Solitons and solitary vortices in continuous and discrete 2D models

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The objective of the talk is to present an overview of fundamental dynamical models for the pattern formation in nonlinear two-dimensional (2D) media. The overview will present both the basic models and their basic solutions, which describe solitary patterns (2D solitons) in them. Physical realizations of the solutions will be considered too.
Both conservative and dissipative models will be reviewed. Different parts of the talk are defined according to basic mechanisms which provide for the stability of the 2D solitons, as the stability is the main issue in this field:
(A) Conservative systems with trapping potentials;
(B) Conservative systems with the cubic-quintic (CQ) nonlinearity;
(C) Dissipative models based on 2D complex Ginzburg-Landau (CGL) equations;
(D): Stable 2D composite solitons in spin-orbit-coupled self-attractive Bose-Einstein condensates in free space.
Part A: Conservative systems with trapping potentials

(1) Introduction. The simplest model which may give rise to solitons and solitary vortices: the 2D nonlinear Schrödinger equation (NLSE), alias the Gross-Pitaevskii equation (GPE):

\[ i \frac{\partial u}{\partial t} = -\frac{1}{2} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - |u|^2 u + W(x, y)u \]

(in optical models, \( t \) is replaced by the propagation distance, \( z \)).

In the presence of the **axially symmetric** trapping potential, \( W = W(r) \), where \( (r, \theta) \) are the polar coordinates in the \((x, y)\) plane, solutions with integer vorticity \( S \) are looked for as

\[ u = \exp(-i \mu t + i S \theta) U(r), \]

with \( \mu < 0 \) and real function \( U(r) \) obeying the stationary equation:

\[ \mu U = -\frac{1}{2} \left( \frac{d^2 U}{dr^2} + \frac{1}{r} \frac{dU}{dr} - \frac{S^2}{r^2} U \right) - U^3 + W(r)U. \]
Localized solutions, with asymptotic forms $U(r) \sim r^S$ at $r \to 0$, and $U(r) \sim \exp(-(-2\mu)^{1/2}r)$, exist even in the absence of the external potential, $W = 0$, but they all are completely unstable. The fundamental solitons [with $S = 0$, alias Townes’ solitons, R.Y. Chiao, E. Garmire & C. H. Townes, Phys. Rev. Lett. 13, 479 (1964)] are unstable against the collapse, and vortical solitons (with $S \geq 1$) are vulnerable to a still stronger instability against azimuthal perturbations which break their axial symmetry and split the vortices into a few separating segments (fundamental solitons, which will later be destroyed by the collapse).
2. Stabilization of single-component vortices
A fundamental issue: how can solitons and vortices be stabilized in physically relevant settings?
The simplest possibility is to use an axially symmetric trapping potential. Typically, it is taken as the harmonic-oscillator potential, \( W(r) = (\frac{\Omega^2}{2}) r^2 \). Here, basic results will be presented as per the following paper:

**Physical Review A 73, 043615 (2006)**

Vortex stability in nearly-two-dimensional Bose-Einstein condensates with attraction

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The basic equation with the **harmonic-oscillator trapping potential** is written as

\[ i \frac{\partial \psi}{\partial t} = \left[ -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2} \Omega_r^2 (x^2 + y^2) + g |\psi|^2 \right] \psi, \]

with \( g < 0 \) (which corresponds to the self-attraction). The norm of solution (\( \sim \) number of atoms in the BEC, or the total power in the optical model) is defined as

\[ N = 2\pi \int_0^\infty |\psi(r)|^2 r^2 \, dr. \]
The main result of the analysis: families of fundamental solitons and vortices are represented by $N(\mu)$ curves, in which **stable subfamilies** are shown by **continuous curves**. For $S = 1$, the **stability region** (its edge is indicated by the arrow) amounts to $\approx 1/3$ of the respective **existence region**:
For the analysis of the stability of these solutions against small perturbations, perturbed solutions were looked for as

$$\psi(x,y,t) = [R(r) + u(r)\exp(\lambda t + iL\theta)$$

$$+ v^*(r)\exp(\lambda^* t - iL\theta)]\exp(iS\theta - i\mu t),$$

(in the application to BEC, the respective linearization is called the **Bogoliubov-de Gennes equations**). As a result, the instability growth rates are calculated in the following form, featuring a **stability window** for \( S = 1 \), but not for \( S = 2 \):
An example of direct simulations of the perturbed evolution of a **stable vortex** (intensity and phase fields are shown):

![Graphs showing the intensity and phase evolution of a stable vortex.](image)

**FIG. 3.** Grey-scale plots illustrating recovery of the perturbed stable vortex with $S=1$ for $\mu=1.4$. (a),(b) Intensity and phase distributions in the initial configuration including random noise. (c),(d) The same in the self-cleaned vortex soliton at $t=120$. The norms of
In the interval of the norm between $\frac{1}{3}$ and 0.43 of the existence region, the vortex with $S = 1$ is semi-unstable, periodically splitting into two fragments and recombining:
3. Another general problem: a possibility of stabilizing two-component states with **hidden vorticity** in a system of two coupled 2D NLSEs with the axisymmetric trapping potential. The presentation is based on the following paper:

PHYSICAL REVIEW A 82, 053610 (2010)

**Hidden vorticity in binary Bose-Einstein condensates**

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The system of the coupled equations is set as:

\[ i \frac{\partial \psi_1}{\partial t} = \left[ -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2} (x^2 + y^2) \right. \]
\[ \left. - (|\psi_1|^2 + \beta |\psi_2|^2) \right] \psi_1, \]

\[ i \frac{\partial \psi_2}{\partial t} = \left[ -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2} (x^2 + y^2) \right. \]
\[ \left. - (|\psi_2|^2 + \beta |\psi_1|^2) \right] \psi_2. \]
The *hidden-vorticity solution* is looked for as the one with **opposite values** of the angular momentum in the two components, so that the **total angular momentum is zero**:

\[
\psi_{1,2}(r, \theta, t) = R(r) \exp \left( i S_{1,2} \theta - i \mu t \right),
\]

\[S_1 = +1, \quad S_2 = -1,\]

with the **common radial amplitude** of both modes, \(R(r)\), satisfying

\[
\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{d^2 R}{dr^2} + \left(2 \mu - \frac{S^2}{r^2} - r^2\right) R + 2(1 + \beta) R^3 = 0.
\]
The stability analysis was based on the use of the **Bogoliubov – de Gennes** equations for perturbed solutions, taken as

\[
\psi_{1,2}(r, \theta, t) = \left[ R(r) + u_{1,2}(r)e^{-i\omega t - iL\theta} + v_{1,2}^*(r)e^{i\omega^* t + iL\theta} \right] \\
\times \exp(-i\mu t + iS_{1,2}\theta),
\]

where \( L \) is integer, and \( \omega \) is the stability eigenvalue, the **instability** occurring if \( \text{Im}\{\omega\} \neq 0 \).
The main result of the analysis – the stability diagram for the hidden-vorticity (HV) modes. Note that both $\beta > 0$ and $\beta < 0$ are included.

FIG. 3. The stability diagram for symmetric HV modes, in the plane of the norm (of one component) and interaction coefficient. Instability areas are labeled by the azimuthal index of the dominating perturbation eigenmode.
Verification of the predicted stability by direct simulations, for $N = 14.10$ and $\beta = +0.2$: 
Verification of the predicted instability (splitting into 2 segments which collapse later) for $N = 13.49$ and $\beta = +0.5$: 

![Streaming media](image-url)
Verification of the predicted instability (splitting into 4 segments which collapse later) for $N = 26.98$ and $\beta = +0.5$:
4. The use of periodic potentials for the stabilization of 2D solitons and vortices: an overview

Periodic potentials, that may be induced by material or photonic (virtual) lattices in optical media, or by optical lattices (OLs) in BECs, can create and/or stabilize various types of localized modes (solitons) which do not exist at all, or are definitely unstable, in the respective uniform media.
A natural link between the description of *continuous media* with periodic potentials and *discrete lattices* is established by the consideration of the *limit case* of a *very strong* periodic (cellular) potential, which, in the *tightly-binding approximation*, effectively splits the wave field into an *array of weakly coupled droplets/filaments*.

This approach leads to the \textit{discrete nonlinear Schrödinger equation} (DNLSE). In the 1D setting, this equation is

\[ i \frac{d\psi_n}{dt} = -\frac{1}{2} (\psi_{n+1} + \psi_{n-1} - 2\psi_n) + g |\psi_n|^2 \psi_n. \]

It can be derived from the continual \textit{Gross-Pitaevskii equation} (GPE), which describes BEC trapped in a \textit{deep optical lattice}:

\[ i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + g |\psi|^2 \psi - \varepsilon \cos \left( \frac{2\pi x}{L} \right) \psi. \]
5. Vortex solitons in continuous periodic media

The fundamental model of the 2D continuous medium equipped with the *periodic (lattice)* potential, in optics and BEC alike, is based on the following continuous **NLSE/GPE** with the self-focusing cubic nonlinearity:

\[
i \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + |u|^2 u - \varepsilon [\cos(2x) + \cos(2y)]u = 0
\]

(the period of the potential is scaled here to be \(\pi\)).

In optics, the *evolutional variable* is \(z\) instead of \(t\).
The possibility of the *stabilization* of both *fundamental solitons* and *solitary vortices* by means of *lattice potentials* was first demonstrated, independently, in the following works:

Stable fundamental solitons found in these works are *single-peak nearly isotropic* objects, slightly deformed by the underlying *square-lattice potential* (this example is shown for the lattice’s strength $\varepsilon = 0.92$, and the integral norm $N = 2\pi$):
Stable objects identified as vortex solitons were actually built as 4-peak complexes. The respective vorticity, $S$, is represented by phase shifts between the wave functions at adjacent peaks: $\Delta \Phi = \pi/2$, hence the total phase circulation around the pivot of the 4-peak complex is $2\pi$, which corresponds to $S = 1$.

Two different types of stable 4-peak vortical modes (with $S = 1$) can be thus built: on-site-centered vortices, and off-site-centered vortices.
A typical example of the stable on-site centered vortex (with norm $N = 2\pi$ and $\varepsilon = 10$). Note the presence of the empty site at the center:
A typical example of the inter-site-centered square-shaped vortex (a densely packed pattern, without an empty site in the middle), for $\varepsilon = 0.5$, is shown by means of a contour plot for the 2D intensity distribution:
6. Discrete vortex solitons
The discovery of the stable vortex complexes, supported by the lattice potentials acting in the continuous media, suggests a possibility of the existence of stable vortex solitons in discrete lattices, obtained, as said above, from the continuum models as the limit case corresponding to a very strong lattice. Such a discrete lattice is described by the 2D DNLSE (recall the evolution variable is $z$ in optics):

$$i \frac{\partial u_{m,n}}{\partial t} = -C \left( u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} - 4u_{m,n} \right)$$

$$- |u_{m,n}|^2 u_{m,n}.$$
Stable solutions to the 2D DNLS, which explicitly represent discrete vortex solitons, were reported by B.A. Malomed and P.G. Kevrekidis, Phys. Rev. E 64, 026601 (2001).
An example of a **stable vortex** at $C = 0.05$ (the intensity and phase distributions are displayed):
Generally, the stationary solutions are sought for as

\[ u_{m,n}(t) = \exp(i\Lambda t)U_{m,n}. \]

The solutions may be normalized by fixing \( \Lambda = 0.32 \) (for instance) and varying \( C \). The basic result is that the vortices are **stable** at

\[ C < C_{cr}^{(S=1)} \approx 0.13. \]

For comparison, the fundamental discrete solitons, with \( S = 0 \), are **stable** in an essentially **broader** interval, \( C < C_{cr}^{(S=0)} \approx 0.29. \)
Experimental results for discrete vortices

Quasi-discrete vortex solitons were created in photorefractive materials with a photonic lattice induced by the illumination of the sample in directions orthogonal to that of the probe beam by counter-propagating pairs of beams, launched in the ordinary polarization (while the probe beam carries the extraordinary polarization):


Characteristic examples of the experimentally observed **stable** localized vortex structures:
8. Stable higher-order ($S > 1$) vortices

In the 2D continuous models including the usual periodic potential, and the cubic self-focusing nonlinearity, stable multi-peak vortex complexes, carrying the vorticity up to $S = 6$, were reported in: H. Sakaguchi and B.A. Malomed, Europhys. Lett. 72, 698 (2005).
Examples of **stable higher-order vortex solitons**:
(a) $S = 2$; (b) $S = 3$; (c,d) $S = 4$: 
9. Vortex solitons in quasi-periodic 2D lattices
For the self-defocusing cubic nonlinearity, stable gap-mode vortex solitons were reported by H. Sakaguchi and B.A. Malomed, Phys. Rev. E 74, 026601 (2006), in the 2D GPE/NLSE with the potential in the form of the five-fold Penrose tiling:

$$i \frac{\partial \phi}{\partial t} = -\frac{1}{2} \nabla^2 \phi + |\phi|^2 \phi - \varepsilon \sum_{n=1}^{5} \cos(k^{(n)} \cdot r) \phi,$$

where five vectors $k^{(n)}$ form a star, with angles $2\pi/5$ between adjacent vectors.
An example of a **stable vortex gap soliton** with $S = 1$ [(a) and (b) display contour plots of $|\Phi(x,y)|$ and $\text{Re}(\Phi(x,y))$]:

![Contour plots](image-url)

“Crater” is a vortex soliton squeezed into a single cell of the underlying lattice potential. In many cases, unlike the multi-peak vortex complexes, such compact vortices are completely unstable.
Nevertheless, in the usual 2D model with the cosinusoidal cellular potential and self-focusing cubic nonlinearit,y a stability region for the “craters” was found, provided that the potential is deep (strong) enough [H. Sakaguchi and B.A. Malomed, Phys. Rev. A 79, 043606 (2009)]. In this picture, $A$ is the depth of the potential:
Moreover, taking the compact *crater-shaped vortices* as *building blocks*, one can arrange them into a *ring*, onto which a *global vorticity*, $S$, may be imprinted. This yields patterns ("*supervortices*”) with *two different vorticities*, namely, *individual vorticity* $s = 1$ of each individual “*crater*”, and *global vorticity* $S$. Therefore, the *supervortices* with $S = +1$ and $S = -1$ are *not* equivalent, if $s = +1$ is fixed: *H. Sakaguchi and B.A. Malomed*, Europhys. Lett. 72, 698 (2005); Phys. Rev. A 79, 043606 (2009).
An example of two *stable non-equivalent supervortices* with global vorticities $S = +1$ (a,b) and $S = -1$ (c) in the 2D GPE/NLSE equation:
Part B: Conservative systems with the cubic-quintic nonlinearity

Besides the use of trapping potentials, the stabilization of 2D fundamental and vortical solitons can be provided, in the uniform space, by a combination of self-focusing cubic and self-defocusing quintic nonlinear terms.
The model equation is written in the normalized form, for the spatial-domain propagation of a stationary optical beam in a bulk medium:

\[
    i \frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + |u|^2 u - |u|^4 u = 0.
\]

Stationary solutions for *vortex solitons* with topological charge \( m \) are looked for as

\[
    u(z, x, y) = R(r) \exp(ikz + im\theta),
\]

with \( R(r) \) satisfying the radial equation:

\[
    -kR + \left( \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - \frac{m^2}{r^2} R \right) + R^3 - R^5 = 0.
\]
At $r \to \infty$, the radial equation becomes asymptotically one-dimensional, which gives rise to the well-known exact soliton solution (Kh.I. Pushkarov, D.I. Pushkarov, and I.V. Tomov, Opt. Quant. Electr. 11, 471 (1979)):

$$U(x) = 2 \sqrt{\frac{k}{1 + \sqrt{1 - 16k/3 \cosh(2\sqrt{k}r)}}},$$

which exists at $k < 3/16$.

Broad 2D solitons are asymptotically equivalent, at $r \to \infty$, to the 1D solitons in the radial direction, therefore 2D solitons of any type may exist solely at $k < 3/16 = 0.1875 \equiv k_{\text{max}}$. 

Numerous theoretical and experimental works demonstrate that the cubic-quintic nonlinearity occurs in real optical media, such as chalcogenide glasses, some organic materials, and suspensions of metallic nanoparticles:

Experimentally, the stability of $(2+1)$D fundamental $(m = 0)$ solitons in an optical cubic-quintic medium was demonstrated in
The stability of fundamental solitons \((m = 0)\) in the framework of this equation is obvious. A nontrivial problem is the stability of vortex solitons against splitting by azimuthal perturbations. For the first time, this possibility was reported, on the basis of direct simulations. in:

Stable azimuthal stationary state in quintic nonlinear optical media

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H. Michinel
Departamento de Física Aplicada, Escola Universitaria de Óptica, Universidade de Santiago de Compostela, E-157 06, Santiago de Compostela, Galicia, Spain
Accurate results for this problem have been reported in the following paper:

DOI: 10.1007/s00332-002-0475-3

Spectrally Stable Encapsulated Vortices for Nonlinear Schrödinger Equations

R. L. Pego¹ and H. A. Warchall²,³
The vortex solitons with topological charge $m$ are stable if their radius is large enough, in intervals $k_{cr} < k < k_{max} \equiv 0.1875$. Values of $k_{cr} \equiv \omega_{cr}$ are collected in the table (the relative width of the stability region is $(k_{max} - k_{cr})/ k_{max} = 0.21$ for $m = 1$, and only $0.04$ for $m = 5$):

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\omega_{cr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1487</td>
</tr>
<tr>
<td>2</td>
<td>0.1619</td>
</tr>
<tr>
<td>3</td>
<td>0.1700</td>
</tr>
<tr>
<td>4</td>
<td>0.1769</td>
</tr>
<tr>
<td>5</td>
<td>0.1806</td>
</tr>
</tbody>
</table>
Profiles of the vortices with different topological charges $m$ at the respective stability boundaries:

Fig. 9. Profiles at stability transition for $m = 1, 2, 3, 4, 5$. 
Thus far, no experimental observation of stable or quasi-stable 2D soliton with embedded vorticity has been reported. Experimental demonstration of such (effectively) stable spatial vortex solitons would be a great achievement.
Part C: localized vortices in dissipative media described by complex Ginzburg-Landau equations (CGLEs).

1. The 2D model in free space

First, we consider the stability of vortex (spiral) 2D solitons in the framework of the CGLE with the cubic-quintic nonlinearity. This class of models was introduced in:

The equation may be generalized as a model of laser cavities. In this case, it includes linear loss, $\delta > 0$ (to secure the stability of the zero background around the soliton), cubic gain, $\varepsilon > 0$, and quintic loss, $\mu > 0$ (to secure the overall stability of the system). The equation also includes the diffusion term $\sim \beta$, which actually does not occur in optical models:

$$iA_z + i\delta \cdot A + (1/2 - i\beta)(A_{xx} + A_{yy}) + (1 - i\varepsilon)|A|^2A - (\nu - i\mu)|A|^4A = 0.$$
The presentation of results for the 2D model will follow a rather old paper on this topic:

PHYSICAL REVIEW E, VOLUME 63, 016605

Stable vortex solitons in the two-dimensional Ginzburg-Landau equation

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(Received 19 June 2000; revised manuscript received 27 September 2000; published 20 December 2000)
Examples of the self-trapping of stable vortical solitons with vorticities $S = 1$ (a) and $S = 2$ (b):

FIG. 3. Formation of spinning solitons from real initial field configurations: (a) $S=1$ and (b) $S=2$. The parameters are $\beta = 0.5$, $\delta = 0.5$, $\nu = 0.1$, $\mu = 1$, and $\varepsilon = 2.5$. The initial field distributions are: (a) $U(r;z=0) = 0.2r \exp\left[-(r/7)^2\right]$, and (b) $U(r;z=0) = 0.02r^2 \exp\left[-(r/12)^2\right]$. 
The illustration of the *spiral form* of the phase field in the stable vortex soliton ($s = 1$ and $2$):

![Gray-scale plots of the input Gaussian field with the vorticity $S=1$](image)

(a) amplitude and (b) phase. The output field at $z = 150$: (c) amplitude and (d) phase. The parameters are $\beta = 0.5$, $\delta = 0.5$, $\nu = 0.1$, $\mu = 1$, and $\varepsilon = 2.5$.

![Gray-scale plots of the output Gaussian field with the vorticity $S=2$](image)

(a) amplitude and (b) phase. The output field was taken at $z = 200$. The same as in Fig. 6, but for the vorticity $S=2$. The output field was taken at $z = 200$. 
2. Stabilization of 2D vortices in the cubic-quintic CGLE with the cellular (lattice) potential

The stabilization of dissipative vortex complexes, built of 4 peaks, of both the on-site-centered and off-site-centered types, in the 2D spatial-domain model of laser cavities, was demonstrated in:

The 2D CGLE with the \textit{cubic-quintic} nonlinearity and \textit{cellular potential} is

\begin{equation*}
\frac{\partial u}{\partial z} = \left[ -\delta + \left( \frac{i}{2} + \beta \right) \nabla^2 + (i + \varepsilon) |u|^2 - (iv + \mu) |u|^4 \right] u \\
+ i V_0 \left[ \cos(2x) + \cos(2y) \right] u,
\end{equation*}

with $\delta > 0$, $\varepsilon > 0$, $\mu > 0$.

As said above, in laser media there is no "diffusion of light", hence $\beta = 0$ will be fixed.

A typical set of \textit{stable 4-peak vortex complexes}, found by varying the \textit{linear-loss factor} $\delta$, while the other parameters are fixed: $\varepsilon = 1.85$, $\mu = 1$, $\nu = 0.1$, and $V_0 = 1$.
On the other hand, if the lattice potential is too weak ($V_0 = 0.15$), the 4-peak complexes are unstable:

\[\delta = 0.31\]  \[\delta = 0.595\]  \[\varepsilon = 1.525\]
Another possibility: *stabilization* of various types of *vortical modes* in the 2D CGLE *without diffusion* and *without trapping potentials*, but with *spatial modulation of the linear loss*:
The model equation (which can also be realized in laser cavities):

\[ iE_z + \left( \frac{1}{2} (E_{xx} + E_{yy}) \right) + (1 - i\varepsilon)|E|^2E - (\nu - i\mu)|E|^4 \]

\[ \times E = -ig(r)E, \quad r = \sqrt{x^2 + y^2}, \]

Here the profile of the modulation of the local linear loss is given by \( g(r) = \gamma + Vr^2 \), with \( \gamma > 0 \) and \( V > 0 \).
The model gives rise to a great variety of stable vortex solitons. An example of self-trapping of a simple stable vortex soliton:
An example of another species of **stable** vortical modes, viz., a **rotating elliptically deformed vortex**:
Another **stable** species: an **eccentric spinning vortex** periodically orbiting around the center:
Still another **stable** species – a **rotating** deformed **crescent-shaped vortex**:
The last **stable** species – an **elliptically-shaped**
“**slanted crater**”: 

![Diagram showing the slanted crater species with different images at various stages.](image-url)
Part D: Stable two-dimensional composite solitons in spin-orbit (SO)-coupled self-attractive Bose-Einstein condensates (BEC) in free space

A result of a collaborative work with:

Hidetsugu Sakaguchi and Ben Li
Interdisciplinary Graduate School of Engineering Sciences, Kyushu University, Fukuoka, Japan
A paper reporting basic results to be presented in this part:

PHYSICAL REVIEW E 89, 032920 (2014)

Creation of two-dimensional composite solitons in spin-orbit-coupled self-attractive Bose-Einstein condensates in free space

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(Received 12 December 2013; published 26 March 2014)

It is commonly known that two-dimensional mean-field models of optical and matter waves with cubic self-attraction cannot produce stable solitons in free space because of the occurrence of collapse in the same setting. By means of numerical analysis and variational approximation, we demonstrate that the two-component model of the Bose-Einstein condensate with the spin-orbit Rashba coupling and cubic attractive interactions gives rise to solitary-vortex complexes of two types: semivortices (SVs, with a vortex in one component and a fundamental soliton in the other), and mixed modes (MMs, with topological charges 0 and ±1 mixed in both components). These two-dimensional composite modes can be created using the trapping harmonic-oscillator (HO) potential, but remain stable in free space, if the trap is gradually removed. The SVs and MMs realize the ground state of the system, provided that the self-attraction in the two components is, respectively, stronger or weaker than the cross attraction between them. The SVs and MMs which are not the ground states are subject to a drift instability. In free space (in the absence of the HO trap), modes of both types degenerate into unstable Townes solitons when their norms attain the respective critical values, while there is no lower existence threshold for the stable modes. Moving free-space stable solitons are also found in the present non-Galilean-invariant system, up to a critical velocity. Collisions between two moving solitons lead to their merger into a single one.
(1) Introduction and objectives

The concept of *emulation* (alias *simulation*) of complex physical effects, known in condensed-matter physics, by much simpler settings available in **BEC** (*matter waves*) and **photonics** (*optical waves*), has drawn a great deal of interest:

A new topic has emerged in the framework of this approach: the emulation of spin-orbit (SO) interactions in semiconductors, such as those accounted for by the Rashba and Dresselhaus terms, by mapping the spinor wave function of electrons into the pseudo-spinor two-component wave function of a binary BEC gas:


The SO coupling is a **linear feature** of the system. Its combination with the natural self-repulsive **cubic nonlinearity** of the atomic BEC gives rise to nonlinear effects, such as delocalized **vortices**:

- C. J. Wu, Mod. Phys. Lett. B **23**, 1 (2009);
- X.-Q. Xu and J. H. Han, Phys. Rev. Lett. **107**, 200401 (2011);
- J. Radic', T. A. Sedrakyan, I. B. Spielman, and V. Galitski, Phys. Rev. A **84**, 063604 (2011);
- H. Sakaguchi and B. Li, Phys. Rev. A **87**, 015602 (2013);
The objective here is to construct **self-trapped** (localized) **stable 2D vortical modes** in the SO-coupled BEC with **attractive SPM and XPM** nonlinearities, in the **free space** (without any trapping potential).
At the first glance, this objective seems absolutely impossible. Formal 2D vortex-soliton solutions (alias the above-mentioned “vortex Townes’ solitons”) of the NLSE with the self-attractive cubic term are well known:


**The theory of spiral laser beams in nonlinear media**

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However, such solitons are subject to the above-mentioned strong instability, against splitting and collapse.

Therefore, as already discussed previously, a problem of fundamental interest is to introduce physically meaningful settings, in which both the fundamental and vortical solitons can be stabilized, against the collapse and splitting.
(2) The model

The system of GPES for the (pseudo-) spinor wave function \((\Phi_+, \Phi_-)\) of the binary BEC coupled by the SO terms of the Rashba type with strength \(\lambda \equiv 1\), coefficient of the SPM self-attraction \(\equiv 1\), coefficient of the XPM inter-component attraction \(\gamma \geq 0\), and (for the time being) the strength of the HO trapping potential \(\Omega\):

\[
i \frac{\partial \Phi_+}{\partial t} = -\frac{1}{2} \nabla^2 \Phi_+ - (|\Phi_+|^2 + \gamma |\Phi_-|^2) \Phi_+
+ \lambda \left( \frac{\partial \Phi_-}{\partial x} - i \frac{\partial \Phi_-}{\partial y} \right) + \frac{1}{2} \Omega^2 (x^2 + y^2) \Phi_+,\]

\[
i \frac{\partial \Phi_-}{\partial t} = -\frac{1}{2} \nabla^2 \Phi_- - (|\Phi_-|^2 + \gamma |\Phi_+|^2) \Phi_-
- \lambda \left( \frac{\partial \Phi_+}{\partial x} + i \frac{\partial \Phi_+}{\partial y} \right) + \frac{1}{2} \Omega^2 (x^2 + y^2) \Phi_-,
\]
(3) Semi-vortex states

The coupled GPEs admit a family of solutions for semi-vortices, with vorticities \( m_+ = 0 \) in one component, and \( m_- = 1 \) in the other. The exact ansatz for these solutions is

\[
\phi_+(x, y, t) = e^{-i\mu t} f_1(r^2), \quad \phi_-(x, y, t) = e^{-i\mu t + i\theta} r f_2(r^2),
\]

where \( \mu \) is the chemical potential, \((r, \theta)\) are the polar coordinates, and real functions \( f_{1,2}(r^2) \) obey the following equations:

\[
\begin{align*}
\mu f_1 + 2 \left[ r^2 \frac{d^2 f_1}{d(r^2)^2} + \frac{df_1}{d(r^2)} \right] + \left( f_1^2 + \gamma r^2 f_2^2 \right) f_1 - 2\lambda \left[ r^2 \frac{df_2}{d(r^2)} + f_2 \right] - \frac{\Omega^2}{2} r^2 f_1 &= 0, \\
\mu f_2 + 2 \left[ r^2 \frac{d^2 f_2}{d(r^2)^2} + 2 \frac{df_1}{d(r^2)} \right] + \left( r^2 f_2^2 + \gamma f_1^2 \right) f_2 + 2\lambda \frac{df_1}{d(r^2)} - \frac{\Omega^2}{2} r^2 f_2 &= 0.
\end{align*}
\]
A numerically found cross-section (along $y = 0$) of a stable semi-vortex, at $y = 0$ and $\Omega = 0$, obtained by means of the imaginary-time integration, as a stationary soliton in the free space:
The numerically found dependence between the total norm of the semi-vortices and their chemical potential demonstrates that (1) the norm of the semi-vortex indeed falls *below the critical value* and (2) there is *no finite minimum (threshold) value* of the norm necessary for the existence of the semi-vortex; (3) the norm is *bounded from above* precisely by the *critical value*; (4) the dependence satisfies the Vakhitov-Kolokolov (VK) criterion, \( \frac{d\mu}{dN} < 0 \), which is a necessary condition for the stability:
Direct simulations demonstrate that the semi-vortices are \textit{completely stable} at \( \gamma < 1 \) (XPM/SPM < 1), but they are \textit{unstable} at \( \gamma > 1 \).

In the limit of \( N \rightarrow N_{\text{critical}} \approx 5.85 \), the semi-vortex \textit{degenerates} into the usual (unstable) \textit{Townes’ soliton} with an \textit{infinitely large chemical potential}, in the first component, leaving the second (formerly vortical) component \textit{empty}:
(4) Mixed modes

Another class of localized states can be constructed in the form of mixed modes, so called because they mix fundamental and vortical terms in each component, namely, $m_1 = (0, -1)$ and $m_2 = (0, +1)$, as per the following ansatz (initial guess):

$$\phi_+ = A_1 \exp(-\alpha_1 r^2) - A_2 r \exp(-i\theta - \alpha_2 r^2),$$

$$\phi_- = A_1 \exp(-\alpha_1 r^2) + A_2 r \exp(+i\theta - \alpha_2 r^2).$$

A typical example of the cross-section of the mixed mode:
The dependence between the chemical potential and norm demonstrates that the **mixed-mode** family also complies with the **VK criterion**, hence it may be stable. In the limit of \( N \to N_{\text{critical}} \approx \frac{2 \cdot 5.85}{(1 + \gamma)} \), the mixed mode **degenerates** into a two-component **Townes’ soliton**, while the vortical terms in both components **vanish**.
Direct simulations demonstrate that the mixed mode is *unstable* at \( \gamma < 1 \), and *stable* at \( \gamma > 1 \), i.e., exactly where the semi-vortex is *stable* or *unstable*, respectively. This *stability switch* between the semi-vortex and mixed mode with the increase of \( \gamma \equiv \text{XPM/SPM} \) is explained by the fact that the semi-vortex and mixed mode are *ground states*, which realize the *minimum* of the system’s energy, \( E \), precisely at \( \gamma < 1 \) and \( \gamma > 1 \), respectively.

\[
E = \iint \left\{ \frac{1}{2}(|\nabla \phi_+|^2 + |\nabla \phi_-|^2) - \frac{1}{2}(|\phi_+|^4 + |\phi_-|^4) \\
- \gamma |\phi_+|^2 |\phi_-|^2 + \frac{\lambda}{2} \left[ \phi^*_+ \left( \frac{\partial \phi_-}{\partial x} - i \frac{\partial \phi_-}{\partial y} \right) \right] \\
+ \phi^*_- \left( - \frac{\partial \phi_+}{\partial x} - i \frac{\partial \phi_+}{\partial y} \right) \right\} dx \, dy,
\]

The dependence of the energy of the semi-vortex ("0") and mixed mode ("01") on $\gamma \equiv \text{XPM/SPM}$, for two different fixed values of the total norm, $N = 3.7$, and $N = 3.0$ (precisely at $\gamma = 1$, both the semi-vortex and mixed mode are stable):
Conclusions of Part D

The main result is that the system of two 2D GPEs with the self-attracting nonlinearity, coupled by the linear SO terms of the Rashba type, gives rise to two families of composite (half-fundamental, half-vortical) solitons: semi-vortices, which are stable, and realize the ground state of the system, at $\gamma \equiv \text{XPM/SPM} < 1$, and mixed modes, which do the same at $\gamma > 1$.

This is the first example of a model in which 2D solitons, supported by the cubic self-focusing in the free space, are stable. This may be explained by the fact that the solitons exist with values of the total norm falling below the critical value necessary for the onset of the collapse. In the limit of the norm approaching the critical value, the solitons degenerate into unstable Townes’ solitons.