An answer to a question of J. G. Thompson
on some generalized characters

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In 1996 J. G. Thompson [2] proved that given any finite group $G$ the
function $\theta_p : G \rightarrow \mathbb{Z}$ defined by

$$\theta_p(g) = |\{h \in G \mid \langle g, h \rangle_p \text{ is abelian}\}|,$$

where $\langle g, h \rangle_p$ denotes a Sylow $p$-subgroup of $\langle g, h \rangle$, is a generalized character. Thompson mentioned that “it seems reasonable to hope that $\theta_p$ is a character”. Certainly, this is the case for groups with abelian Sylow subgroups or for nilpotent groups. Unfortunately, we will show that $\theta_p$ does not need to be a character even for supersolvable groups with a normal Sylow $p$-subgroup.

Let $E$ be the extraspecial 3-group of order $3^3$ and exponent 3. Let $a$ be
the automorphism of $E$ that centralizes $Z(E)$ and inverts the elements of $E/Z(E)$. Put $E = \langle x, y \rangle$, $z = [x, y]$ and let $G$ be the semidirect product of $\langle a \rangle$ and $E$. A routine but tedious calculation shows that $[\theta_3, \lambda] = -8$, where $\lambda$ is the non-principal linear character of $G$. Therefore $\theta_3$ is not a character.

At the end of [2], Thompson defines the functions $\theta_{\text{solv}}$ and $\theta_{\text{nilp}}$ by

$$\theta_{\text{solv}}(g) = |\{h \in G \mid \langle g, h \rangle \text{ is solvable}\}|$$

and

$$\theta_{\text{nilp}}(g) = |\{h \in G \mid \langle g, h \rangle \text{ is nilpotent}\}|.$$  

These functions are generalized characters and Thompson asserts that “they are quite possibly characters”. However, this is false again. If we take $G = A_5$ then one can check that $\theta_{\text{solv}} = 22\chi_1 + 8\chi_2 + 6\chi_3 - 4\chi_4 - 4\chi_5$ (we are using the notation of the Appendix of [1] for the character table of $A_5$).

Finally, the semidirect product $G$ of the Frobenius group of order 72 which has an elementary abelian kernel of order 9 and a quaternion complement acted on by an automorphism of order 3 shows that $\theta_{\text{nilp}}$ does not need to be a character (because $[\theta_{\text{nilp}}, \chi] = -2$, where $\chi$ is the rational irreducible character of $G$ of degree 2).

References
