HIGH ACADEMIC ACHIEVEMENT AND MATHEMATICAL GIFTEDNESS IN SECONDARY SCHOOL STUDENTS: ARE THERE DIFFERENCES?

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A significant number of mathematics teachers believe that the students getting the highest marks are the ones with mathematical giftedness, so, when asked for their potential mathematically gifted pupils, they nominate the higher academic achievers. An important question to be investigated is in which ways are both kinds of students different. We present results from a study aimed at looking for characteristics differentiating secondary school higher academic achievers and potential mathematically gifted students. We focus on students' conceptions of mathematics and the quality of their mathematical reasoning. We designed a teaching experiment, to look for behavioral characteristics of each kind of students, and carried out a case study with two higher academic achievers and three potential mathematically gifted students participating in the experiment. The data gathered show some similarities and quite many differences between higher academic achievers and potential mathematically gifted students' performance in solving problems and interest in mathematics.

INTRODUCTION

A widespread misconception among mathematics teachers is that the mathematically gifted (MG) students are high academic achievers (HAA) in mathematics. However, some authors conclude that this is not always true and, even, some MG students suffer school failure in mathematics (Bicknell, 2008; Diezmann, & Watters, 2002). These authors consider that differential characteristics of MG students are, among others, their exceptional reasoning ability and their preference for working abstractly. But they also raise the issue of the negative influence on the MG students of lack of support by their teachers and, in general, the society's anti-intellectual beliefs. Sheffield (2017) discusses some false myths about MG students, analyzes them, and proposes ways to counteract them.

Likewise, being HAA in mathematics does not imply being a MG student. This misconception is present in society in general and mathematics teachers in particular. It is also present in Spain but, as far as we know, this kind of comparative study has never been made in Spanish schools. Then, it is of interest to describe and compare the behaviors of Spanish students considered as HAA or MG students, to identify behavioral differences between them. This would allow teachers to adequately attend to each kind of students and enhance the talent of MG students. Matthews and Farmer (2008) present an analysis of students' qualifications to look for relationships between the performance of MG students in general mathematics courses and the Algebra I course. Their results show that MG was not strongly related to achievement in Algebra I, i.e., some MG students did not get good marks in the subject, the teaching methodology in Algebra I being a main reason for that. Brandl (2011) analyzes the characteristics of German MG and HAA students and raises several differences, some of them relevant to our study, as we will show below.

The problem of identifying MG students is not trivial. Mathematics education literature demonstrates that problem-solving and problem-posing are two main tools to identify MG students. However, a single experiment with a few problems is not sufficient to be sure that a student has really MG. We have to test students' mathematical talent in a diversity of mathematical contexts and with a diversity of types of problems. This takes time and, quite often, we cannot do it with sufficient detail. We talk about *potential* MG students when students show partial or initial signs of MG, to differentiate them from students who have sustainedly demonstrated their MG (Jablonski & Ludwig, 2022; Pitta-Pantazi & Leikin, 2018; Sheffield, 2017).

In this context, our research objectives are to identify possible differences among Spanish HAA and potential MG students in i) their ways of solving mathematical tasks, ii) their feelings about mathematics and learning mathematics, and iii) the evolution of the two above mentioned objectives from the beginning to the end of the teaching experiment.

THEORETICAL FRAMEWORK

To accomplish the mentioned objectives, we present differential characteristics between HAA and MG students which are the ground for the exploration of students' behavior in our experiment. Brandl (2011) presents characteristics of HAA students in contrast with characteristics of MG students. Our study is grounded by a selection of those characteristics, the ones that are adequate to the particularities of our study. According to Brandl (2011), HAA students usually:

- like mathematics lessons to be done in detail and precisely by their teacher,
- are motivated (mainly intrinsically) to be intellectually active,
- have extreme requirements on themselves, with internal pressure to get the best marks,
- dislike missing anything in class, so they ever pay attention to the teacher,
- are teacher-oriented, they expect teachers to know the contents and teach them adequately, and
- like to write good exams, to get the highest marks.

And potential MG students usually:

- have mathematical intuition, sensing the overall train of thoughts where the procedure leads them,
- are satisfied with highly abstract contents,
- are creative when solving problems, looking for relationships, or answering questions,
- find solutions to problems that were not expected by the teacher.
- show curiosity about reasons for properties or new relationships,
- are able to intuitively "see" inner mathematical connections,

METHODOLOGY: A CASE STUDY TEACHING EXPERIMENT

We carried out an experimental case study in a public secondary school in Spain. Aiming to collect data about secondary school students in a diversity of grades, we selected 5 students from grades 7, 9, 10, and 11, after their nominations by the mathematics teachers (Bicknell, : we identified two HAA students in grades 7 (HAA1, 12-13 years old) and 10 (HAA2, 15-16 years old), based on their excellent grades in mathematics; and we identified three potential MG students in grades 9 (PMG1, 14-15 years old) and 11 (PMG2 and PMG3, 16-17 years old), based on their habit of making unusual questions about mathematical contents, their easy in learning mathematics, and their ways of solving problems in the classes. The data we collected were students' answers to the two recorded interviews and oral and written solutions to the tasks.

We designed a teaching experiment for those students, based on a non-standard teaching methodology, where the first author was the researcher-teacher. The analysis of changes in students' perceptions was based on two individual interviews carried out at the beginning and the end of the experiment. The questions were:

- ◊ What is your opinion about mathematics? Define the subject using one word.
- ◊ What mathematics content do you like the most?
- ◊ How would you describe the activities in mathematics?
- ◊ Do you think you have a special ability in mathematics? Why?
- ◊ Do you like solving mathematical problems?

The final interviews included the previous questions plus two new questions:

- ◊ What is your opinion about the classes we have done?
- ◊ Would you like to work more frequently on this kind of activities?

Based on the information students provided us in the initial interview, about the mathematical contents they liked the most, we designed the tasks. We selected, for each student, mathematical contents of his/her interest, aiming to induce students' interest, commitment, and engagement towards the experiment. We designed personalized sets of tasks, that could be intended for enrichment (based on extra-curricular contents) or deepening (of the ordinary curricular contents). The final design of the tasks consisted of three styles (an example of each style and be found in the Appendix):

Standard tasks for enrichment or deepening, which include several increasingly difficult questions. The mathematical contents depended on whether they were aimed at enrichment or deepening. We designed this style of tasks for students PMG1 (on number theory and equations), PMG2 (on functions), and PMG3 (on plane and spatial Euclidean geometry with GeoGebra). The tasks were solved verbally or with paper and pencil.

Story tasks, which have a story structure and include several questions based on the mathematical contents hidden in the story. We designed tasks of this kind for student HAA1 (on arithmetic problem-solving and logic). The tasks were solved verbally or with paper and pencil.

Experimental tasks, which are similar to the standard tasks but designed to be mostly solved experimentally by using GeoGebra. We designed this kind of tasks for students HAA2 (on probability and statistics), PMG1 (on number theory and equations), and PMG3 (on plane and spatial Euclidean geometry).

We also planned two ways of solving the tasks, depending on the teacher's support, to provide the students with a working environment that would make them feel comfortable and allow them to show their mathematical abilities:

Work with teaching support, when the student could ask for help and the teacher provided it. During resolutions, any new content was explained.

Working without teacher support, when the student solved the activities autonomously at home, without the possibility of asking the teacher. As new knowledge was incorporated into some tasks, in some sessions the teacher introduced it.

Table 1 presents a summary of the different tasks solved by the students and the ways they solved each task (HAA1-T1 is the first task solved by the student HAA1).

Tasks	**Styles of tasks**	**Way of solution**
HAA1-T1, HAA2-T2	Without support	Written
HAA1-T2, HAA2-T1	With support	Written
HAA1-T3, HAA2-T3	With support	Verbally
PMG2-T3, PMG3-T1	Without support	Written
PMG2-T1, PMG3-T2	With support	Verbally
PMG2-T2, PMG3-T3	With support	Written
PMG1-T1, PMG1-T1, PMG1-T3	With support	Written and verbally

Table 1: Characteristics of the tasks solved by each student

RESULTS

Interest in the subject of mathematics

All students in the sample showed a higher interest in mathematics at the end of the teaching experiment, which suggests that these rich tasks were motivating for all of them. For instance, in the initial interview, student PMG3 said that the subject of mathematics was "interesting but not very interesting, mechanical and sometimes boring" and that it was not his favorite subject. In the final interview, he said that mathematics now seemed more "entertaining and challenging" and that "he had never thought that he could learn by doing problems". We see that the student moved from seeing mathematics as a static and monotonous subject to a dynamic and interesting one.

There was also an improvement in students' perception of mathematics, evidenced when we asked them for a word describing the subject. In the interviews before and after the teaching, they answered:

HAA1: number	HAA2: exercises	PMG1: equations	PMG2: functions	PMG3: equations
HAA1: interesting	HAA2: fun	PMG1: connections	PMG2: unlimited	PMG3: visual

Students first described mathematics using terms related to contents and learning methods that are not representative of mathematics as a science. On the contrary, in the end, they used terms more diverse and representative of mathematics. This shows that students' perception of mathematics is directly related to the teaching methodology they experience at the school.

Furthermore, their answers after the experiment showed an increment in their commitment to studying mathematics. For example, PMG1 stated that "studying what interests me helps me be able to study things that I don't like that much". HAA1 said that, despite being tired of class activities, she "wanted to solve the problems" that we posed her.

Differences in performance among HAA and potential MG students

We have not noticed any significant difference between the two HAA students, since they shared the most relevant characteristics of their reasoning and behavior: the need to carry out the work in a clean and clear way, the need for approval of their work by the teacher, the need to know their mistakes and how to solve them, the request for extra activities to prepare the sessions, the self-demand during all the sessions, and the rigidity of most of their processes of reasoning. Furthermore, the two HAA students did not show characteristics of MG throughout the experiment.

We have noticed some differences between the three potential MG students, since they showed different mathematical mastery: the mathematical giftedness of PMG1 was evidenced mainly by his ease of generalization and abstraction, natural formulation of extensions of the posed tasks, ease of connecting and integrating knowledge, and use of creative reasoning. PMG2 evidenced mathematical giftedness, although clearly lower than PMG1; he showed the ease of integrating new knowledge and a tendency to find different ways to solve a task, but he never showed mastery in generalization. PMG3 showed a great capacity for visualization, which he used to generalize his reasoning and justify it orally. On the other hand, he showed a shortage in the quality of his reasoning when he had to write it down.

The students showed different levels of self-demand, since the HAA students wanted the teacher to get a good evaluation of their outcomes, while the potential MG students wanted to correctly solve the tasks, without caring about the teacher's evaluation. The HAA students were very dependent on their teacher's support, in contrast to the little need for such support by the potential MG students.

There were also differences in the quality of students' reasoning: the potential MG students tended to provide poorly structured but concept-rich reasoning, experimenting with various solutions, and choosing the most appropriate one; on the contrary, unlike the HAA students tended to provide much more structured and rigid reasoning, sometimes showing difficulties in initiating solutions.

CONCLUSIONS

It is convenient to consider students' preferences in the classes, since it provides them with an optimal workspace where students feel more comfortable, so it is beneficial to them. The analysis of data from our teaching experiment showed that different students may like diverse teaching methodologies (in our case, working with or without teaching support and exposing ideas in written or orally) and that organizing the teaching according to the student's preferences helps them to be more productive and successful in learning mathematics.

The differences between HAA and potential MG students can be detected by analyzing their solutions to enriching and motivating tasks. These differences, presented above, coincide with those exposed by Brandl (2011) and Szabo (2015). This shows that these differentiating characteristics are independent of the type of instruction and are rather inherent to what MG and HAA entail, respectively.

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References

Bicknell, B. (2008). Who are the mathematically gifted? Student, parent, and teacher perspectives. *Journal of the Korean Society of Mathematical Education Series D: Research in Mathematical Education, 13*(1), 63–73.

Brandl, M. (2011). High attaining versus (highly) gifted pupils in mathematics: a theoretical concept and an empirical survey. In E. Pytlak, M. Swoboda, & T. Rowland (Eds.), *Proceedings of the 7th Congress of the European Society for Research in Mathematics Education (CERME7)* (pp. 1044-1055). ERME.

Diezmann, C. M., & Watters, J. J. (2002). Summing up the education of mathematically gifted students. In B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.), *Proceedings of the 25th Annual Conference of the MERGA* (pp. 219-226). MERGA.

Jablonski, S., & Ludwig, M. (2022). Examples and generalizations in mathematical reasoning – A study with potentially mathematically gifted children. *Journal of Mathematics Education, 13*(4), 605–630. https://doi.org/10.22342/jme.v13i4.pp605-630

Matthews, M. S., & Farmer, J. L. (2008). Factors affecting the Algebra I achievement of academically talented learners. Journal of Advanced Academics, 19(3), 472-501. https://doi.org/10.4219/jaa-2008-810

Pitta-Pantazi, D., & Leikin, R. (2018). Mathematical potential, creativity and talent. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger, & K. Ruthven (Eds.), *Developing research in mathematics education. Twenty years of communication, cooperation and collaboration in Europe* (pp. 115–127). Routledge. https://doi.org/10.4324/9781315113562

Sheffield, L. J. (2017). Dangerous myths about "gifted" mathematics students. *ZDM Mathematics Education, 49*(1), 13-23. https://doi.org/10.1007/s11858-016-0814-8

Szabo, A. (2015). Mathematical problem-solving by high achieving students: Interaction of mathematical abilities and the role of the mathematical memory. In K. Krainer, & N. Vondrová (Eds.), *Proceedings of the 9th Congress of the European Society for Research in Mathematics Education* (pp. 1087–1093). ERME.

APPENDIX

Example of standard tasks

Task PMG2-T1. We will study how to solve functional equations. Functional equations are a type of equation in which the unknown is not a number or set of numbers, but a function or set of functions. With this in mind, try to answer the following questions:

1. Let the functional equation $x^2 - 2f(x) = f(1/x)$, with $x \in \mathrm{R}-\{0\}$.

- If we assume that x is a concrete value x_0, what can you say about the equation?
- What is the relationship between $f(x)$ and $f(1/x)$?
- Calculate all the solutions of this functional equation. Justify your answer.

2. Find all functions $f(x) :]0, \infty[\rightarrow]0, \infty[$ that satisfy the following equation:

$$x - f(x) = 1/f(x) - 1/x, \text{ with } x \in]0, \infty[$$

Example of story tasks

Task HAA1-T1. Dear Detective,

My name is Clotilde Colboc, head of the research team at the Musée d'Orsay in Paris. We have been told that you are one of the most brilliant minds in Europe at solving enigmas and, at the Orsay, we need your help.

Our problem is mathematical because it involves the famous Greek Archimedes. In the inheritance of one of France's largest fortunes, relatives discovered in a display case a fragment of parchment written in ancient Greek by Archimedes. The manuscript was lent to us for research and display in the museum.

You can imagine our surprise when we discovered that this was his last work and that there was an unsolved enigma of the ancient mathematician. The text is this:

Although it is neither a discovery nor a matter for the Academy, this last work of mine is a gift for all lovers of Mathematics. The gift, in the form of an enigma, is to find 3 ways to solve this sum if each letter corresponds to a different digit [Arquímedes is the name of the mathematician in Spanish].

$$\begin{array}{cccc} & A & R & Q \\ + & U & I & M \\ \hline E & D & E & S \end{array}$$

In this research institute, we do not know how to solve it. Therefore, we send you the problem and ask you to solve it and tell us each combination and the reasoning you have followed to solve it. Also please let us know if there are more than 3 ways to solve it.

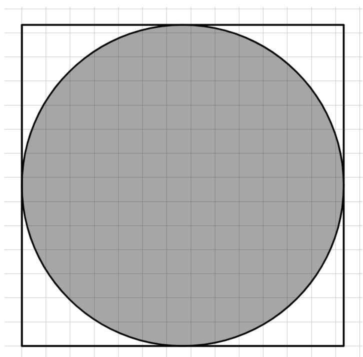
I hope you are interested in collaborating with us and helping us. Best regards.

Dr. Clotilde Colboc

Example of experimental tasks

Task HAA2-T3. In this session, we will work on the Law of Large Numbers, which states that, when the number of observations of a random phenomenon is very large, the relative frequency of an event almost certainly converges to its probability. Making use of this law, there are methods that allow to calculate areas and relations between areas. One of these is the Monte Carlo method. To use this method, we will consider that choosing a point in a bounded region is a uniform probabilistic event, i.e., all points have the same probability of being chosen.

1. According to this fact, we consider that the intersections of the lines in the following figure are points to be chosen.



Answer the following questions:

- Calculate the ratio of the area of the square to the area of the circle using the method. Does the result resemble the theoretical relationship? Why do you think this is?

- Can you get a better approximation of the relationship? Reason your answer.

- Can you approximate the value of π with the approximation of the ratio between areas? If so, approximate it.

2. Now, repeat the questions above with the following figure.