Homogeneous products of characters

by

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1 Introduction

I. M. Isaacs [2] has conjectured that if the product of two faithful irreducible characters of a solvable group is irreducible, then the group is cyclic. In this note we discuss the following conjecture, which generalizes Isaacs conjecture.

**Conjecture A.** Suppose that $G$ is solvable and that $\psi, \varphi \in \text{Irr}(G)$ are faithful. If $\psi \varphi = m \chi$ where $m$ is a positive integer and $\chi \in \text{Irr}(G)$ then $\psi$ and $\varphi$ are fully ramified with respect to $\mathbb{Z}(G)$.

Other ways to state the conclusion of this conjecture are that $\varphi, \psi$ and $\chi$ vanish on $G - \mathbb{Z}(G)$ or that $\varphi(1) = \psi(1) = \chi(1) = |G : \mathbb{Z}(G)|^{1/2}$ (by Problem 6.3 of [2]). In particular, if $m = 1$, these equalities yield $\varphi(1) = 1$ and since it is faithful, we deduce that $G$ is cyclic. So Conjecture A is indeed a strong form of Isaacs conjecture.

Among other results, Isaacs proved that a counterexample to his conjecture has Fitting height at least 4 (see Theorem A of [3]). We can prove Conjecture A for nilpotent groups.

**Theorem B.** Conjecture A holds for $p$-groups.

Using Theorem B we can prove Conjecture A for $p$-special characters (see [1] for their definition and basic properties).

**Theorem C.** Let $G$ be a $p$-solvable group and suppose that the product of two faithful $p$-special characters is a multiple of a $p$-special character. Then $G$ is a $p$-group.

Theorem C is an easy consequence of the following elementary, but perhaps surprising, result.
Theorem D. Let $\varphi$ be a faithful irreducible character of a finite group $G$ and assume that $\psi \in \text{Irr}(G)$. Write $\varphi \psi = m\Delta$, where $\Delta$ is a (not necessarily irreducible) character of $G$. If $\Delta(1) \leq \min\{\varphi(1), \psi(1)\}$, then $\Delta(x) = 0$ for all $x \in G - Z(G)$.

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2 Proof of Theorem B

We begin work toward a proof of Theorem B. We need two elementary lemmas.

Lemma 2.1. Let $\chi \in \text{Irr}(G)$, where $G$ is a $p$-group. Suppose $Z \subseteq Y \triangleleft G$, where $Z \triangleleft G$ and $|Y : Z| = p$. If $Z \subseteq Z(\chi)$ and $Y \nsubseteq Z(\chi)$, then $\chi$ vanishes on $Y - Z$.

Proof. Let $\lambda$ be the unique (linear) irreducible constituent of $\chi_Z$. Then every irreducible constituent of $\chi_Y$ is an extension of $\lambda$, and in particular is linear. Because $Y \nsubseteq Z(\chi)$, the number of distinct linear constituents of $\chi_Y$ is a power of $p$ exceeding 1, and so is at least $p$. It follows that the irreducible constituents of $\chi_Y$ are all of the extensions of $\lambda$, and they all occur with equal multiplicity, as $Y \triangleleft G$. Since the sum of these extensions is $\lambda^Y$, that sum vanishes on $Y - Z$ and the result follows.

Lemma 2.2. Let $\epsilon$ and $\delta$ be $p$th roots of unity, where $p$ is an odd prime.
If $\delta \neq 1$, then
\[
\left| \sum_{i=0}^{p-1} \epsilon^i \delta^{i(i-1)/2} \right| = \sqrt{p}.
\]

Proof. Write $\epsilon = \delta^k$, where $0 \leq k < p$. Then the $i$th term of the sum is $\delta^{ki + i(i-1)/2}$. Since $p \neq 2$, we can write this in the form $\delta^{ai^2 + bi}$ for suitable integer constants $a$ and $b$, where $1 \leq a < p$ and $0 \leq b < p$. Let $\tau = \delta^a$. The $i$th term of our sum is then $\tau^{i^2 + 2ci}$ for some constant $c$. If we multiply the sum by $\tau^{c^2}$, the $i$th term becomes $\tau^{(i+c)^2}$. Since $i + c$ runs over the same set of values (mod $p$) as $i$, we can rewrite our sum as $\sum \tau^{i^2}$. This is the well known Gauss sum, with absolute value $\sqrt{p}$ (see, for instance, p. 84 of [4]).

The next result is Theorem B.

**Theorem 2.3.** Assume that $G$ is a finite $p$-group, for some prime $p$. Assume further that $\varphi$ and $\psi$ are faithful irreducible characters of $G$ whose product is a multiple of an irreducible character. Then $\varphi$ and $\psi$ vanish on $G - Z(G)$.

**Proof.** Let $\varphi \psi = m \chi$, for some positive integer $m$ and an irreducible character $\chi$ of $G$. We argue by induction on $|G|$. So assume that $G$ is a minimal counterexample. Clearly $G$ is not abelian. So the center $Z(G)$ of $G$ is a cyclic proper subgroup of $G$, since $G$ has a faithful irreducible character $\varphi$.

**Step 1.** $G$ has an elementary abelian normal subgroup of order $p^2$.

Assume that every normal abelian subgroup of $G$ is cyclic. Then 4.3 of [5] yields that $G$ is dihedral or semidihedral of order $\geq 16$ or (generalized) quaternion. If $G \cong Q_8$, the result is clear. Thus, we may assume that
\[ |G| \geq 16. \] Since \( \varphi \) and \( \psi \) lie over the unique non-principal irreducible character of \( Z(G) \), \( \chi \) lies over \( 1_{Z(G)} \). Also, it is clear that \( \varphi(1) = \psi(1) = 2 \) and they vanish on \( G - G' \). It follows that \( \chi \) is not linear, i.e., \( \chi(1) = 2 \) and \( m = 2 \). Pick \( x \in Z_2(G) - Z(G) \). We have that

\[ 4 = 2|\chi(x)| = |\varphi(x)||\psi(x)| < 4, \]

because \( x \in Z(\chi) \) but \( x \notin Z(\varphi) \). This contradiction proves Step 1.

We fix an elementary abelian normal subgroup \( A \) of \( G \) of order \( p^2 \).

Then \( Z = A \cap Z(G) \) is the cyclic group \( \Omega_1(Z(G)) \) of order \( p \). Furthermore \( A = K \times Z \), for some subgroup \( K \) of \( G \) of order \( p \). Put \( C = C_G(A) = C_G(K) \). Note that \( |G : C| = p \).

**Step 2.** \( \varphi_C \) and \( \psi_C \) are reducible and each has a unique irreducible constituent with kernel containing \( K \).

The center \( Z(C) \) certainly contains \( A \) and thus is not cyclic. Hence \( C \) has no irreducible faithful character. Therefore \( \varphi_C \) and \( \psi_C \) reduce. Because \( C \) has index \( p \) in \( G \), both \( \varphi_C \) and \( \psi_C \) equal the sum of \( p \) distinct irreducible constituents that form a single \( G \)-orbit. Let \( \varphi_1 \) be an irreducible constituent of \( \varphi_C \). Note that \( A \cap \ker \varphi_1 \) is nontrivial since \( A \subseteq Z(C) \) and \( A \) is noncyclic. Also, this intersection does not contain \( Z \) since \( Z \triangleleft G \) and \( \varphi \) is faithful. It follows that \( A \cap \ker \varphi_1 \) is one of the \( p \) subgroups of order \( p \) in \( A \) other than \( Z \), and we note that these subgroups form a \( G \)-conjugacy class. We can thus replace \( \varphi_1 \) by a \( G \)-conjugate and assume that \( K = A \cap \ker \varphi_1 \). Similarly, \( \psi_C \) is reducible and has an irreducible constituent, say \( \psi_1 \), with kernel containing \( K \).

Since \( K = A \cap \ker \varphi_1 \) and \( K \) has \( p \) distinct conjugates in \( G \), it follows that the subgroups \( A \cap \ker \varphi_i \) are distinct as \( \varphi_i \) runs over the \( p \) irreducible
constituents of $\varphi_C$. This establishes the uniqueness for $\varphi_1$ and a similar argument works for $\psi_1$.

We now fix irreducible constituents $\varphi_1$ and $\psi_1$ of $\varphi_C$ and $\psi_C$ respectively, such that $K \subseteq \text{Ker} \varphi_1$ and $K \subseteq \text{Ker} \psi_1$.

**Step 3.** $\varphi$ and $\psi$ vanish on $G - C$ and $\chi$ is faithful. Also, $\chi_C$ is reducible and $\varphi_1 \psi_1 = (m/p) \chi_1$, where $\chi_1$ is the unique irreducible constituent of $\chi_C$ with kernel containing $K$.

According to Clifford’s theorem, $\varphi = \varphi_1^G$ and $\psi = \psi_1^G$, and thus $\varphi, \psi$ vanish on $G - C$. Hence $\chi$ vanishes on $G - C$. It follows that $\chi_C$ is reducible, and thus is the sum of $p$ distinct irreducible constituents. Since $K$ is in the kernel of both $\varphi_1$ and $\psi_1$, we see that $\varphi_1 \psi_1$ is a sum (with multiplicities) of irreducible constituents of $\chi_C$ having $K$ in their kernel.

Let $\psi_2$ be an irreducible constituent of $\psi_C$ different from $\psi_1$. Then $K$ is not in the kernel of $\psi_2$, and so it is not in the kernel of $\varphi_1 \psi_2$. It follows that $K$ is in the kernel of some irreducible constituent $\chi_1$ of $\chi_C$ but $K$ is not in the kernel of all of the conjugates of $\chi_1$.

If $Z \subseteq \text{Ker} \chi$ then $A = ZK \subseteq \text{Ker} \chi_1$, and since $A < G$, we see that $A$ is contained in the kernel of every irreducible constituent of $\chi_C$, which is not the case. Thus $Z \not\subseteq \text{Ker} \chi$. This, along with the fact that $Z(G)$ is cyclic, implies that $\chi$ is faithful.

Therefore, the same argument we gave in Step 2 for $\varphi$, implies that $\chi_1$ is the unique irreducible constituent of $\chi_C$ with kernel containing $K$. It follows that $\varphi_1 \psi_1 = m_1 \chi_1$ for some integer $m_1$. Comparison of degrees yields $(\varphi(1)/p)(\psi(1)/p) = m_1(\chi(1)/p)$. Since $\varphi(1)\psi(1) = m\chi(1)$, we deduce that $m_1 = m/p$. 

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Step 4. $p \neq 2$.

Otherwise $|Z| = 2$ and $Z$ has a unique nonprincipal irreducible character. In this case, both $\varphi$ and $\psi$ lie above this nonprincipal character. Hence $Z \subseteq \text{Ker } \varphi \psi$. Then $Z \subseteq \text{Ker } \chi$, which is not the case.

Let $V/K = Z(C/K)$ and write $Y = A Z(G)$. Note that $Y < G$ and that $Y \subseteq Z(C) \subseteq V$.

Step 5. $V > Y$.

Note that $Y = KZ(G)$ and assume that $V = KZ(G)$. Let $K_1 = \text{Ker } \varphi_1$. If $K_1 > K$, then $(K_1/K) \cap Z(C/K) > 1$, and thus $K_1 \cap KZ(G) = K_1 \cap V > K$. It follows that $K_1 \cap Z(G) > 1$, and thus $Z \subseteq K_1$ as $Z(G)$ is cyclic. This is not the case, however, since $Z \not\subseteq \text{Ker } \varphi_1$. We conclude that $K_1 = K$.

Similarly we show that $\text{Ker } \psi_1 = K$. Hence $\varphi_1, \psi_1$ are inflated from unique faithful characters $\tilde{\varphi}_1$ and $\tilde{\psi}_1$, respectively of the factor group $C/K$. In addition, $\chi_1$ is also inflated from a unique character $\tilde{\chi}_1$ of $C/K$ and satisfies $\varphi_1 \psi_1 = m_1 \chi_1$. By the minimality of $G$, we conclude that $\varphi_1$ and $\psi_1$ vanish on $C - V$.

In this situation, where $V = Y$, we see that $V < G$, and thus all irreducible constituents of $\varphi_C$ vanish on $C - V$. We conclude that $\varphi$ vanishes on $G - V$. But $|V : Z(G)| = p$ and $V \not\subseteq Z(\varphi)$. By Lemma 2.1, therefore, $\varphi$ vanishes on $V - Z(G)$, and hence on $G - Z(G)$. Similarly, $\psi$ vanishes on $G - Z(G)$, and this is a contradiction since $G$ is a counterexample.

Step 6. $Z(C) > Y$.
Certainly, \( Y = A \mathbb{Z}(G) \subseteq \mathbb{Z}(C) \) and we suppose that equality occurs. Since \( V > Y \), we can choose a subgroup \( U \) such that \( Y \subseteq U \subseteq V \) and \( |U : Y| = p \). We have 1 < \([C, U] \subseteq [C, V] \subseteq K\), and thus \([C, U] = K\). In particular, we see that \( U \subseteq \mathbb{Z}(\varphi_1) \) and so all values of \( \varphi_1 \) on \( U \) are nonzero.

Now let \( \varphi_2 \) be any irreducible constituent of \( \varphi_C \) other than \( \varphi_1 \). We argue \( U \not\subseteq \mathbb{Z}(\varphi_2) \) since otherwise, \( K = [C, U] \subseteq \text{Ker} \varphi_2 \), which is not the case. But \( Y \subseteq \mathbb{Z}(C) \), and \( |U : Y| = p \), so Lemma 2.1 implies that \( \varphi_2 \) vanishes on \( U - Y \). Since \( \varphi_2 \) is an arbitrary constituent of \( \varphi_C \) other than \( \varphi_1 \), it follows that if \( u \in U - Y \), then

\[
\varphi(u) = \sum_{i=1}^{p} \varphi_i(u) = \varphi_1(u) \neq 0.
\]

Similarly, \( \psi(u) = \psi_1(u) \neq 0 \) and also \( \chi(u) = \chi_1(u) \neq 0 \). We now have

\[
m\chi_1(u) = m\chi(u) = \varphi(u)\psi(u) = \varphi_1(u)\psi_1(u) = (mp)\chi_1(u)
\]

and this is a contradiction.

We choose \( W \triangleleft G \) with \( Y \subseteq W \subseteq \mathbb{Z}(C) \) and \( |W : Y| = p \). We also fix elements \( g \in G - C \) and \( w \in W - Y \) and we write \([w, g] = a \) and \([a, g] = z\). Then we can show

**Step 7.** \( 1 \neq a \in A \), \( 1 \neq z \in Z \) and \( w^{g^i} = wa^iz^{(i-1)/2} \).

Since \( |W : Y| = p \), we have \( W/Y \subseteq \mathbb{Z}(G/Y) \), and thus \([W, G] \leq Y\). Hence \( a = [w, g] \) is an element of \( Y \) and thus \( a^p \in \mathbb{Z}(G) \). Also, \( |Y : \mathbb{Z}(G)| = p \), and similarly we get \( z \in \mathbb{Z}(G) \). Then \( a^p = (a^p)^q = (az)^p = a^pz^p \). We conclude that \( z \in Z = \Omega_1(\mathbb{Z}(G)) \).

Because \( w^g = wa \) and \( a^g = az \) we can easily calculate that \( w^{g^i} = wa^iz^{(i-1)/2} \) for integers \( i \) with \( 1 \leq i \leq p \). Note that \( g^p \in C \) while \( w \in \)
$W \leq Z(C)$. So $w^p = w$. Also, since $p \neq 2$ and $z^p = 1$, we see that $z^{p(p-1)/2} = 1$. It follows that $w = w^p = wa^p$ and so $a^p = 1$. Since $a \in Y$ and $A = \Omega_1(Y)$, we have $a \in A$, as wanted.

Finally, we must show that $a \neq 1$ and $z \neq 1$. If $a = 1$ then $w \in Z(C)$ is centralized by $g$, and thus $w \in Z(G)$ contradicting the way $w$ was picked. Also if $z = 1$, then $g$ centralizes $a \in Z(C)$, and thus $a \in Z(G)$.

Hence $a \in A \cap Z(G) = Z$. Note that since $A$ is not central in $G$ we have $1 < [A,G] < G$. It follows that $[A,G] = Z$ and thus $[Y,G] = Z$. Hence $[W,g] \subseteq Z$ since $W = Y\langle w \rangle$. But $W$ is abelian, and it follows that $|W : C_W(g)| \leq |Z| = p$. This is a contradiction, however since $C_W(g) = Z(G)$ has index $p^2$ in $W$.

**Step 8.** We have a contradiction.

Since $W \subseteq Z(C)$, there exists a linear character $\alpha \in \text{Lin}(W)$ such that $(\varphi_1)_W = \varphi_1(1)\alpha$. Furthermore, as $A \subseteq W$ we can write $\alpha(a) = \epsilon$ and $\alpha(z) = \delta$, where $\epsilon$ and $\delta$ are $p$th roots of unity and $\delta \neq 1$ since $z \neq 1$ and $\varphi$ is faithful. We see now that

$$\varphi(w) = \sum_{i=0}^{p-1} \varphi_1(w^g) = \sum_{i=0}^{p-1} \varphi_1(1)\alpha(w)\alpha(a^i)\alpha(z^{i(i-1)/2}) = \varphi_1(w)A,$$

where $A = \sum_{i=0}^{p-1} \epsilon^i \delta^{i(i-1)/2}$. By Lemma 2.2, therefore, we have $|A| = \sqrt{p}$.

We also have similar formulas $\psi(w) = \psi_1(w)B$ and $\chi(w) = \chi_1(w)D$, where $|B| = |D| = \sqrt{p}$. Also $\chi_1(w) \neq 0$ since $w \in Z(C)$.

We have

$$m\chi_1(w)D = m\chi(w) = \varphi(w)\psi(w) = \varphi_1(w)\psi_1(w)AB = (m/p)\chi_1(w)AB,$$

and thus $AB = pD$. But this is not consistent with $|A| = |B| = |D| = \sqrt{p}$, and the proof is complete. $\square$
3 Proof of Theorems C and D

In order to prove Theorem D, we need the following easy lemma.

**Lemma 3.1.** Let $n$ be a positive integer and $a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n$ complex numbers. If both $\sum_{i=1}^{n} |a_i|^2$ and $\sum_{i=1}^{n} |b_i|^2$ do not exceed $\sum_{i=1}^{n} |c_i|^2$, then there exists $j \in \{1, \ldots, n\}$ such that $|c_j|^2 \geq |a_j||b_j|$.

**Proof.** Write $S = \sum_{i=1}^{n} |c_i|^2$ and assume that $|c_j|^2 < |a_j||b_j|$ for all $j$. Then

$$S = \sum_{i=1}^{n} |c_i|^2 < \sum_{i=1}^{n} |a_i||b_i| \leq (\sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2)^{1/2} = S,$$

a contradiction. The second inequality is Cauchy-Schwarz’s inequality. □

**Proof of Theorem D.** Set $S = \{x \in G - Z(G) \mid \Delta(x) \neq 0\}$. We want to prove that $S$ is the empty set. Assume not and we will work to find a contradiction. Using the orthogonality relations, we have that

$$\sum_{x \in G} |\Delta(x)|^2 = \sum_{x \in G} \Delta(x)\overline{\Delta(x)} = \left( \sum_{i=1}^{n} m_i^2 \right)|G|,$$

where $\Delta = m_1\chi_1 + \cdots + m_n\chi_n$ with $\chi_i \in \text{Irr}(G)$. Similarly,

$$\sum_{x \in Z(G)} |\Delta(x)|^2 = \left( \sum_{i=1}^{n} m_i^2 \chi_i(1)^2 \right)|Z(G)|.$$

We deduce that

$$\sum_{x \in S} |\Delta(x)|^2 = \left( \sum_{i=1}^{n} m_i^2 \right)|G| - \left( \sum_{i=1}^{n} m_i^2 \chi_i(1)^2 \right)|Z(G)|.$$
Now,

\[ \sum_{x \in S} |\psi(x)|^2 \leq |G| - \sum_{x \in \text{Z}(G)} |\psi(x)|^2 \leq |G| - |\Delta(1)|^2 |\text{Z}(G)| \]

\[ = |G| - \left( \sum_{i=1}^{n} m_i \chi_i(1) \right)^2 |\text{Z}(G)| \leq |G| - \left( \sum_{i=1}^{n} m_i^2 \chi_i(1)^2 \right) |\text{Z}(G)| \]

\[ \leq \sum_{x \in S} |\Delta(x)|^2. \]

In the same way, \( \sum_{x \in S} |\varphi(x)|^2 \leq \sum_{x \in S} |\Delta(x)|^2. \) Now, we can use Lemma 3.1 to deduce that there exists \( x \in S \) such that \( |\Delta(x)|^2 \geq |\varphi(x)||\psi(x)|. \)

Thus

\[ \frac{|\varphi(x)|^2|\psi(x)|^2}{m^2} = |\Delta(x)|^2 \geq |\varphi(x)||\psi(x)| \]

and we deduce that \( |\varphi(x)||\psi(x)| \geq m^2. \)

On the other hand, we have that \( |\psi(x)| \leq \psi(1) = m\Delta(1)/\varphi(1) \) and, since \( x \notin \text{Z}(\varphi) \), \( |\varphi(x)| < \varphi(1) = m\Delta(1)/\psi(1) \). Thus,

\[ |\varphi(x)||\psi(x)| < \varphi(1)\psi(1) = m^2\Delta(1)^2/\varphi(1)\psi(1) \leq m^2, \]

by hypothesis. This is a contradiction. \( \square \)

Theorem C is an immediate consequence of the following result.

**Theorem 3.2.** Let \( G \) be a finite group and suppose that \( \varphi, \psi \in \text{Irr}(G) \) are faithful and \( \varphi \psi = m\chi \) with \( \chi \in \text{Irr}(G) \). Then for every proper nilpotent subgroup \( H \) of \( G \), \( \varphi_H, \psi_H \) or \( \chi_H \) is not irreducible.

**Proof.** Assume that all three restrictions to a proper nilpotent subgroup \( H \) are irreducible. We can apply Theorem B and deduce that \( \chi(1) = \varphi(1) = \psi(1) \). By Theorem D, we have that \( \chi \) is fully ramified with respect to \( \text{Z}(G) \) and it follows that \( \chi_H \) is not irreducible, a contradiction. \( \square \)
Finally, we prove Theorem C, which we restate.

**Corollary 3.3.** Let $G$ be a $p$-solvable group and suppose that the product of two faithful $p$-special characters is a multiple of a $p$-special character. Then $G$ is a $p$-group.

**Proof.** Note that the restriction of $p$-special characters to a Sylow $p$-subgroup is irreducible (see [1]). Apply Theorem 3.2. □

**References**


