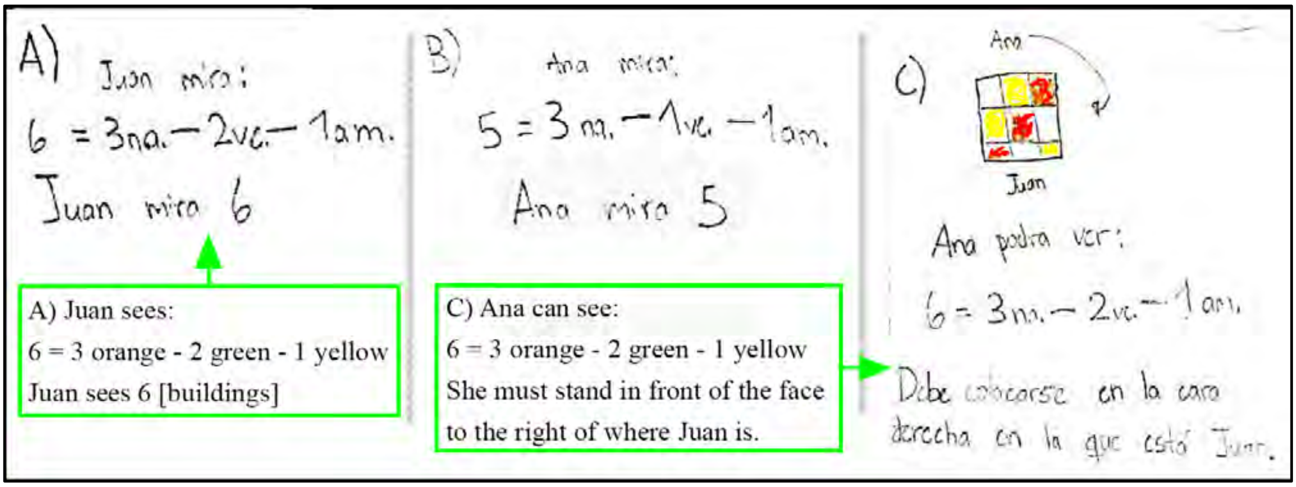
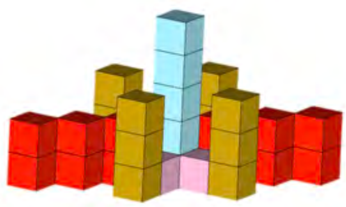


Figure 2: Student E.E63031's solution to the problem of phase 3, grade 4

To answer question c), students had to do recognition of *positions-in-space*, to choose a position taking as the reference either *themselves as an observer* (DV4.1) or *another fixed object* in the figure (DV4.2). The first option was observed in 18,7% of the students and the second one in 37,5% of them. In the solutions to this problem, all students scoring high used some of these descriptors to solve the questions, but none of the students scoring low did it. Then, this pair of descriptors may well differentiate students with high visualization ability and thus potential MG students. The student in Figure 2 used the descriptors DV1, DV3.1, DV3.2, and DV4.2.

Type B problems: identify the quantity and/or type of objects that form a complex structure

In the park there is a sculpture made of columns of cubes, the columns of the same height have the same color. If someone stands at the top of the light blue column, which is the center of the sculpture, they will see the sculpture symmetrical both comparing front and behind and comparing right and left.



a) How many cubes in total were used to build the sculpture?

b) What is the difference between the number of pink cubes and blue cubes?

c) What is the difference between the number of red cubes and brown cubes?

Figure 3: Problem of phase 2, grade 5

When solving the problem of Figure 3, students used *visual identification* (DV1) to extract information from the figure, identify the polygonal shapes that form the visible faces of each cube, and identify the cubes as simple objects and as parts of the columns. This descriptor was used by almost all students (94.4%), as it was not difficult for them to interpret the information. Once the elements of the figure were identified, the student used the *conservation of perception* to identify the cubes and count them. All the students scoring high used DV3.1 and DV3.2 to count the cubes and solve the problem. Also, 80% of the students scoring low evidenced DV3.1, so it was easy for them to identify the cubes that have a visible part or are deformed by the perspective. On the contrary, none of the students with low scores was able to identify the non-visible cubes (DV3.2). Figure 4 shows the solution by a student who drew separate representations of each element (columns) of the structure. Thus, the answer showed the three descriptors mentioned, because, after having identified the elements of the sculpture (DV1), the student used DV3.1 and DV3.2 to identify the partially and completely hidden cubes and count them.

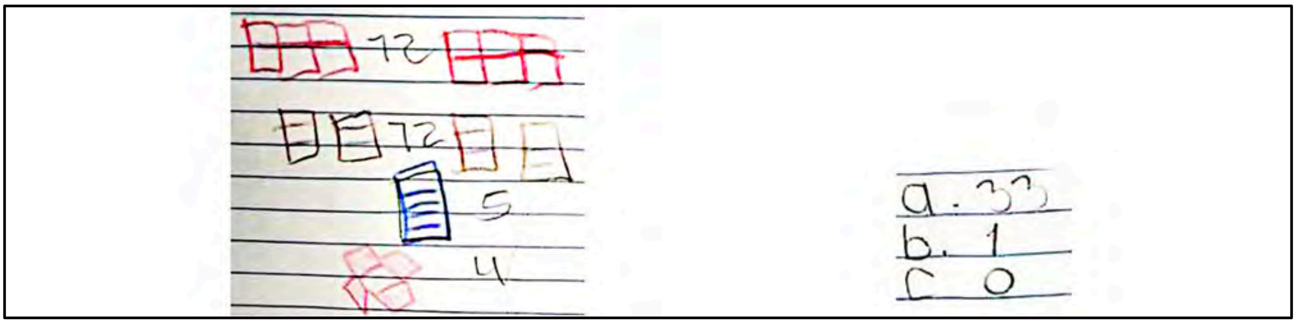


Figure 4: Student E.E52002's solution to the problem of phase 2, grade 5

Type C problems: mentally disassemble or rearrange the elements that make up a structure

Martín's friends are playing with cubes that are held together with magnets. They have built the four colored shapes above and Martín has built the two larger shapes, A and B.

a) Is it possible, from Figure A, only by removing some cubes, to obtain each of the figures built by your friends? Justify your answer.
b) Is the above possible from Figure B? Justify your answer.

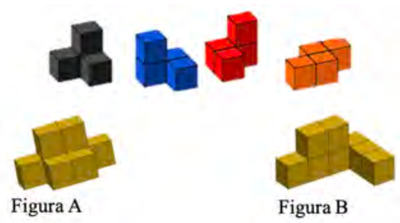


Figure 5: Problem of phase 3, grade 6

When solving the problem in Figure 5, students found it easy to identify the cubes that form each structure in the statement, with their faces and edges. Therefore, almost all students (81.25%) used *visual identification* (DV1) and, therefore, this descriptor did not help to differentiate the best visualizers of grade 6, although, in other problems of lower grades, it did discriminate them very well. To determine the shapes of the colored structures and compare them with figures A and B, it is necessary to identify some cubes of each structure that are partially or completely hidden, making use of the *conservation of perception*. In this problem, almost all high-scoring students (85.7%) showed evidence of having used DV3.1 and DV3.2. Furthermore, 57.14% of the low-scoring students evidenced the use of DV3.1. Then, DV3.1 do not help to differentiate the best visualizers in this problem. On the other hand, DV3.2 helped more to differentiate students with higher visualization capacity, because it was used by fewer low-scoring students (42.8%). When asked to compare several figures, students used *visual discrimination* (DV6) to compare some structures with others, by identifying visual similarities and differences between them. This descriptor was evidenced by almost all high-scoring students (85.7%) and only 14.28% of low-scoring students, so DV6 helped to differentiate students with higher visualization ability. Figure 6 presents the response of a student who showed these four descriptors. Consequently, among the descriptors necessary to solve this problem, DV6 is the one discriminating best the students with higher visualization capacity.

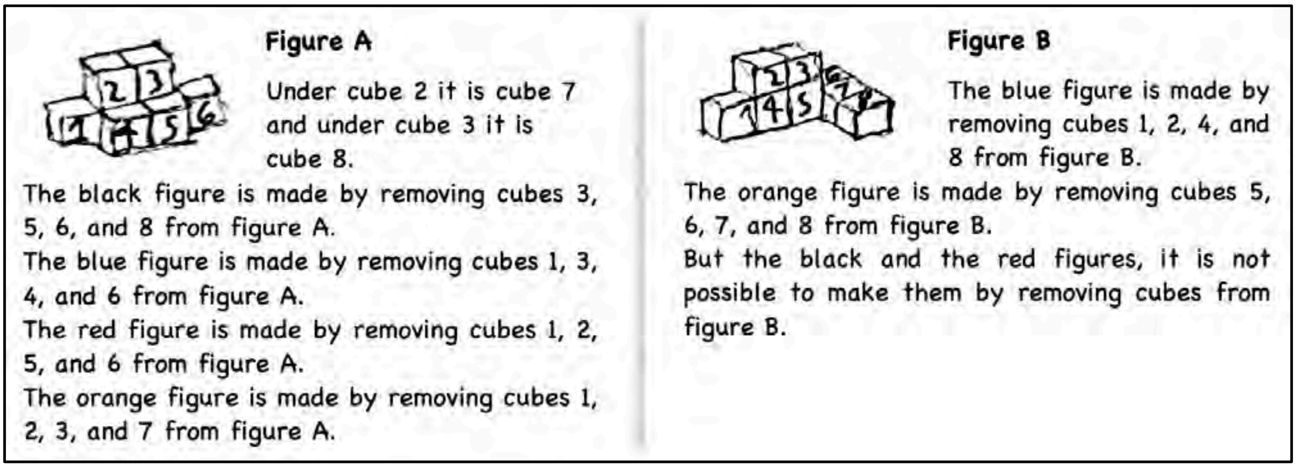


Figure 6: Student E.E63031’s solution to the problem of phase 3, grade 6

CONCLUSIONS

Researchers in mathematics education have identified several capacities related to MG. The ability to visualize is necessary to achieve a complete mathematical education of students, although there is debate as to whether it is a characteristic ability of MG. In our study, we experimented with a set of 13 problems of three different types, administered in the OLCOMEP olympiad to participants in grades 2 and 4 to 6 of primary school. To correctly solve these problems, students had to use a variety of visualization skills. We have shown examples of the three types of problems posed, analyzed some solutions to those problems, and reported which visualization descriptors proved to be good identifiers of high visualization ability and potential MG students, as a large majority of the students who used them solved the problems requiring them well and obtained high scores in the olympiad.

Our study cannot provide generalizable answers, due to the small sample size, but it raises an important avenue for future research, consisting in analyzing: how the characteristics of different types of visualization problems influence the success of students with high and low visualization abilities; which visualization skills are more adequate to discriminate students who have high visualization abilities and, therefore, who are potential MG students.

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