An answer to a question of J. G. Thompson on some generalized characters

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In 1996 J. G. Thompson [2] proved that given any finite group G the function $\theta_p : G \to \mathbb{Z}$ defined by

$$\theta_p(g) = |\{h \in G \mid \langle g, h\rangle_p \text{ is abelian}\}|,$$

where $\langle g, h\rangle_p$ denotes a Sylow p-subgroup of $\langle g, h\rangle$, is a generalized character. Thompson mentioned that "it seems reasonable to hope that θ_p is a character". Certainly, this is the case for groups with abelian Sylow subgroups or for nilpotent groups. Unfortunately, we will show that θ_p does not need to be a character even for supersolvable groups with a normal Sylow p-subgroup.

Let E be the extraspecial 3-group of order 3^3 and exponent 3. Let a be

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the automorphism of E that centralizes $Z(E)$ and inverts the elements of $E/Z(E)$. Put $E = \langle x, y\rangle$, $z = [x, y]$ and let G be the semidirect product of $\langle a\rangle$ and E. A routine but tedious calculation shows that $[\theta_3, \lambda] = -8$, where λ is the non-principal linear character of G. Therefore θ_3 is not a character.

At the end of [2], Thompson defines the functions $\theta_{\rm solv}$ and $\theta_{\rm nilp}$ by

$$\theta_{\rm solv}(g) = |\{h \in G \mid \langle g, h\rangle \text{ is solvable}\}|$$

and

$$\theta_{\rm nilp}(g) = |\{h \in G \mid \langle g, h\rangle \text{ is nilpotent}\}|.$$

These functions are generalized characters and Thompson asserts that "they are quite possibly characters". However, this is false again. If we take $G = A_5$ then one can check that $\theta_{\rm solv} = 22\chi_1 + 8\chi_2 + 6\chi_3 - 4\chi_4 - 4\chi_5$ (we are using the notation of the Appendix of [1] for the character table of A_5). Finally, the semidirect product G of the Frobenius group of order 72 which has an elementary abelian kernel of order 9 and a quaternion complement acted on by an automorphism of order 3 shows that $\theta_{\rm nilp}$ does not need to be a character (because $[\theta_{\rm nilp}, \chi] = -2$, where χ is the rational irreducible character of G of degree 2).

References

[1] I. M. Isaacs, "Character Theory of Finite Groups", Dover, New York, 1994.

[2] J. G. Thompson, Some generalized characters, *J. Algebra*, **179** (1996), 889–893.