Homogeneous products of characters

by

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1 Introduction

I. M. Isaacs [2] has conjectured that if the product of two faithful irreducible characters of a solvable group is irreducible, then the group is cyclic. In this note we discuss the following conjecture, which generalizes Isaacs conjecture.

Conjecture A. Suppose that G is solvable and that $\psi, \varphi \in \mathrm{Irr}(G)$ are faithful. If $\psi\varphi = m\chi$ where m is a positive integer and $\chi \in \mathrm{Irr}(G)$ then ψ and φ are fully ramified with respect to $\mathbf{Z}(G)$.

Other ways to state the conclusion of this conjecture are that φ, ψ and χ vanish on $G - \mathbf{Z}(G)$ or that $\varphi(1) = \psi(1) = \chi(1) = |G : \mathbf{Z}(G)|^{1/2}$ (by Problem 6.3 of [2]). In particular, if $m = 1$, these equalities yield $\varphi(1) = 1$ and since it is faithful, we deduce that G is cyclic. So Conjecture A is indeed a strong form of Isaacs conjecture.

Among other results, Isaacs proved that a counterexample to his conjecture has Fitting height at least 4 (see Theorem A of [3]). We can prove Conjecture A for nilpotent groups.

Theorem B. Conjecture A holds for p-groups.

Using Theorem B we can prove Conjecture A for p-special characters (see [1] for their definition and basic properties).

Theorem C. Let G be a p-solvable group and suppose that the product of two faithful p-special characters is a multiple of a p-special character. Then G is a p-group.

Theorem C is an easy consequence of the following elementary, but perhaps surprising, result.

Theorem D. Let φ be a faithful irreducible character of a finite group G and assume that $\psi \in \mathrm{Irr}(G)$. Write $\varphi\psi = m\Delta$, where Δ is a (not necessarily irreducible) character of G. If $\Delta(1) \le \min\{\varphi(1), \psi(1)\}$, then $\Delta(x) = 0$ for all $x \in G - \mathbf{Z}(G)$.

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2 Proof of Theorem B

We begin work toward a proof of Theorem B. We need two elementary lemmas.

Lemma 2.1. *Let $\chi \in \mathrm{Irr}(G)$, where G is a p-group. Suppose $Z \subseteq Y \lhd G$, where $Z \lhd G$ and $|Y : Z| = p$. If $Z \subseteq \mathbf{Z}(\chi)$ and $Y \not\subseteq \mathbf{Z}(\chi)$, then χ vanishes on $Y - Z$.*

Proof. Let λ be the unique (linear) irreducible constituent of χ_Z. Then every irreducible constituent of χ_Y is an extension of λ, and in particular is linear. Because $Y \not\subseteq \mathbf{Z}(\chi)$, the number of distinct linear constituents of χ_Y is a power of p exceeding 1, and so is at least p. It follows that the irreducible constituents of χ_Y are all of the extensions of λ, and they all occur with equal multiplicity, as $Y \trianglelefteq G$. Since the sum of these extensions is λ^Y, that sum vanishes on $Y - Z$ and the result follows. $\square$

Lemma 2.2. *Let ϵ and δ be pth roots of unity, where p is an odd prime.*

If $\delta \neq 1$, then

$$\left|\sum_{i=0}^{p-1} \epsilon^i \delta^{i(i-1)/2}\right| = \sqrt{p}\,.$$

Proof. Write $\epsilon = \delta^k$, where $0 \leq k < p$. Then the ith term of the sum is $\delta^{ki+i(i-1)/2}$. Since $p \neq 2$, we can write this in the form δ^{ai^2+bi} for suitable integer constants a and b, where $1 \leq a < p$ and $0 \leq b < p$. Let $\tau = \delta^a$. The ith term of our sum is then τ^{i^2+2ci} for some constant c. If we multiply the sum by τ^{c^2}, the ith term becomes $\tau^{(i+c)^2}$. Since $i+c$ runs over the same set of values (mod p) as i, we can rewrite our sum as $\sum \tau^{i^2}$. This is the well known Gauss sum, with absolute value $\sqrt{p}$ (see, for instance, p. 84 of [4]). $\square$

The next result is Theorem B.

Theorem 2.3. *Assume that G is a finite p-group, for some prime p. Assume further that φ and ψ are faithful irreducible characters of G whose product is a multiple of an irreducible character. Then φ and ψ vanish on $G - \mathbf{Z}(G)$.*

Proof. Let $\varphi\psi = m\chi$, for some positive integer m and an irreducible character χ of G. We argue by induction on $|G|$. So assume that G is a minimal counterexample. Clearly G is not abelian. So the center $\mathbf{Z}(G)$ of G is a cyclic proper subgroup of G, since G has a faithful irreducible character φ.

Step 1. G has an elementary abelian normal subgroup of order p^2.

Assume that every normal abelian subgroup of G is cyclic. Then 4.3 of [5] yields that G is dihedral or semidihedral of order ≥ 16 or (generalized) quaternion. If $G \cong Q_8$, the result is clear. Thus, we may assume that

$|G| \geq 16$. Since φ and ψ lie over the unique non-principal irreducible character of $\mathbf{Z}(G)$, χ lies over $1_{\mathbf{Z}(G)}$. Also, it is clear that $\varphi(1) = \psi(1) = 2$ and they vanish on $G - G'$. It follows that χ is not linear, i.e, $\chi(1) = 2$ and $m = 2$. Pick $x \in \mathbf{Z}_2(G) - \mathbf{Z}(G)$. We have that

$$4 = 2|\chi(x)| = |\varphi(x)||\psi(x)| < 4,$$

because $x \in \mathbf{Z}(\chi)$ but $x \notin \mathbf{Z}(\varphi)$. This contradiction proves Step 1.

We fix an elementary abelian normal subgroup A of G of order p^2. Then $Z = A \cap \mathbf{Z}(G)$ is the cyclic group $\Omega_1(\mathbf{Z}(G))$ of order p. Furthermore $A = K \times Z$, for some subgroup K of G of order p. Put $C = \mathbf{C}_G(A) = \mathbf{C}_G(K)$. Note that $|G : C| = p$.

Step 2. φ_C and ψ_C are reducible and each has a unique irreducible constituent with kernel containing K.

The center $\mathbf{Z}(C)$ certainly contains A and thus is not cyclic. Hence C has no irreducible faithful character. Therefore φ_C and ψ_C reduce. Because C has index p in G, both φ_C and ψ_C equal the sum of p distinct irreducible constituents that form a single G-orbit. Let φ_1 be an irreducible constituent of φ_C. Note that $A \cap \operatorname{Ker} \varphi_1$ is nontrivial since $A \subseteq \mathbf{Z}(C)$ and A is noncyclic. Also, this intersection does not contain Z since $Z \lhd G$ and φ is faithful. It follows that $A \cap \operatorname{Ker} \varphi_1$ is one of the p subgroups of order p in A other than Z, and we note that these subgroups form a G-conjugacy class. We can thus replace φ_1 by a G-conjugate and assume that $K = A \cap \operatorname{Ker} \varphi_1$. Similarly, ψ_C is reducible and has an irreducible constituent, say ψ_1, with kernel containing K.

Since $K = A \cap \operatorname{Ker} \varphi_1$ and K has p distinct conjugates in G, it follows that the subgroups $A \cap \operatorname{Ker} \varphi_i$ are distinct as φ_i runs over the p irreducible constituents of φ_C. This establishes the uniqueness for φ_1 and a similar argument works for ψ_1.

We now fix irreducible constituents φ_1 and ψ_1 of φ_C and ψ_C respectively, such that $K \subseteq \operatorname{Ker} \varphi_1$ and $K \subseteq \operatorname{Ker} \psi_1$.

Step 3. φ and ψ vanish on $G - C$ and χ is faithful. Also, χ_C is reducible and $\varphi_1 \psi_1 = (m/p)\chi_1$, where χ_1 is the unique irreducible constituent of χ_C with kernel containing K.

According to Clifford's theorem, $\varphi = \varphi_1^G$ and $\psi = \psi_1^G$, and thus φ, ψ vanish on $G - C$. Hence χ vanishes on $G - C$. It follows that χ_C is reducible, and thus is the sum of p distinct irreducible constituents. Since K is in the kernel of both φ_1 and ψ_1, we see that $\varphi_1\psi_1$ is a sum (with multiplicities) of irreducible constituents of χ_C having K in their kernel.

Let ψ_2 be an irreducible constituent of ψ_C different from ψ_1. Then K is not in the kernel of ψ_2, and so it is not in the kernel of $\varphi_1\psi_2$. It follows that K is in the kernel of some irreducible constituent χ_1 of χ_C but K is not in the kernel of all of the conjugates of χ_1.

If $Z \subseteq \operatorname{Ker} \chi$ then $A = ZK \subseteq \operatorname{Ker} \chi_1$, and since $A \lhd G$, we see that A is contained in the kernel of every irreducible constituent of χ_C, which is not the case. Thus $Z \nsubseteq \operatorname{Ker} \chi$. This, along with the fact that $\mathbf{Z}(G)$ is cyclic, implies that χ is faithful.

Therefore, the same argument we gave in Step 2 for φ, implies that χ_1 is the unique irreducible constituent of χ_C with kernel containing K. It follows that $\varphi_1\psi_1 = m_1\chi_1$ for some integer m_1. Comparison of degrees yields $(\varphi(1)/p)(\psi(1)/p) = m_1(\chi(1)/p)$. Since $\varphi(1)\psi(1) = m\chi(1)$, we deduce that $m_1 = m/p$.

Step 4. $p \neq 2$.

Otherwise $|Z| = 2$ and Z has a unique nonprincipal irreducible character. In this case, both φ and ψ lie above this nonprincipal character. Hence $Z \subseteq \operatorname{Ker} \varphi\psi$. Then $Z \subseteq \operatorname{Ker} \chi$, which is not the case.

Let $V/K = \mathbf{Z}(C/K)$ and write $Y = A\mathbf{Z}(G)$. Note that $Y \lhd G$ and that $Y \subseteq \mathbf{Z}(C) \subseteq V$.

Step 5. $V > Y$.

Note that $Y = K\mathbf{Z}(G)$ and assume that $V = K\mathbf{Z}(G)$. Let $K_1 = \operatorname{Ker} \varphi_1$. If $K_1 > K$, then $(K_1/K) \cap \mathbf{Z}(C/K) > 1$, and thus $K_1 \cap K\mathbf{Z}(G) = K_1 \cap V > K$. It follows that $K_1 \cap \mathbf{Z}(G) > 1$, and thus $Z \subseteq K_1$ as $\mathbf{Z}(G)$ is cyclic. This is not the case, however, since $Z \nsubseteq \operatorname{Ker} \varphi_1$. We conclude that $K_1 = K$.

Similarly we show that $\operatorname{Ker} \psi_1 = K$. Hence φ_1, ψ_1 are inflated from unique faithful characters $\bar{\varphi}_1$ and $\bar{\psi}_1$, respectively of the factor group C/K. In addition, χ_1 is also inflated from a unique character $\bar{\chi}_1$ of C/K and satisfies $\bar{\varphi}_1\bar{\psi}_1 = m_1\bar{\chi}_1$. By the minimality of G, we conclude that φ_1 and ψ_1 vanish on $C - V$.

In this situation, where $V = Y$, we see that $V \lhd G$, and thus all irreducible constituents of φ_C vanish on $C - V$. We conclude that φ vanishes on $G - V$. But $|V : \mathbf{Z}(G)| = p$ and $V \nsubseteq \mathbf{Z}(\varphi)$. By Lemma 2.1, therefore, φ vanishes on $V - \mathbf{Z}(G)$, and hence on $G - \mathbf{Z}(G)$. Similarly, ψ vanishes on $G - \mathbf{Z}(G)$, and this is a contradiction since G is a counterexample.

Step 6. $\mathbf{Z}(C) > Y$.

Certainly, $Y = A\mathbf{Z}(G) \subseteq \mathbf{Z}(C)$ and we suppose that equality occurs. Since $V > Y$, we can choose a subgroup U such that $Y \subseteq U \subseteq V$ and $|U : Y| = p$. We have $1 < [C, U] \subseteq [C, V] \subseteq K$, and thus $[C, U] = K$. In particular, we see that $U \subseteq \mathbf{Z}(\varphi_1)$ and so all values of φ_1 on U are nonzero.

Now let φ_2 be any irreducible constituent of φ_C other than φ_1. We argue $U \not\subseteq \mathbf{Z}(\varphi_2)$ since otherwise, $K = [C, U] \subseteq \operatorname{Ker} \varphi_2$, which is not the case. But $Y \subseteq \mathbf{Z}(C)$, and $|U : Y| = p$, so Lemma 2.1 implies that φ_2 vanishes on $U - Y$. Since φ_2 is an arbitrary constituent of φ_C other than φ_1, it follows that if $u \in U - Y$, then

$$\varphi(u) = \sum_{i=1}^{p} \varphi_i(u) = \varphi_1(u) \neq 0.$$

Similarly, $\psi(u) = \psi_1(u) \neq 0$ and also $\chi(u) = \chi_1(u) \neq 0$. We now have

$$m\chi_1(u) = m\chi(u) = \varphi(u)\psi(u) = \varphi_1(u)\psi_1(u) = (m/p)\chi_1(u)$$

and this is a contradiction.

We choose $W \lhd G$ with $Y \subseteq W \subseteq \mathbf{Z}(C)$ and $|W : Y| = p$. We also fix elements $g \in G - C$ and $w \in W - Y$ and we write $[w, g] = a$ and $[a, g] = z$. Then we can show

Step 7. $1 \neq a \in A$, $1 \neq z \in Z$ and $w^{g^i} = wa^i z^{i(i-1)/2}$.

Since $|W : Y| = p$, we have $W/Y \subseteq \mathbf{Z}(G/Y)$, and thus $[W, G] \leq Y$. Hence $a = [w, g]$ is an element of Y and thus $a^p \in \mathbf{Z}(G)$. Also, $|Y : \mathbf{Z}(G)| = p$, and similarly we get $z \in \mathbf{Z}(G)$. Then $a^p = (a^p)^g = (az)^p = a^p z^p$. We conclude that $z \in Z = \Omega_1(\mathbf{Z}(G))$.

Because $w^g = wa$ and $a^g = az$ we can easily calculate that $w^{g^i} = wa^i z^{i(i-1)/2}$ for integers i with $1 \leq i \leq p$. Note that $g^p \in C$ while $w \in$

$W \leq \mathbf{Z}(C)$. So $w^{g^p} = w$. Also, since $p \neq 2$ and $z^p = 1$, we see that $z^{p(p-1)/2} = 1$. It follows that $w = w^{g^p} = wa^p$ and so $a^p = 1$. Since $a \in Y$ and $A = \Omega_1(Y)$, we have $a \in A$, as wanted.

Finally, we must show that $a \neq 1$ and $z \neq 1$. If $a = 1$ then $w \in \mathbf{Z}(C)$ is centralized by g, and thus $w \in \mathbf{Z}(G)$ contradicting the way w was picked. Also if $z = 1$, then g centralizes $a \in \mathbf{Z}(C)$, and thus $a \in \mathbf{Z}(G)$. Hence $a \in A \cap \mathbf{Z}(G) = Z$. Note that since A is not central in G we have $1 < [A, G] \lhd G$. It follows that $[A, G] = Z$ and thus $[Y, G] = Z$. Hence $[W, g] \subseteq Z$ since $W = Y\langle w \rangle$. But W is abelian, and it follows that $|W : \mathbf{C}_W(g)| \leq |Z| = p$. This is a contradiction, however since $\mathbf{C}_W(g) = \mathbf{Z}(G)$ has index p^2 in W.

Step 8. We have a contradiction.

Since $W \subseteq \mathbf{Z}(C)$, there exists a linear character $\alpha \in \mathrm{Lin}(W)$ such that $(\varphi_1)_W = \varphi_1(1)\alpha$. Furthermore, as $A \subseteq W$ we can write $\alpha(a) = \epsilon$ and $\alpha(z) = \delta$, where ϵ and δ are pth roots of unity and $\delta \neq 1$ since $z \neq 1$ and φ is faithful. We see now that

$$\varphi(w) = \sum_{i=0}^{p-1} \varphi_1(w^{g^i}) = \sum_{i=0}^{p-1} \varphi_1(1)\alpha(w)\alpha(a^i)\alpha(z^{i(i-1)/2}) = \varphi_1(w)A\,,$$

where $A = \sum_{i=0}^{p-1} \epsilon^i \delta^{i(i-1)/2}$. By Lemma 2.2, therefore, we have $|A| = \sqrt{p}$. We also have similar formulas $\psi(w) = \psi_1(w)B$ and $\chi(w) = \chi_1(w)D$, where $|B| = |D| = \sqrt{p}$. Also $\chi_1(w) \neq 0$ since $w \in \mathbf{Z}(C)$.

We have

$$m\chi_1(w)D = m\chi(w) = \varphi(w)\psi(w) = \varphi_1(w)\psi_1(w)AB = (m/p)\chi_1(w)AB\,,$$

and thus $AB = pD$. But this is not consistent with $|A| = |B| = |D| = \sqrt{p}$, and the proof is complete. $\square$

3 Proof of Theorems C and D

In order to prove Theorem D, we need the following easy lemma.

Lemma 3.1. *Let n be a positive integer and $a_1, \dots, a_n, b_1, \dots, b_n, c_1 \dots, c_n$ complex numbers. If both $\sum_{i=1}^n |a_i|^2$ and $\sum_{i=1}^n |b_i|^2$ do not exceed $\sum_{i=1}^n |c_i|^2$, then there exists $j \in \{1, \dots, n\}$ such that $|c_j|^2 \geq |a_j||b_j|$.*

Proof. Write $S = \sum_{i=1}^n |c_i|^2$ and assume that $|c_j|^2 < |a_j||b_j|$ for all j. Then

$$S = \sum_{i=1}^n |c_i|^2 < \sum_{i=1}^n |a_i||b_i| \leq (\sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2)^{1/2} = S,$$

a contradiction. The second inequality is Cauchy-Schwarz's inequality. $\square$

Proof of Theorem D. Set $\mathcal{S} = \{x \in G - \mathbf{Z}(G) \mid \Delta(x) \neq 0\}$. We want to prove that $\mathcal{S}$ is the empty set. Assume not and we will work to find a contradiction. Using the orthogonality relations, we have that

$$\sum_{x \in G} |\Delta(x)|^2 = \sum_{x \in G} \Delta(x)\overline{\Delta(x)} = (\sum_{i=1}^n m_i^2)|G|,$$

where $\Delta = m_1\chi_1 + \cdots + m_n\chi_n$ with $\chi_i \in \mathrm{Irr}(G)$. Similarly,

$$\sum_{x \in \mathbf{Z}(G)} |\Delta(x)|^2 = (\sum_{i=1}^n m_i^2 \chi_i(1)^2)|\mathbf{Z}(G)|.$$

We deduce that

$$\sum_{x \in \mathcal{S}} |\Delta(x)|^2 = (\sum_{i=1}^n m_i^2)|G| - (\sum_{i=1}^n m_i^2 \chi_i(1)^2)|\mathbf{Z}(G)|.$$

Now,

$$\begin{aligned}\sum_{x\in\mathcal{S}}|\psi(x)|^2 &\leq |G| - \sum_{x\in\mathbf{Z}(G)}|\psi(x)|^2 \leq |G| - \Delta(1)^2|\mathbf{Z}(G)| \\ &= |G| - (\sum_{i=1}^{n} m_i\chi_i(1))^2|\mathbf{Z}(G)| \leq |G| - (\sum_{i=1}^{n} m_i^2\chi_i(1)^2)|\mathbf{Z}(G)| \\ &\leq \sum_{x\in\mathcal{S}}|\Delta(x)|^2.\end{aligned}$$

In the same way, $\sum_{x\in\mathcal{S}}|\varphi(x)|^2 \leq \sum_{x\in\mathcal{S}}|\Delta(x)|^2$. Now, we can use Lemma 3.1 to deduce that there exists $x \in \mathcal{S}$ such that $|\Delta(x)|^2 \geq |\varphi(x)||\psi(x)|$. Thus

$$\frac{|\varphi(x)|^2|\psi(x)|^2}{m^2} = |\Delta(x)|^2 \geq |\varphi(x)||\psi(x)|$$

and we deduce that $|\varphi(x)||\psi(x)| \geq m^2$.

On the other hand, we have that $|\psi(x)| \leq \psi(1) = m\Delta(1)/\varphi(1)$ and, since $x \notin \mathbf{Z}(\varphi)$, $|\varphi(x)| < \varphi(1) = m\Delta(1)/\psi(1)$. Thus,

$$|\varphi(x)||\psi(x)| < \varphi(1)\psi(1) = m^2\Delta(1)^2/\varphi(1)\psi(1) \leq m^2,$$

by hypothesis. This is a contradiction. □

Theorem C is an immediate consequence of the following result.

Theorem 3.2. *Let G be a finite group and suppose that $\varphi, \psi \in \mathrm{Irr}(G)$ are faithful and $\varphi\psi = m\chi$ with $\chi \in \mathrm{Irr}(G)$. Then for every proper nilpotent subgroup H of G, φ_H, ψ_H or χ_H is not irreducible.*

Proof. Assume that all three restrictions to a proper nilpotent subgroup H are irreducible. We can apply Theorem B and deduce that $\chi(1) = \varphi(1) = \psi(1)$. By Theorem D, we have that χ is fully ramified with respect to $\mathbf{Z}(G)$ and it follows that χ_H is not irreducible, a contradiction.

□

Finally, we prove Theorem C, which we restate.

Corollary 3.3. *Let G be a p-solvable group and suppose that the product of two faithful p-special characters is a multiple of a p-special character. Then G is a p-group.*

Proof. Note that the restriction of p-special characters to a Sylow p-subgroup is irreducible (see [1]). Apply Theorem 3.2. □

References

[1] D. Gajendragadkar, A characteristic class of characters of finite π-separable groups, *J. Algebra* **59** (1979), 237–259.

[2] M. Isaacs, "Character Theory of Finite Groups", Dover, New York, 1994.

[3] M. Isaacs, Irreducible products of characters, *J. Algebra* **223** (2000), 630–646.

[4] S. Lang, "Algebraic Number Theory", Springer-Verlag, New York, 1986.

[5] M. Suzuki, "Group Theory II", Springer-Verlag, New York, 1986.