A RESTRICTION ON THE POSITION OF NORMAL SUBGROUPS IN NILPOTENT GROUPS

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1. Introduction

The study of groups by imposing conditions on the set of their normal subgroups is a theme which has recursively appeared in group theory since early in its history, both for finite and infinite groups. It is well-known, for example, the structure of the groups in which every subgroup is normal, a result that dates back to Dedekind. Another property which is frequently used in group theory is that a finite p-group all of whose abelian normal subgroups are cyclic must be either cyclic itself or a 2-group of maximal class. It is also very interesting, but perhaps not so well-known, the following generalization of this result due to Thompson: if $p > 2$ and G is a finite p-group in which every abelian normal subgroup can be generated by r elements then every subgroup of G can be generated by $r(r+1)/2$ elements. Finiteness conditions related to normal subgroups are also of great importance, such as the minimal and maximal conditions. A comprehensive account of the properties of the groups satisfying one of these conditions can be found in Chapter 5 of [7].

In this paper we survey recent research on a new kind of restriction on the normal subgroups of a group, a subject to which we have contributed actively. An elementary fact in group theory is that any subgroup of a group G which either contains G' or is contained in $Z(G)$ must be normal. But what can we say about the structure of G if these are the only normal subgroups it has? Surprisingly enough, this question does not seem to have been raised until very recently and the answer to it, at least for nilpotent groups, will be one of the issues we will touch on in this paper. More generally, we will deal with nilpotent groups whose normal subgroups either contain G' or do not separate more than a fixed distance from the centre of the group. In other words, we will assume there exists $n \in \mathbb{N} \cup \{\infty\}$ such that $|NZ(G) : Z(G)| < n$ for every $N \trianglelefteq G$ not containing G'. As we will show, this restriction has important consequences about the structure of the group, and in particular about the index of the centre of G.

Let us briefly explain how this paper is organized. In the second section we comment on the character-theoretical problem that motivated our dealing with this kind of restrictions, and present the first results we proved, which apply to finite p-groups. We will devote the third section to a substantial generalization of our study due to I.M. Isaacs, still within the world of finite p-groups. Isaacs' ideas suggested to us abandoning finite p-groups and working more generally with (infinite) nilpotent groups. The results obtained in this direction are collected in the fourth section of the paper. Finally, in the last section we give a list of open problems for the reader who gets interested in this area of research.

2. Origin of the problem

Incidentally, our interest in these questions arose when we were dealing with a problem about characters in finite groups. It is well-known that the degree of a complex irreducible character of a finite group G cannot exceed $|G : Z(G)|^{1/2}$. If this value is attained then G is called of *central type*. As proved by R.B. Howlett and I.M. Isaacs in [3], groups of central type are necessarily soluble. If we ask further that the only character degrees of G are 1 and $|G : Z(G)|^{1/2}$ then the conclusion is much stronger: G is nilpotent of class 2 and all but one of the Sylow subgroups of G are abelian. In these circumstances, we say that G is a *group with two extreme character degrees*. By the last remark, the study of these groups may be reduced to the case of p-groups, and we wonder how we can characterize p-groups with two extreme character degrees in purely group-theoretical terms. This is the main purpose of our paper [1]. One of these characterizations, which is yet another instance of a still not well understood duality between characters and conjugacy classes in finite groups, is the following: the groups under consideration are just the p-groups with two extreme conjugacy class lengths, namely 1 and $|G'|$.

On the other hand, if G has two extreme character degrees, it is straightforward to see that a normal subgroup N of G must satisfy either that $G' \leq N$ or that $N \leq Z(G)$. If the normal subgroups of a group G fulfil this restriction, we say that G satisfies the *strong condition on normal subgroups*. Is the converse true? It turns out that this is the case, provided that $|G : Z(G)|$ is a square, a condition that is obviously necessary for G to have two extreme character degrees, but is not a consequence of the strong condition on normal subgroups. Thus we have the following theorem.

Theorem 2.1. *Let G be a finite p-group such that $|G : Z(G)|$ is a square. Then G is a group with two extreme character degrees if and only if G satisfies the strong condition on normal subgroups.*

The proof of the 'if part' of this theorem relies on a couple of results giving detailed information about the structure of finite p-groups with the strong condition on normal subgroups. The first one is related to groups of class 2.

Theorem 2.2. *Let G be a finite p-group of class 2 satisfying the strong condition on normal subgroups. Then $\exp G/Z(G) = \exp G' = p$.*

Once we have this result, it is quite easy to prove Theorem 2.1 for groups of class 2. To see this, let $\chi \in \mathrm{Irr}(G)$ be non-linear. Then $G' \not\leq \operatorname{Ker}\chi$ and, by the strong condition, $\operatorname{Ker}\chi \leq Z(G)$. It follows that $Z(\chi)/Z(G)$ is a cyclic group. Now G being of class 2 has two consequences: on the one hand, the last theorem yields that $|Z(\chi) : Z(G)| = 1$ or p; on the other hand, it is a basic result in character theory that $|G : Z(\chi)| = \chi(1)^2$ is a square. Since $|G : Z(G)|$ is also a square by hypothesis, we necessarily have that $Z(\chi) = Z(G)$ and $\chi(1) = |G : Z(G)|^{1/2}$, as desired.

In view of the previous discussion, in order to complete the proof of Theorem 2.1, it suffices to take into account this second result from [1], which requires a thorough analysis of the groups in question.

Theorem 2.3. *Let G be a finite p-group satisfying the strong condition on normal subgroups. Then the nilpotency class of G is at most 3, and if it equals 3 then necessarily $|G : Z(G)| = p^3$.*

After having proved Theorem 2.1, it is natural to ask next what happens when the index of the centre of the p-group G is not a square, say $|G : Z(G)| = p^{2n+1}$. Of course, if we stick to the definition we have given of a group with two extreme character degrees then there are no such groups in this setting. However, since the biggest possible degree of an irreducible character of G is p^n, it seems reasonable to extend our definition and say that a group has two extreme character degrees in this case if the only degrees arising are 1 and p^n. Again, we can give a characterization of these groups in terms of their normal subgroups, but nevertheless it is not exactly having the strong condition on normal subgroups.

Theorem 2.4. *Let G be a finite p-group such that $|G : Z(G)|$ is not a square. Then the following two conditions are equivalent:*

(i) *G has two extreme character degrees.*
(ii) *For any normal subgroup N of G, either $G' \leq N$ or $|NZ(G) : Z(G)| \leq p$.*

If (ii) above holds for a finite p-group G, we say that G satisfies the *weak condition on normal subgroups.* The proof of Theorem 2.4 follows in a similar way to that of Theorem 2.1. Again, it is straightforward to see that (i) implies (ii), and the converse can be easily proved for groups of class 2 with the help of the following result, in the same spirit of Theorem 2.2.

Theorem 2.5. *Let G be a finite p-group of class 2 satisfying the weak condition on normal subgroups. Then $\exp G/Z(G) = \exp G' = p$ or p^2. Moreover, in the latter case $G/Z(G) \cong C_{p^2} \times C_{p^2}$ and $G' \cong C_{p^2}$.*

In particular, if G is a finite p-group of class 2 satisfying the weak condition on normal subgroups and $|G : Z(G)|$ is not a square then $\exp G/Z(G) = p$, and we can argue much in the same way as we did for groups with the strong condition. Now, the result that allows us to reduce to the class 2 case is the following one, which plays for the weak condition the role of Theorem 2.3 for the strong condition.

Theorem 2.6. *Let G be a finite p-group satisfying the weak condition on normal subgroups. Then the nilpotency class of G is at most 4, and if the class is greater than 2 then $|G : Z(G)| = p^3, p^4$ or p^6 for odd p and $|G : Z(G)| = 2^3$ or 2^4 for $p = 2$.*

Observe that the only way for $|G : Z(G)|$ not to be a square if the class exceeds 2 is that $|G : Z(G)| = p^3$, and in this case it is clear that the only character degrees are 1 and p. Hence Theorem 2.4 is a consequence of the last two theorems. We stress that the hardest part of the work is done in the proof of Theorem 2.6.

On the other hand, Theorems 2.3 and 2.6 have an importance of their own, since they provide a significant restriction on the structure of a finite p-group with either the strong or the weak condition on normal subgroups if the class is greater than 2, namely that the index of the centre must be quite small. (Note that this is not the case when the class is 2: all p-groups with derived subgroup of order p satisfy the strong condition, and the index of the centre cannot be bounded in this case, as extraspecial groups show.) Even more, sometimes it is not only the index of the centre that we can bound, but also the order of the whole group. This happens when the nilpotency class takes the maximum possible value in the corresponding family of groups, as stated in the following result from our work [1].

Theorem 2.7. *Let G be a finite p-group.*

(i) *If G satisfies the strong condition on normal subgroups and has class 3 then $|G| \leq p^5$. Furthermore, if $p = 2$ then $|G| = 2^4$.*

(ii) *If G satisfies the weak condition on normal subgroups and has class 4 then $|G| \leq p^6$. Furthermore, if $p = 2$ then $|G| = 2^5$.*

The bounds we provide above are sharp, and show that the group is very close to a p-group of maximal class. In fact, for $p = 2$ they actually yield that the group has maximal class. For odd p this is not necessarily true, but we can see that $G/Z(G)$ is of maximal class. As it turns out, this is the first step in the proof of the previous theorem.

We note that G. Silberberg has also proved that a finite p-group G with the strong condition and class 3 satisfies that $|G : Z(G)| = p^3$ and $|G| \leq p^5$, see [8].

Summarizing, we could say that the underlying moral in all of these results is that the groups with one of the above conditions on normal subgroups face more and more restrictions as the nilpotency class grows: when the class is 2 the group can separate as much as we want from its centre, this is no longer true when the class is bigger than 2, then there comes a limit value of the class that bounds even the order of the group, and after this limit value the groups of this kind no longer exist.

3. Isaacs' generalization

Prompted by the structural results we have described in the previous section, Isaacs considered in [4] a natural generalization of the strong and weak conditions on normal subgroups for finite p-groups, by asking that all normal subgroups either contain the derived subgroup or separate boundedly from the centre. Then his objective is to prove that, provided that the class is greater than 2, the index of the centre is bounded in these groups, thus generalizing Theorems 2.3 and 2.6. In fact, he does even better, since he shows that it suffices to impose the condition not on all normal subgroups but only on those containing the centre. It is convenient to introduce the following two invariants.

Definition. Let G be a finite p-group.

(i) Let $\mathcal{N}$ denote the collection of all normal subgroups N of G such that $G' \not\leq N$. Then we define $a(G)$ to be the minimum value of $a \geq 0$ such that $|NZ(G) : Z(G)| < p^a$ for all $N \in \mathcal{N}$.

(ii) Similarly, let $\mathcal{N}^*$ denote the collection of all normal subgroups N of G such that $G' \not\leq N$ and $Z(G) \leq N$. Then we define $b(G)$ to be the minimum value of $b \geq 0$ such that $|NZ(G) : Z(G)| < p^b$ for all $N \in \mathcal{N}^*$.

When there is no risk of confusion we write simply a and b instead of $a(G)$ and $b(G)$. Observe that $a(G) = 0$ if and only if $\mathcal{N}$ is empty, that is, if and only if G is abelian. In the same way, $b(G) = 0$ if and only if the class of G is at most 2. Also, $a(G) \leq 1$ is equivalent to G satisfying the strong condition on normal subgroups and $a(G) \leq 2$ is equivalent to the weak condition.

Since $\mathcal{N}^* \subseteq \mathcal{N}$, we always have that $b(G) \leq a(G)$. On the other hand, if K is a normal subgroup of G then $a(G/K) \leq a(G)$ and $b(G/K) \leq b(G)$. If $a(G) > 1$ then it is not difficult to see that $a(G/K) < a(G)$ for any $K \trianglelefteq G$ containing $\Omega_1(Z(G))$. This provides the grounds for the use of inductive arguments.

The first of Isaacs' results bounds the exponent of G' in terms of $a(G)$, thus extending Theorems 2.2 and 2.5 to arbitrary values of a and of the nilpotency class. We give a slightly sharpened version of it, more precisely we describe in part (ii) of the theorem all the exceptions to the general rule stated in (i). For the sake of completeness and to maintain the parallelism with our previous theorems, we also include a similar result for the exponent of $G/Z(G)$, which can be proved analogously.

Theorem 3.1. *Let G be a finite p-group. Then one of the following cases holds:*

(i) $\exp G' \leq p^a$ *and* $\exp G/Z(G) \leq p^a$.

(ii) *G is a 2-group of maximal class of order greater than 8, and* $\exp G' = \exp G/Z(G) = 2^{a+1}$.

The main result of Isaacs' in [4] is the next one, giving a bound for the index of the centre in terms of a or b.

Theorem 3.2. *If G is a finite p-group of class greater than 2 then $|G : Z(G)| \leq p^{3b}$. In particular, $|G : Z(G)| \leq p^{3a}$.*

The following construction, also due to Isaacs, shows that the two bounds above are sharp. Let F be a finite field with q elements, where $q = p^e$ is a power of a prime $p > 2$. Then the set

$$E = \left\{ \begin{pmatrix} 1 & x & x(x-1)/2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \mid x \in F \right\}$$

is a group under matrix multiplication, and it acts naturally on the space F^3 of row vectors. Let G denote the corresponding semidirect product. It is not difficult to see that any normal subgroup of G is placed between two consecutive members of the lower central series, which coincides with the upper central series. This group is nilpotent of class 3, and $|G : G'| = q^2$ and $|G' : \gamma_3(G)| = |\gamma_3(G)| = q$. It follows that $a(G) = b(G) = e$ and that $|G : Z(G)| = q^3 = p^{3e}$.

It is not a coincidence that the group in the example above has class 3. In fact, it is easy to see that Isaacs' proof of Theorem 3.2 can be used to obtain a bound for $|G : Z(G)|$ including the class of the group. This bound, which we give below, improves the general bound if the class is greater than 3 and reflects the phenomenon we have mentioned before: increasing the class of the group imposes stronger restrictions on the group structure.

Theorem 3.3. *Let G be a finite p-group of class $c > 2$. Then $|G : Z(G)| \leq p^{3b-2(c-3)}$.*

It is now clear that any group G satisfying $|G : Z(G)| = p^{3b}$ must have class 3.

If G is a finite p-group of class c then $G' \not\leq Z_{c-2}(G)$ and consequently $|Z_{c-2}(G) : Z(G)| < p^b$. Hence $c \leq b + 2$ and the class is bounded when we fix the value of b. The smallest bound we can get in the last theorem is achieved when the class takes its biggest possible value, that is $b + 2$, and reads $|G : Z(G)| \leq p^{b+2}$. For this value of the class the bound is sharp, as shown by any p-group of maximal class of order p^{b+3}.

4. Infinite groups

While our results about the groups with either the strong or weak condition on normal subgroups can be considered as a by-product of our interest in providing group-theoretical characterizations of a character-theoretical property of groups, the generalization in Isaacs' paper put the main focus on the group-theoretical side of the problem. This opened the door to further generalizations, and in our work [2] we set out to extend the study of this kind of restrictions on normal subgroups to the realm of infinite groups. We introduce the following definition.

Definition. Let n be a positive integer or ∞. We say that a group G *satisfies condition* $\mathfrak{C}_n$, or that G is a $\mathfrak{C}_n$*-group*, if either $G' \le N$ or $|NZ(G):Z(G)| < n$ for every $N \trianglelefteq G$.

Thus we want to study the infinite groups satisfying condition $\mathfrak{C}_n$ for some n, which can be understood as a finiteness condition for the group, and want to derive information about the group structure, mainly about the index of the centre in the whole group. Since the $\mathfrak{C}_1$-groups are simply the abelian groups, we may well assume that $n > 1$. On the other hand, all simple groups clearly satisfy condition $\mathfrak{C}_2$ (which, by the way, is equivalent to the strong condition on normal subgroups). Hence if we want to get interesting results in our study, we have to restrict somehow the class of groups we want to explore. By analogy with the finite group case, where we only dealt with p-groups, we will focus our attention on the class of nilpotent groups. In fact, our study also covers residually nilpotent groups, since any residually nilpotent $\mathfrak{C}_\infty$-group is actually nilpotent. To see this, suppose that G is a $\mathfrak{C}_\infty$-group, residually nilpotent and non-abelian. Then $\gamma_3(G) < G'$ and consequently $|\gamma_3(G)Z(G) : Z(G)| < \infty$. Hence there exists $n \in \mathbb{N}$ such that $\gamma_i(G)Z(G) = \gamma_n(G)Z(G)$ for all $i \ge n$. It follows that $\gamma_{i+1}(G) = \gamma_{n+1}(G)$ for all $i \ge n$ and, since G is residually nilpotent, this implies the nilpotency of G.

We first consider the case when n is a positive integer. If G is a nilpotent $\mathfrak{C}_n$-group of class c then we may argue as at the end of the last section to prove that $c < 3 + \log_2 n$. So the nilpotency class of nilpotent $\mathfrak{C}_n$-groups is n-bounded. (We will say that a certain invariant associated to a family of groups is n-bounded if it can be uniformly bounded by a function of n for all groups in the family. We may speak similarly of boundedness in terms of more than one parameter.) Now Isaacs' result suggests that the index of the centre might be n-bounded for the nilpotent $\mathfrak{C}_n$-groups of class exceeding 2, but this is not exactly the case. For instance, any finite p-group G of class 3 satisfying the strong condition is a $\mathfrak{C}_2$-group, but there is no absolute bound for $|G : Z(G)| = p^3$ as p ranges over all primes. Surprisingly, the next theorem says that everything goes as expected if we just avoid a small set of exceptions closely related to this simple example.

Theorem 4.1. *Let n be a positive integer and G a nilpotent $\mathfrak{C}_n$-group of class greater than 2. Then G is central-by-finite and, more precisely:*

(i) *If G is not of the form $P \times Q$, with P a finite p-group of class 3 satisfying the strong condition and Q an abelian p'-group, then the index of the centre of G is n-bounded.*
(ii) *For the exceptions in (i), the index of the centre is p^3.*

It follows from Theorem 2.7 that the order of a p-group of class 3 satisfying the strong condition is either p^4 or p^5. Hence the exceptions pointed out in the last theorem are all classified.

Let us now consider condition $\mathfrak{C}_\infty$. We have that central-by-finite groups are $\mathfrak{C}_\infty$-groups. All the previous results point to the converse being also true but, as happened in the last theorem, there may be problems with the groups of class 3. For example, let $C = \langle x \rangle$ be an infinite cyclic group and let D be the direct product of three copies of Prüfer's C_{p^∞} group. Then C acts on D by means of

$(a, b, c)^x = (a, ab, bc)$ and the corresponding semidirect product is a nilpotent $\mathfrak{C}_\infty$-group of class 3 with infinite central index. Fortunately, the groups of class greater than 3 have a tame behaviour, as shown by the following theorem.

Theorem 4.2. *Let G be a nilpotent group of class greater than 3. Then G is a $\mathfrak{C}_\infty$-group if and only if it is central-by-finite.*

It seems to us that, contrary to what happened in Theorem 4.1, a complete classification of the $\mathfrak{C}_\infty$-groups of class 3 which are not central-by-finite is out of reach. Instead, we have contented ourselves with providing several independent characterizations of the $\mathfrak{C}_\infty$-groups of class 3 which *are* central-by-finite. We say that a group G is *Prüfer-free* if there are no normal subgroups $N \leq K$ of G such that $K/N \cong C_{p^\infty}$.

Theorem 4.3. *Let G be a nilpotent $\mathfrak{C}_\infty$-group of class 3. Then the following conditions are equivalent:*

(i) *G is central-by-finite.*
(ii) *G' is finite.*
(iii) *$G/Z_2(G)$ is either finitely generated, a torsion group or a Prüfer-free group.*

We digress momentarily from our focus on nilpotent groups to give the following result for residually finite groups, analogous to Theorem 4.2, which is very easy to prove.

Theorem 4.4. *Let G be a residually finite group. Then G is a $\mathfrak{C}_\infty$-group if and only if it is central-by-finite.*

Finally, we wonder whether we may obtain results similar to Theorem 3.1 about the exponents of $G/Z(G)$ and G'. If the class of G is greater than 3 and G satisfies condition $\mathfrak{C}_n$, we have just seen that $G/Z(G)$ is then finite, and furthermore of n-bounded order if n is finite. It is clear that one cannot get an absolute bound for the exponent of $G/Z(G)$ for $\mathfrak{C}_\infty$-groups, and of course the exponent is n-bounded for $\mathfrak{C}_n$-groups when n is finite, but we will not try to obtain accurate bounds, in the same way as we did not for the index of the centre in Theorem 4.1. As for G', by the so-called Schur-Baer Theorem (see Theorem 2.4.1 in [5]) we deduce that G' is finite if G satisfies condition $\mathfrak{C}_n$ and has class greater than 3, and in fact of n-bounded order if n is finite. Again we will not consider the problem of giving actual bounds for the exponent of G'.

What happens if G satisfies condition $\mathfrak{C}_n$ and its class does not exceed 3? We know that $G/Z(G)$ need not be finite (or n-bounded) but, can we bound its exponent? Is it at least a torsion group? What is the situation like with G'? There is not much we can say if the class is 2. For instance, if we let $\mathbb{Q}$ act on $\mathbb{Q} \times \mathbb{Q}$ via $(a, b)^x = (a, b + ax)$ then the corresponding semidirect product is a torsion-free group of class 2 that satisfies the strong condition on normal subgroups. For groups of class 3, $G/Z(G)$ may well be a torsion-free group, see the example right before Theorem 4.2. Still the behaviour of G' is a bit better, as shown by the theorem below.

Theorem 4.5. *Let G be a $\mathfrak{C}_\infty$-group of class 3. Then G' is a torsion group.*

For $n < \infty$, we cannot assure analogously that the exponent of G' is n-bounded when G is a $\mathfrak{C}_n$-group of class 3. Again, the exceptions in Theorem 4.1 provide a counterexample.

5. Open problems

We finish this survey by pointing out several directions for further work on this field of research. With this purpose, we list below some open problems we consider interesting.

Problem 1. Isaacs has proved that the bound $|G : Z(G)| \leq p^{3b}$ holds for any finite p-group of class greater than 2 and that the bound is sharp for $p > 2$. However, Theorem 2.6 shows that the bound can be improved for $p = 2$ in the particular case of the weak condition on normal subgroups. We pose the problem of finding an optimum bound for 2-groups for arbitrary b.

Problem 2. If p is odd and G is a finite p-group satisfying the weak condition and class greater than 2 then we know from Theorem 2.6 that $|G : Z(G)|$ is either p^3, p^4 or p^6, so it cannot take the value p^5. Do similar gaps appear in Isaacs' bounds for $|G : Z(G)|$ in terms of a and b?

Problem 3. We have seen in Theorem 3.3 that Isaacs' bound can be refined to include the nilpotency class c of the group, getting that $|G : Z(G)| \leq p^{3b-2(c-3)}$. From Isaacs' example we know that this bound is sharp for $p > 2$ and $c = 3$, and p-groups of maximal class show that the same happens when the class takes its biggest possible value, that is, $b + 2$. Is it possible, for every $3 < c < b + 2$, to find examples of class c for which equality holds in the previous bound? If equality does not hold, can we provide examples of arbitrarily high order?

Problem 4. As seen in Theorem 2.7, the order of a finite p-group with either the strong or the weak condition and biggest possible class is bounded. In [1] we proved that this is also the case for groups with the weak condition of class 3 and for which $|G : Z(G)| = p^6$. We conjecture that the order of the group is bounded whenever equality holds in the bound $|G : Z(G)| \leq p^{3b-2(c-3)}$.

Problem 5. Isaacs' proof of Theorem 3.2 shows that the invariant b is more adequate than a for the use of inductive arguments. Furthermore, since $b \leq a$ it provides better bounds. In [4], Isaacs asks how much bigger than b can a be in a finite p-group of class exceeding 2. He constructs examples of groups for which $a > b$, but it is not clear whether one can even get $a > b + 1$. On the other hand, it is a consequence of Theorem 3.2 and the definition of a that $a < 3b$ for finite p-groups of class greater than 2.

Problem 6. It is possible to relate the invariant a of a finite p-group G with the degrees of its complex irreducible characters, although not in such a precise way as for groups with either the strong or the weak condition. If we write $|G : Z(G)| = p^{2n+e}$, with $e = 0$ or 1, then the maximum possible degree of an irreducible character of G is p^n. Let p^m be the smallest degree of G, apart from 1. Then the difference $n - m$ is a way of measuring how close G is to having extreme degrees. Theorems 2.1 and 2.4 say that this difference is 0 if and only if $a \leq 1$ when $e = 0$, and if and only if $a \leq 2$ when $e = 1$. How does $n - m$ relate to a in general? We conjecture that $(a - 1 - e)/2 \leq n - m \leq a - 1 - e$ holds for any $a > 1$. It is easy to prove that $n - m \geq (a - 1 - e)/2$ is always true, and also that $n - m \leq a - 1 - e$ if G has class 2. (See the survey paper [6] on characters of p-groups by the second author.) On the other hand, both for the lower and the upper bound for $n - m$, there are examples with arbitrary $a > 1$ for which equality holds.

Problem 7. The results in the last section apply to (residually) nilpotent groups. Can they somehow be extended to other classes of generalized nilpotent groups? Is there anything we can say under a condition of type $\mathfrak{C}_n$ for soluble groups or even for some class of generalized soluble groups?

Problem 8. The origin of this area of research has produced an asymmetry between the role of G' and $Z(G)$ in the restrictions we are imposing on the normal subgroups of G. It would be interesting to generalize the results in this paper to groups whose normal subgroups satisfy that either $|G'N : N| < m$ or $|NZ(G) : Z(G)| < n$ for some fixed $m, n \in \mathbb{N} \cup \{\infty\}$.

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