CHARACTERS OF p-GROUPS AND SYLOW p-SUBGROUPS

ALEXANDER MORETÓ[1]

Departamento de Matemáticas, Universidad del País Vasco, Apartado 644, 48080 Bilbao (Spain)
e-mail: mtbmoqua@lg.ehu.es

1 Introduction

The aim of this note is to present some problems and also partial results in some cases, mainly on characters of p-groups. (In the last section we deal with a problem that consists in obtaining information about characters of a Sylow p-subgroup of an arbitrary group from information about the characters of the whole group.) This survey is far from being exhaustive. The topics included are strongly influenced by the author's interests in the last few years. There seems to be an increasing interest in the character theory of p-groups and we hope that this expository paper will encourage more research in the area. In the sixties I. M. Isaacs and D. S. Passman [17, 18] wrote two important papers that initiated the study of the degrees of the irreducible complex characters of finite groups (henceforth referred to as character degrees). The study of the influence of the set of character degrees on the structure of a group was taken up again in the eighties, in large part due to B. Huppert and his school. In particular, this has led to several papers dealing with the character degrees of important families of p-groups since the nineties (see [6, 8, 12, 28, 30, 32, 33, 34, 35, 36, 37]). Here we are mostly concerned with character degrees, but instead of studying particular families of p-groups, we intend to obtain general structural properties of groups according to their character degrees. Other problems on characters of p-groups appear in [25].

The notation is standard. All the groups considered are finite. We write $\mathrm{cd}(G)$ to denote the set of character degrees of a group G, $b(G)$ the maximum of the character degrees of G, $\mathrm{cs}(G)$ the set of conjugacy class sizes, and $c(G)$ and $\mathrm{dl}(G)$ the nilpotence class and derived length of G, respectively. The terms of the ascending Fitting series of a group G will be denoted $F_i(G)$ and the Fitting subgroup $F(G)$. If P is a p-group we write $\Omega_i(P)$ to denote the subgroup of P generated by the elements of order $\leq p^i$.

2 Bounding the derived length and the nilpotence class

Taketa proved that if G is a monomial group (in particular, a p-group) then $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$. An important problem in character theory of finite solvable groups is the Isaacs-Seitz conjecture, which asserts that the derived length of any solvable group G is bounded by $|\mathrm{cd}(G)|$. Despite the fact that this conjecture is not proved yet, it is widely believed that the "right" bound for $\mathrm{dl}(G)$ in terms of $|\mathrm{cd}(G)|$ is logarithmic. This is the case for several families of p-groups, like the Sylow

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subgroups of the symmetric groups or the Sylow p-subgroups of the general linear groups in characteristic p.

A related problem was studied by Isaacs and Knutson in [15]. If N is a normal subgroup of G, we write $\mathrm{cd}(G|N)$ to denote the set of character degrees of G whose kernel does not contain N, i.e., the set of degrees of the irreducible characters of G that "'say something about N". With this notation, they obtained bounds for $\mathrm{dl}(N)$ in terms of $|\,\mathrm{cd}(G|N)|$.

In an impressive series of papers, T. M. Keller (see [21, 22, 23]) has reduced the problem of finding a logarithmic bound for the derived length of a solvable group in terms of the cardinality of the set of character degrees to the following conjecture.

Conjecture 1 Let G be a solvable group. Then there exist constants C_1 and C_2 such that

$$\mathrm{dl}(F(G)) \leq C_1 \log |\,\mathrm{cd}(G|F(G))| + C_2.$$

The reason for the inclusion of this conjecture in this survey is that its proof should not be much harder than that of the following conjecture.

Conjecture 2 Let P be a p-group. Then $\mathrm{dl}(P)$ is bounded logarithmically in terms of $|\,\mathrm{cd}(P)|$.

In other words, Keller's work comes close to reducing to p-groups the problem of replacing Taketa's bound by a logarithmic one. Unfortunately, the p-group case seems to be extremely hard. For instance, it is not known the answer to the following well-known question.

Question 2.1 Does there exist a p-group of derived length 4 with 4 character degrees?

M. C. Slattery has proved that the set of degrees of such a p-group cannot be $\{1, p, p^2, p^3\}$ [38].

Of course, one cannot hope to obtain any bounds for the nilpotence class of a p-group in terms of the number of character degrees. (It is well-known that there exist p-groups P of maximal class of arbitrarily large order with an abelian subgroup of index p and, therefore, $\mathrm{cd}(P) = \{1, p\}$.) However, if we fix the set $\mathcal{S}$ of character degrees then, sometimes, we can obtain bounds for the nilpotence class. In the following, $\mathcal{S}$ denotes a finite set of powers of p containing 1. We say that $\mathcal{S}$ is class bounding if there exists a constant C (depending on $\mathcal{S}$) such that $c(P) \leq C$ for any p-group P with $\mathrm{cd}(P) = \mathcal{S}$. In 1968 Isaacs and Passman [18] proved that if $|\mathcal{S}| = 2$ then $\mathcal{S}$ is class bounding if and only if p does not belong to $\mathcal{S}$. Later, in 1994, Slattery [39] found sets of arbitrarily large size that are class bounding within the class of metabelian groups. However, apart from the result of Isaacs and Passman on sets of cardinality 2, there were no more theorems asserting that a given set $\mathcal{S}$ is class bounding (or non-class-bounding) until 2001. In [16] Isaacs and the present author proved, among other results, that if $\mathcal{S} \subseteq \{1, p^a, \ldots, p^{2a}\}$ then $\mathcal{S}$ is class bounding. All the class bounding sets $\mathcal{S}$ found in that paper have the property that $p \notin \mathcal{S}$. In [19] Isaacs and Slattery prove that this a necessary condition for a set $\mathcal{S}$ to be class bounding. However, an example constructed in [20] shows that this is not a sufficient condition. In [20] Jaikin-Zapirain and the author also find more class bounding sets. We refer the reader to [16, 19] and [20] for the detailed results and to [20] for some specific questions related to this problem.

Question 2.2 Which are the class bounding sets?

A complete answer to this question seems to be out of the scope of the known methods. We do not know even what to conjecture. I am inclined to think that the probability that a set is class bounding is 0, in the sense that

$$\lim_{n\to\infty} \frac{\#\{\mathcal{S} \mid \mathcal{S} \text{ is class bounding and } \max(\mathcal{S}) \le p^n\}}{\#\{\mathcal{S} \mid \max(\mathcal{S}) \le p^n\}} = 0,$$

but there is little evidence for this.

We close this section with a question that relates the two problems discussed here.

Question 2.3 Is it true that if $\mathrm{cd}(P) = \mathcal{S}$, $|\mathcal{S}| = 3$ and $p \notin \mathcal{S}$, then P is metabelian?

On the one hand, an affirmative answer to this question would give a new proof of part of Theorem D of [16]. On the other hand, it would provide another situation where Taketa's bound is not best possible.

3 Minimal characters and normal subgroups

We write $m(P)$ to denote the minimal degree of the non-linear irreducible characters of P (for convenience, we write $m(P) = 1$ if P is abelian). It is well-known that this number has a strong influence on the structure of the group P (see, for instance, Problem 5.14 of [11] and [26]). In particular, we want to stress the influence of $m(P)$ in the last problem discussed in the previous section (see [20]). We introduce a new invariant associated to any p-group. Write $|G : Z(G)| = p^{2n+e}$, where $e \in \{0, 1\}$. We define $m_1(P) = n - \log_p m(P)$, i.e., $m(P) = p^{n-m_1(P)}$.

With this notation, the groups studied in [4] are exactly those that satisfy $m_1(P) = 0$. One of the main results of that paper characterizes such groups in terms of their normal subgroups. More precisely, we prove the following. (See Theorems B and C of [4].)

Theorem 3.1
1. *Assume that* $e = 0$. *Then* $m_1(P) = 0$ *if and only if* P *satisfies the strong condition on normal subgroups.*
2. *Assume that* $e = 1$. *Then* $m_1(P) = 0$ *if and only if* P *satisfies the weak condition on normal subgroups.*

We recall from [4] the definition of the strong (weak) condition. We say that a p-group P satisfies the strong (weak) condition on normal subgroups if for every $N \trianglelefteq P$, either $P' \le N$ or $N \le Z(P)$ (either $P' \le N$ or $|NZ(P) : Z(P)| \le p$).

The proof of Theorem 3.1 requires a careful study of the groups satisfying these properties. This analysis leads G. A. Fernández-Alcober and the author to group-theoretical properties of the groups under consideration which, we think, have some interest by themselves. For instance, in Theorem F of [4] we obtain a bound for the index of the center of the groups with any of these properties and class ≥ 3.

These definitions and part of Theorem F of [4] have been generalized by Isaacs [14]. Isaacs defines an invariant $a(P)$ associated to any finite p-group P as follows: $a(P) = a$ is the minimum integer such that if N is a normal subgroup of P and $|NZ(P) : Z(P)| \geq p^a$ then $P' \leq N$. Observe that $a(P) = 0$ if and only if P is abelian, $a(P) = 1$ if and only if P satisfies the strong condition and is not abelian and $a(P) = 2$ if and only if P satisfies the weak condition but not the strong condition. Therefore, Theorem 3.1 says that if $e = 0$, then $m_1(P) = 0$ if and only if $a(P) \leq 1$, while if $e = 1$, then $m_1(P) = 0$ if and only if $a(P) \leq 2$. We think that it should be possible to generalize this result too. Our aim now is to present some results that suggest this.

We begin by proving that if $m_1(P)$ is small then $a(P)$ is also small.

Proposition 3.2 *With the notation above, if $m_1(P) \leq m$ for some $a \in \mathbb{N}$, then $a(P) \leq 2m + 1 + e$.*

Proof Let N be a normal subgroup of P such that $P' \not\leq N$. Since P/N is not abelian and $\mathrm{cd}(P/N) \subseteq \mathrm{cd}(P)$, we have that

$$p^{2(n-m)} \leq |P/N : Z(P/N)| \leq |P : NZ(P)| \leq |P : Z(P)| = p^{2n+e},$$

and it follows that $|NZ(P) : Z(P)| \leq p^{2m+e}$. Therefore $a(P) \leq 2m + 1 + e$. □

The bound obtained here is best possible, as direct products of p-groups of maximal class with non-trivial abelian p-groups show. This result generalizes the easy part of Theorem 3.1. However, in this general setting the converse is not true, as the following example shows.

Example 3.3 Let $P = \langle x, y \mid x^{p^n} = y^{p^{2n}} = 1, y^x = y^{1+p^n}\rangle$ for $n \in \mathbb{N}$. Then $P' = Z(P) = \langle y^{p^n}\rangle$, $|P : Z(P)| = p^{2n}$ and $N = \langle x^p, y^{p^{n+1}}\rangle$ is a normal subgroup of P such that $P' \not\leq N$ and $|NZ(P) : Z(P)| = p^{n-1}$. Now it is possible to prove that $a(P) = n$ and $m_1(P) = n - 1$.

Now we prove that if P is a group of class 2 and $a(P)$ is small, then $m_1(P)$ is small. We first need a lemma, which is a generalization of [4, Theorem D]. In order to prove it it is enough to mimic the proof of Theorem D of [4], so we will just give a sketch of it.

Lemma 3.4 *Let P be a p-group of class 2. If $a(P) > 1$, then $\exp P/Z(P) \leq p^{a(P)}$. Furthermore, if $\exp P/Z(P) = p^{a(P)}$ then $P/Z(P) \cong C_{p^{a(P)}} \times C_{p^{a(P)}}$ and $P' \cong C_{p^{a(P)}}$.*

Proof Let M be a maximal subgroup of P' and let N be a normal subgroup of P maximal with respect to the property $P' \cap N = M$. Let $K/N = Z(P/N)$. Since the derived subgroup of P/N is of order p, P/K is elementary abelian. By the choice of N, we have that K/N is cyclic, so $K/NZ(P)$ is cyclic. Since $P' \not\leq N$, $|NZ(P) : Z(P)| \leq p^{a(P)-1}$. Now the bound for the exponent of $P/Z(P)$ follows from the fact that this abelian group has at least two cyclic factors of maximal order.

If $\exp P/Z(P) = p^{a(P)}$ the previous argument shows that $P/Z(P)$ is a direct product of two cyclic groups of order $p^{a(P)}$ and, perhaps, some cyclic factors of smaller order. We want to prove that in fact all the cyclic factors are of order $p^{a(P)}$. In this case, we can choose $M = \Omega_{a(P)-1}(P')$. Let $T/Z(P) = \Omega_{a(P)-1}(P/Z(P))$. Then $\exp[T,P] = \exp T/Z(P) = p^{a(P)-1}$ and consequently $[T,P] \leq M$. Hence $T \leq K$. We know that $K/NZ(P)$ is cyclic and $|NZ(P) : Z(P)| \leq p^{a(P)-1}$. Therefore,

$$|\Omega_{a(P)-1}(P/Z(P))| \leq p^{2a(P)-2}.$$

It follows that $P/Z(P) \cong C_{p^{a(P)}} \times C_{p^{a(P)}}$ and $P' \cong C_{p^{a(P)}}$. □

The previous example shows that the bound obtained in this lemma cannot be improved. Of course, the order of $P/Z(P)$ cannot be bounded in terms of $a(P)$ when $c(P) = 2$. This lemma proves that, at least, it is possible to bound the exponent.

Theorem 3.5 *Let P be a p-group of class 2. If $a(P) = a > 1$ then $m_1(P) \leq a - 1 - e$.*

Proof Since $c(P) = 2$, $\chi(1)^2 = |P : Z(\chi)|$ for any $\chi \in \mathrm{Irr}(P)$, by Theorem 2.31 of [11]. Let $K = \mathrm{Ker}\,\chi$. If χ is non-linear, then $|KZ(P) : Z(P)| \leq p^{a-1}$. We also know that $Z(\chi)/K$ is cyclic and $\exp P/Z(P) \leq p^{a-e}$ by the previous lemma. It follows that $|Z(\chi) : Z(P)| \leq p^{2a-1-e}$, so

$$|P : Z(\chi)| \geq p^{2n-2a+1+2e}.$$

Since this index is a square, we have that $|P : Z(\chi)| \geq p^{2(n-a+1+e)}$, and we deduce that $\chi(1) \geq p^{n-a+1+e}$, as we wanted to prove. □

The bound in this theorem cannot be improved, as Example 3.3 shows. However, I think that the hypothesis on the class can be removed.

Conjecture 3 Let P be a p-group. Then $m_1(P) \leq a(P)-1-e$ whenever $a(P) > 1$.

4 Miscellaneous questions

In the last years a number of similar results for character degrees and conjugacy class sizes have been obtained. However, the reason for the existence of this parallelism, if any, is still unknown. Isaacs [10] proved that any set of powers of p containing 1 occurs as the set of character degrees of a p-group of class ≤ 2.

The analog result for conjugacy class sizes has been obtained by J. Cossey and T. Hawkes [2]. Although it is easy to find sets of character degrees (resp. conjugacy class sizes) that impose restrictions on the class sizes (resp. character degrees), Fernández-Alcober and the author [5] proved that there is not any relation between the cardinalities of these sets, i.e., given any two integers m and n greater than 1, there exists a p-group with m character degrees and n conjugacy class sizes. A complete answer to the following question, which was first raised in [5], seems to be very difficult.

Question 4.1 Determine the pairs (A, B) of sets of powers of p containing 1 such that there exists a p-group P with $\mathrm{cd}(P) = A$ and $\mathrm{cs}(P) = B$.

The particular case $|A| = |B| = 2$ is being studied by Cossey and Hawkes [7].

We recall that a group G is normally monomial if any irreducible character of G is induced from a linear character of a normal subgroup. It was thought that, perhaps, the derived length of normally monomial groups was bounded by some constant. However, in [24] L. Kovacs and C. R. Leedham-Green constructed normally monomial p-groups of derived length the integer part of $\log_2(p+1)$. As far as I know, the answer to the following question is still unknown.

Question 4.2 Does there exist a function f such that if P is a normally monomial p-group then $\mathrm{dl}(P) \leq f(p)$?

A classical problem in character theory is to determine what kind of group theoretical information can be determined from the knowledge of the character table. For instance, R. Brauer [1] asked whether or not the derived length of a solvable group can be read off from the character table. This question was answered negatively even for p-groups by S. Mattarei [29], but the following question remains open.

Question 4.3 Do there exist p-groups P_1 and P_2 with the same character table and $\mathrm{dl}(P_1) \geq \mathrm{dl}(P_2) + 2$?

5 Character degrees of Sylow p-subgroups

There are some known results that give information on the structure of a finite group in terms of the p-parts of the irreducible complex (or Brauer) characters (see, for instance, [27]). In this section we deal with the following problem.

Conjecture 4 Let G be a finite group and write $e_p(G)$ to denote the exponent of the largest p-part of the irreducible complex characters of G. Then $e_p(P)$ is bounded by some function of $e_p(G)$, where $P \in \mathrm{Syl}_p(G)$.

If $e_p(G) = 0$ then P is abelian (by Ito-Michler's theorem), so we will assume $e_p(G) > 0$ in the remaining. This might lead one to think that, in fact, $e_p(P) \leq e_p(G)$ but this is false even for solvable groups, as the following example of Isaacs shows.

Example 5.1 Let V be a row vector space of dimension 3 over the finite field with two elements. Consider the natural action of the subgroup H of $GL(3,2)$ consisting of the matrices

$$\begin{pmatrix} GL(2,2) & \mathbf{0} \\ * \quad * & 1 \end{pmatrix}$$

on V. If $G = VH$ then $e_2(G) = 1$ and $e_2(P) = 2$, where P is a Sylow 2-subgroup of G. There are similar examples for $p = 3$.

As far as I know, it might be true that $e_p(P) \leq e_p(G)$ for $p \geq 5$ and G solvable and that $e_p(P) \leq 2e_p(G)$ for any prime and any group. This would imply that if G is solvable then $\mathrm{dl}(P) \leq 2e_p(G) + 1$, using Taketa's theorem. This last inequality is the main theorem of [9].

We are able to prove the bound $e_p(P) \leq e_p(G)$ for metanilpotent groups (in particular for supersolvable groups). However this bound does not hold for groups of Fitting height 3 as Isaacs' example shows. For solvable groups we obtain a bound for $e_p(P)$ in terms of $e_p(G)$ and the Fitting height of G.

We begin with an elementary lemma.

Lemma 5.2 *Let N be a normal subgroup of G. Then $e_p(N) \leq e_p(G)$. In particular, if G has a normal Sylow p-subgroup P, then $e_p(P) \leq e_p(G)$.*

Proof This is an immediate consequence of Clifford theory. □

Theorem 5.3 *Let G be a solvable group and R the smallest normal subgroup of G such that G/R is nilpotent. Assume that R is p-nilpotent. Then $e_p(P) \leq e_p(G)$ for any $P \in \mathrm{Syl}_p(G)$.*

Proof We may assume that $PR > R$ (otherwise the result follows by induction and Lemma 5.2). Since G/R is nilpotent and PR/R is a Sylow p-subgroup of G/R, we have that $PR \trianglelefteq G$ and, again by induction and Lemma 5.2, we may assume that $G = PR$. Then G is p-nilpotent, and the result follows. □

Corollary 5.4 *Let G be a metanilpotent group and $P \in \mathrm{Syl}_p(G)$. Then $e_p(P) \leq e_p(G)$.*

Our next aim is to bound $e_p(P)$ in terms of the Fitting height $h(G)$ and $e_p(G)$. The key is the following lemma, which gives a bound for the p-part of the order of the quotient of two consecutive terms of the ascending Fitting series $F_{i+1}(G)/F_i(G)$ when $F_i(G)/F_{i-1}(G)$ is a p'-group.

Lemma 5.5 *Let G be a solvable group and assume that $O_p(G) = 1$. Then*

$$|F_2(G)/F(G)|_p \leq p^{2e_p(G)-1}.$$

Proof By Gaschutz's theorem (see [27, Theorem 1.12]) $G/F(G)$ acts faithfully on the abelian group $F(G)/\Phi(G)$. Therefore it also acts faithfully on $\mathrm{Irr}(F(G)/\Phi(G))$. Let $P_1/F(G) \in \mathrm{Syl}_p(F_2(G)/F(G))$ and consider the faithful action of the p-group

$P_1/F(G)$ on the p'-group $\mathrm{Irr}(F(G)/\Phi(G))$. We may assume that $|P_1/F(G)| > 1$. By [13], there exists $\lambda \in \mathrm{Irr}(F(G)/\Phi(G))$ such that $|I_{P_1/F(G)}(\lambda)| < |P_1/F(G)|^{1/2}$. Since $P_1 \trianglelefteq G$,

$$p^{e_p(G)} \geq |P_1/F(G) : I_{P_1/F(G)}(\lambda)| > |P_1/F(G)|^{1/2}$$

and the result follows. □

Let G be a group such that $O_p(G) = 1$. We define the following series

$$1 = P_0 \triangleleft N_1 \triangleleft P_1 \triangleleft N_2 \triangleleft P_2 \triangleleft \cdots \triangleleft G,$$

where N_{i+1}/P_i is the largest nilpotent subgroup of G/P_i and P_i/N_i is the largest normal p-subgroup of G/N_i. Let $s_p(G)$ be the minimum integer such that $G = N_s$ or $G = P_s$. It is clear that $l_p(G) \leq s_p(G) \leq h(G) \leq 2s_p(G)$. Observe that if there are just two primes dividing $|G|$, then this series is exactly the ascending p', p-series.

Corollary 5.6 *Let G be a solvable group and $P \in \mathrm{Syl}_p(G)$. Then $|P : O_p(G)| \leq p^{s_p(G/O_p(G))(2e_p(G)-1)}$.*

Proof It is enough to apply repeatedly the previous lemma. □

Now we are ready to bound $e_p(P)$ in terms of $e_p(G)$ and $s_p(G)$.

Theorem 5.7 *Let G be a solvable group and $P \in \mathrm{Syl}_p(G)$. Then*

$$e_p(P) \leq (2s_p(G/O_p(G)) + 1)e_p(G) - s_p(G).$$

Proof By Lemma 5.2, $e_p(O_p(G)) \leq e_p(G)$ and, by the previous corollary, $|P : O_p(G)| \leq p^{s_p(G)(2e_p(G)-1)}$. Therefore

$$b(P) \leq |P : O_p(G)| b(O_p(G)) \leq p^{s_p(G)(2e_p(G)-1)+e_p(G)}$$

and the result follows. □

Corollary 5.8 *Let G be a group of order $p^a q^b$ and $P \in \mathrm{Syl}_p(G)$. Then*

$$e_p(P) \leq C_1 e_p(G) \log e_p(G) + C_2.$$

Proof It is enough to observe that $s_p(G/O_p(G)) = l_p(G/O_p(G))$ and apply [27, p. 194] to the bound in Theorem 5.7. □

It seems hard to find a counterexample to the conjecture, since it would be necessary to construct an infinite family of groups with Fitting height going to infinity and at least 3 primes dividing the order of each of the groups.

There is a similar conjecture, due to G. Navarro [31].

Conjecture 5 Let G be a finite group. Then

$$\prod_{p\in\pi(G)} b(G_p) \leq b(G),$$

where G_p is a Sylow p-subgroup of G.

Despite the similar flavor of these conjectures, it does not seem easy to apply results obtained for one of them to the other.

We close by remarking that it is not difficult to prove the conjugacy-class analog of Conjecture 4 for solvable groups. (We believe that the solvability hypothesis is not necessary, however.) On the other hand, the analog of Conjecture 5 is completely false even for solvable groups (see [3]).

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