An answer to two questions of Brewster and Yeh on M-groups

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1 Introduction

Let χ be a (complex) irreducible character of a finite group. Recall that χ is monomial if there exists a linear character $\lambda \in \mathrm{Irr}(H)$, where H is some subgroup of G, such that $\chi = \lambda^G$. A group is an M-group if all its irreducible characters are monomial. In 1992, B. Brewster and G. Yeh [1] raised the following two questions.

Question A. Let M and N be normal subgroups of a group G. Assume that $(|G : M|, |G : N|) = 1$ and that M and N are M-groups. Does this imply that G is an M-group?

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Question B. Let M and N be normal subgroups of a group G. Assume that $(|M|, |N|) = 1$ and that G/M and G/N are M-groups. Does this imply that G is an M-group?

Our main result here is:

Theorem 1.1. *Both Question A and Question B have a negative answer.*

Brewster and Yeh mentioned in [1] that they had found an example showing that Question B has a negative answer and that it would appear in a subsequent article because it relied on techniques different from those of [1]. However, as far as we know, the answer to this question has not been published. For completeness, we consider that it is convenient to include it here.

There is another reason why we think that the group that answers Question B is interesting. One of the most striking problems in character theory of solvable groups is whether or not normal subgroups of odd order M-groups need to be M-groups. Recently, in a 240-page long thesis, M. Loukaki [8] has provided an affirmative answer to this problem for groups whose order is divisible by just two different primes. After such a difficult proof, it is natural to ask whether or not there is a different approach to the problem that allows for a simpler proof. In view of Theorem 5.1 of [7], another way to attack this problem would be to prove that if all the Brauer characters of an odd $\{p, q\}$-group are monomial then the group is monomial. The group that settles Question B is also an example of an odd order $\{p, q\}$-group where all the Brauer characters are monomial and yet the group is not an M-group. It was originally built by E. C. Dade for this purpose, and we are indebted to him for allowing us to use it here.

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2 Proofs

All groups considered are finite. The notation is standard and follows [5]. Recall that if a group A acts coprimely on a group G, $\mathrm{Irr}_A(G)$ is the set of A-invariant irreducible characters of G. It follows from Corollary 8.16 of [5], for instance, that if $\chi \in \mathrm{Irr}_A(G)$, then there exists $\hat{\chi} \in \mathrm{Irr}(AG)$ that extends χ. Recall also that $\chi \in \mathrm{Irr}_A(G)$ is said to be A-primitive if it cannot be induced from any A-invariant character of any proper A-invariant subgroup of G. We begin with the following general lemma.

Lemma 2.1. *Let a π-group A on a π'-group G. Assume that $\chi \in \mathrm{Irr}_A(G)$ and let $\hat{\chi}$ be an extension of χ to AG. If χ is A-primitive, then $\hat{\chi}$ is primitive.*

Proof. Let U be a subgroup of AG and $\varphi \in \mathrm{Irr}(U)$ such that $\varphi^{AG} = \hat{\chi}$. Since

$$|AG : U|\varphi(1) = \varphi^{AG}(1) = \hat{\chi}(1) = \chi(1)$$

is a π'-number, we deduce that U contains a conjugate of A (by Schur-Zassenhaus theorem). If U contains A^g for some $g \in G$, then $\varphi^{g^{-1}} \in \mathrm{Irr}(U^{g^{-1}})$ induces $\hat{\chi}$ and $U^{g^{-1}}$ contains A. Thus, we may assume that U contains A. Now, $U \cap G$ is an A-invariant subgroup of G (because $U \cap G \trianglelefteq U$). Since $(|U : U \cap G|, \varphi(1)) = 1$, Corollary 11.29 of [5] yields that $\varphi_{U \cap G} \in \mathrm{Irr}_A(U \cap G)$. By Problem 5.2 of [5], we have that $\varphi_{U \cap G}$ induces χ. By hypothesis, $U \cap G = G$ and we conclude that $U = AG$, as desired. $\square$

It is also easy to prove the converse, but we will not need it here. In the next result, we show that the answer to Question A is "no". Some of the ideas involved in the construction of the example that solves this question appeared in Theorem A of [10].

Theorem 2.2. *Let p and q be odd primes satisfying $q \equiv -1 \pmod{p}$. Then there exists a group $\Gamma = AE$ that is a split extension of a cyclic group A of order $2p$ by a normal extraspecial q-group E such that Γ has non-linear primitive characters but all its proper Hall subgroups are M-groups.*

Proof. It was proved in [12] that the group of automorphisms of an odd order extraspecial q-group E that act trivially on $Z(E)$ is isomorphic to $\mathrm{Sp}(2n, q)$, where $|E| = q^{2n+1}$. In particular, since $|\mathrm{Sp}(2, q)| = |\mathrm{SL}(2, q)| = (q-1)q(q+1)$ and p is an odd prime divisor of $q+1$, there exists a cyclic group P of order p that acts faithfully on an extraspecial group F of order q^3 and exponent q, in such a way that P centralizes $Z(F)$ and acts faithfully and irreducibly on $F/Z(F)$. Let $E = F_1F_2$ be the central product of two copies F_1 and F_2 of F with $Z(F)$ amalgamated and let P act on F_i, for $i = 1, 2$, as it acts on F. Let an involution ι act on F_1 centralizing $Z(F_1)$ and inverting the elements of $F_1/Z(F_1)$ and let ι act trivially on F_2 and on P. Put $A = P\langle\iota\rangle$ and $\Gamma = AE$.

We have to check that PE and $\langle\iota\rangle E$ are M-groups but Γ has non-linear primitive characters. First, we prove that PE is an M-group. Since $|PE : E| = p$, the linear characters of E either extend or induce to PE and in either case they are monomial. By Theorem 7.5 of [3], the non-linear characters of E have degree q^2. We have that $|PE : Z(PE)| = pq^4$ and that $\chi(1)^2 \leq |PE : Z(PE)|$ for all $\chi \in \mathrm{Irr}(PE)$ (by Lemma 7.4 of [3]). Therefore, the non-linear characters of E extend to irreducible characters of PE. Since the actions of P on F_1 and F_2 are isomorphic, P leaves invariant some maximal abelian subgroup B of E. Now, the characters of PE of degree q^2 lie over linear characters of PB, and we conclude that they are induced from linear characters of PB. Thus, PE is an M-group.

It is clear that $\langle \iota \rangle E$ is supersolvable, so by Theorem 6.22 of [5] it is an M-group.

Finally, we claim that Γ has non-linear primitive characters. Let χ be a non-linear irreducible character of E. By the lemma, it suffices to show that χ is A-primitive. It is easy to see that the unique proper A-invariant subgroups of E that properly contain $Z(E)$ are F_1 and F_2. Since χ is not linear, it lies over some non-principal character of $Z(E)$, so it lies over non-linear characters of F_1 and F_2 (because $F_1' = F_2' = Z(E)$). This means that χ is not induced from any character of either F_1 or F_2 (by degrees). We conclude that χ is A-primitive, as we wanted to show. $\square$

Before building the group that answers Question B, we need one lemma.

Lemma 2.3. *Let a p-group P act faithfully and irreducibly on a p'-group Q and put $G = P \ltimes Q$. Then G acts faithfully and irreducibly on some elementary abelian p-group.*

Proof. Let n be the smallest integer such that G is isomorphic to a subgroup of $\mathrm{GL}(n,p)$. Of course, we may assume that $n > 1$. We have that G acts faithfully on V, where V stands for the natural module for $\mathrm{GL}(n,p)$. We may assume that there exists a non-trivial proper irreducible G-submodule W of V. By the choice of n, G does not act faithfully on W and since the kernel K of the action of G on W is a normal subgroup of G, we conclude that $Q \leq K$. Hence the p-group G/K acts faithfully and irreducibly on the p-group W. It follows that $K = G$, i.e, G acts trivially on W. Thus $C_V(G) > 1$.

It is clear that G acts faithfully on $V/C_V(G)$ (otherwise, arguing as before, we would have that Q acts trivially on $V/C_V(G)$ and, by coprime action, Q would also act trivially on V and this is a contradiction). This means that G is isomorphic to a subgroup of $\mathrm{GL}(m, p)$ for some $m < n$ and this contradicts the choice of n. $\square$

Next, we present Dade's construction. If N is a normal subgroup of a group G and $\psi \in \mathrm{Irr}(N)$ we write $\mathrm{Irr}(G|\psi)$ to denote the set of irreducible characters of G that lie over ψ, i.e, $\mathrm{Irr}(G|\psi) = \{\chi \in \mathrm{Irr}(G) \mid [\chi_N, \psi] \neq 0\}$.

Theorem 2.4. *Let p and q be odd primes satisfying $q \equiv -1 \pmod p$. There exists a $\{p, q\}$-group G that is not an M-group such that $G/O_p(G)$ and $G/O_q(G)$ are M-groups.*

Proof. Let PE be as in Theorem 2.2. The group P acts faithfully and irreducibly on the elementary abelian group $\overline{E} = E/F_2$. So by Lemma 2.3 there exists an elementary abelian p-group A on which $P \ltimes \overline{E}$ acts faithfully and irreducibly. We inflate the action of $P \ltimes \overline{E}$ on A to one of PE on A. Put $G = (P \ltimes E) \ltimes A$. Note that $O_p(G) = A$ and $O_q(G) = F_2$.

First, we claim that $G/O_p(G)$ is an M-group. In order to see this, it is enough to observe that $G/O_p(G) \cong PE$ and that it was already proved in Theorem 2.2 that PE is an M-group.

Next, we claim that $G/O_q(G)$ is an M-group. But $G/O_q(G) \cong (P \ltimes \overline{E}) \ltimes A$ is a semidirect product of the cyclic group P acting on a group with abelian Sylow subgroups. Using Theorem 6.23 of [5], we conclude that $G/O_q(G)$ is an M-group.

Now, we just need to show that G has some non-monomial character. Let $\alpha \in \mathrm{Irr}_P(A)$ be any non-principal P-invariant irreducible character of A (it exists because both $\mathrm{Irr}(A)$ and P are p-groups). Since PE acts irreducibly on A it also acts irreducibly on $\mathrm{Irr}(A)$ (by Proposition 12.1 of [9]) and we conclude that the inertia group of α in $\overline{E}$ is a proper subgroup of $\overline{E}$. But since this inertia subgroup is P-invariant and P acts irreducibly on $\overline{E} \cong F_1/Z(F_1)$, we conclude that it has to be trivial. Thus, $I_G(\alpha) = PF_2A$. Let $\hat{\alpha}$ be an extension of α to PF_2A (such an extension exists by Theorem 6.26 and Corollaries 6.27 and 11.22 of [5]). Since P acts irreducibly on $F_2/Z(F_2)$, we have that PF_2 has no subgroups of index q. Thus any character of PF_2 of degree q is primitive. Let $\xi \in \mathrm{Irr}(PF_2A/A)$ be any such character. Now, by Gallagher's Theorem (Corollary 6.17 of [5]), $\xi\hat{\alpha} \in \mathrm{Irr}(PF_2A|\alpha)$. We have that $(\xi\hat{\alpha})(1) = q$ and PF_2A has no subgroups of index q, so $\xi\hat{\alpha}$ is primitive. By Clifford's correspondence (Theorem 6.11 of [5]), $\chi = (\xi\hat{\alpha})^G \in \mathrm{Irr}(G)$. However, χ is not monomial (for instance, by Theorem 10.1 of [6]). $\square$

Since in any group $O_p(G)$ is contained in the kernel of every p-Brauer character (see Lemma 2.32 of [11]), it is clear all the Brauer characters of this group are monomial. In the situation of Question B, it is possible to prove the following. The proof is an easy consequence of some results on π-special and factorable characters. For a review of their basic properties we refer the reader to [2, 4] or Chapter 40 of [3].

Theorem 2.5. *Let M and N be normal subgroups of a group G. Assume that $(|M|, |N|) = 1$ and that G/M and G/N are M-groups. Then any primitive character of G is linear.*

Proof. Since G/M and G/N are solvable and $M \cap N = 1$, we have that G is solvable. Let $\alpha \in \mathrm{Irr}(G)$ be primitive and let π be the set of prime divisors of $|M|$. By Corollary 2.7 of [4], α factors as the product of a primitive π-special character α_π and a primitive π'-special character $\alpha_{\pi'}$. By Corollary 4.2 of [2], $M \leq \mathrm{Ker}\,\alpha_{\pi'}$. Since G/M is an M-group, we deduce that $\alpha_{\pi'}$ is linear. In the same way, α_π is linear. Therefore, α is linear, as we wanted to prove. $\square$

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