# DERIVED LENGTH AND CHARACTER DEGREES OF SOLVABLE GROUPS

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ABSTRACT. We prove that the derived length of a solvable group is bounded in terms of certain invariants associated to the set of character degrees and improve some of the known bounds. We also bound the derived length of a Sylow p-subgroup of a solvable group by the number of different $p$-parts of the character degrees of the whole group.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

It has been known for some time that the derived length of a solvable group is bounded in terms of the number of character degrees. (See [9] for the first result in this direction.) In a different direction, it was proved by M. Isaacs in [8] that the derived length of a Sylow $p$-subgroup of a solvable group $G$ is bounded by a linear function of the largest $p$-part of the character degrees of $G$. (This result was extended to $p$-solvable groups in [2] and improved to a logarithmic bound in [22].) Motivated by these results here we bound the derived length of a Sylow $p$-subgroup of a solvable group in terms of the number of different $p$-parts $|\operatorname{cd}(G)_p|$ of the set $\operatorname{cd}(G)$ of character degrees of $G$.

**Theorem A.** Let $G$ be a solvable group and $P \in \operatorname{Syl}_p(G)$. Put $n = |\operatorname{cd}(G)_p|$. Then the derived length of $P$ is bounded by a real-valued function $f(n)$ such that $f(n) = O(n \log n)$.

Thus, we obtain an almost linear bound. In general it is not true that $\operatorname{dl}(P) \le |\operatorname{cd}(G)_p|$ as shown by Example 6.1 of [8]. The bound obtained in [9] for the derived length of a group by $|\operatorname{cd}(G)|$ was linear and no bounds of a better order of magnitude have been found. However, it is believed that the "right" bound is logarithmic and a lot of work has been recently done in this direction by T. Keller, who has essentially reduced the problem to $p$-groups. (See [13, 14, 15].) We believe that it should also be possible to improve the bound in Theorem A to a logarithmic bound, but for that one would need to solve first the $p$-group case of the original problem. We also conjecture that it is possible to drop the solvability hypothesis.

In this note, we also obtain bounds for the derived length of a group in terms of other invariants associated to the set of character degrees. Given an integer $n = p_1^{a_1} \ldots p_t^{a_t}$ written as a product of different primes, we write $\omega(n)$ to denote $a_1 + \cdots + a_t$ and $\tau(n) = \max\{a_i \mid i = 1, \ldots, t\}$. Given a group $G$, $\omega(G) = \max\{\omega(\chi(1)) \mid \chi \in \operatorname{Irr}(G)\}$ and similarly $\tau(G) = \max\{\tau(\chi(1)) \mid \chi \in \operatorname{Irr}(G)\}$. Note that for any integer $n$, $\tau(n) \le \omega(n)$, so that $\tau(G) \le \omega(G)$ for any $G$. B. Huppert

[5] proved that $\mathrm{dl}(G) \leq 2\omega(G)$ if $\omega(G) > 1$. This linear bound was improved by A. Feyziogly [1], who proved a slightly better linear bound. On the other hand, U. Leisering and O. Manz (see [16] or Theorem 17.12 of [21]) proved a (somewhat worse) linear bound for the derived length of a group in terms of $\tau(G)$. We prove the following.

**Theorem B.** Let $G$ be a solvable group. Then there exist (absolute) constants $C_1$ and $C_2$ such that $\mathrm{dl}(G) \leq C_1 \log \tau(G) + C_2$. In particular, $\mathrm{dl}(G) \leq C_1 \log \omega(G) + C_2$.

As the Sylow $p$-subgroups of $\mathrm{GL}(n,p)$ show (see [6]), this bound is asymptotically best possible. We will see that the analog result for conjugacy class sizes also holds.

Another problem on character degrees that has been studied recently is the so-called one-prime hypothesis. We say that the set of character degrees of a group $G$ satisfies the one-prime hypothesis if for every $a < b \in \mathrm{cd}(G), \omega(a,b) \leq 1$, i.e, if the greatest common divisor of any two different members of $\mathrm{cd}(G)$ is either 1 or a prime number. More generally, we say that $G$ satisfies the $n$-**primes hypothesis** if for every $a < b \in \mathrm{cd}(G)$, $\omega(a,b) \leq n$. M. Lewis [17] proved that a group with the one-prime hypothesis has at most 14 character degrees. More recently, J. Hamblin [3] has found a bound for the number of character degrees of a group satisfying the two-primes hypothesis. It is natural to conjecture that the number of character degrees of a group with the $n$-primes hypothesis is bounded in terms of $n$. On the other hand, it was proved in [18, 12] and [19], that the derived length of a group with the one-prime hypothesis is at most 5 and that this bound is best possible. With the help of the results of these papers, Lewis [20] has been able to prove that a group with the one-prime hypothesis cannot have more than 9 character degrees, which is the best possible bound. Here we find a bound for the derived length of a group with the $n$-primes hypothesis.

**Theorem C.** Let $G$ be a solvable group that satisfies the $n$-primes hypothesis. Then the derived length of $G$ is bounded by a linear function of $n$.

This result might be useful to find a bound for the number of character degrees of a group with the $n$-primes hypothesis, but we have not succeeded in finding it. It also seems likely that there is a logarithmic bound for the derived length.

Our proofs of Theorems A, B and C are more applications of the recent results of [22]. We will also make use of the recent results of [11] and [15].

It is known that if $P$ is a $p$-group then the Taketa inequality $\mathrm{dl}(P) \leq |\mathrm{cd}(P)|$ holds. As already mentioned, it is believed that a logarithmic bound should exist. However, this problem is extremely difficult. M. Slattery [24] proved that if $\mathrm{cd}(P) = \{1, p, p^2, \ldots, p^n\}$ and $n \geq 3$, then $\mathrm{dl}(P) \leq n$, improving by one on Taketa's bound. We show that the logarithmic bound holds in a quite more general situation than Slattery's.

**Theorem D.** Let $P$ be a $p$-group and write $\mathrm{cd}(P) = \mathcal{A} \cup \mathcal{B}$, where $p^n$ the largest member of $\mathcal{A}$, $p^{2n} < m$ for every $m \in \mathcal{B}$ and $|\mathcal{B}| = k$. Then $\mathrm{dl}(P) \leq E_1 \log n + E_2 + k$ for some (absolute) constants $E_1$ and $E_2$..

In particular, if $\mathcal{B}$ is empty and $\mathcal{A} = \{1, p, p^2, \ldots, p^n\}$, we obtain a logarithmic bound in Slattery's situation. This result will be used in the proof of Theorem B.

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## 2. THE INVARIANTS $\omega(G)$ AND $\tau(G)$ AND THE $n$-PRIMES HYPOTHESIS

In this section we prove Theorems B, C and D. We also prove the analog of Theorem B for class sizes. We begin with the proof of Theorem D. Following [11], if $N$ is a normal subgroup of a group $G$, we set $\mathrm{cd}(G|N) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G), N \not\leq \mathrm{Ker}\, \chi\}$.

*Proof of Theorem D.* Let $\Delta$ be the sum of the irreducible characters of $P$ whose degree belongs to $\mathcal{A}$ and $K = \mathrm{Ker}\, \Delta$. Note that $K$ is the intersection of the kernels of the characters whose degree belongs to $\mathcal{A}$. In particular, $\mathrm{cd}(P|K) \subseteq \mathcal{B}$ and $\mathcal{A} \subseteq \mathrm{cd}(P/K)$. By the definition of $K$, $\Delta$ is a faithful character of $P/K$ and by a theorem of W. Burnside and R. Brauer (see Theorem 4.3 of [10]), all the irreducible characters of $P/K$ appear among the irreducible constituents of the powers of $\Delta$. Since any member of $\mathcal{B}$ exceeds $p^{2n}$, we deduce that $\mathrm{cd}(P/K) = \mathcal{A}$ and hence $\mathrm{cd}(P|K) = \mathcal{B}$.

Since $K$ is nilpotent, Corollary 3.3 of [11] tells us that

$$\mathrm{dl}(K) \leq |\mathrm{cd}(P|K)| = |\mathcal{B}| = k,$$

so all we need to show is that $\mathrm{dl}(P/K)$ is logarithmically bounded in terms of $n$. Write $H = P/K$. By Theorem 12.26 of [10], we have that $H$ has an abelian subgroup whose index in $H$ does not exceed $p^{4n}$. It is proved in Theorem 5.1 of [23] that if a group has an abelian subgroup of index $m$, then is has a characteristic abelian subgroup of index at most $m^2$. In our case, we have that $H$ has a normal abelian subgroup whose index does not exceed $p^{8n}$. Now it follows from a classical result of P. Hall (see Satz III.7.11 of [4]) that the derived length of $H$ is logarithmically bounded in terms of $|H|$, as desired. $\square$

The following (probably known) lemma will be necessary in the proofs of Theorems B and C.

**Lemma 2.1.** *Let $G$ be a solvable group. Then there exist real numbers $D_1$ and $D_2$ such that $\mathrm{dl}(G) \leq D_1 \log \tau(|G|) + D_2$.*

*Proof.* By Gaschutz's Theorem (see Satz III.4.2 and III.4.5 of [4]), $G/F(G)$ acts faithfully and completely reducibly on $F(G)/\Phi(G)$. Thus $G/F(G)$ is isomorphic to a subgroup of a direct product of linear groups of degree at most

$$\tau(|F(G)/\Phi(G)|) \leq \tau(|G|).$$

By Theorem 3.9 of [21], there exist constants $E_1$ and $E_2$ such that $\mathrm{dl}(G/F(G)) \leq E_1 \log \tau(|G|) + E_2$. By the above mentioned result of Hall,

$$\mathrm{dl}(F(G)) \leq F_1 \log \tau(|F(G)|) + F_2 \leq F_1 \log \tau(|G|) + F_2.$$

The result follows. $\square$

Now, we are ready to prove Theorem B.

*Proof of Theorem B.* Write $n = \tau(G)$. It was proved in Theorem C of [22] that $|G : F_{10}(G)|$ divides the degree of some irreducible character, where $F_{10}(G)$ is the 10-th term of the ascending Fitting series of $G$. It follows from Lemma 2.1 that the derived length of $G/F_{10}(G)$ is logarithmically bounded in terms of $\tau(|G/F_{10}(G)|)$, so it is logarithmically bounded in terms of $n$.

It follows easily from Clifford theory (see for instance Theorem 6.2 of [10]) that $\tau(P) \leq \tau(G)$ whenever $P$ is a Sylow subgroup of $F_{i+1}(G)/F_i(G)$ for any $i$. In

particular, if $i = 0, \ldots, 9$ and $P$ is a Sylow $p$-subgroup of $F_{i+1}(G)/F_i(G)$ we have that all the character degrees of $P$ divide $p^n$. Now we can apply Theorem D to deduce the the derived length of $P$ is logarithmically bounded in terms of $n$. The result follows. $\qquad \square$

Next, we sketch the proof of the conjugacy class analog. For any group $G$, we write $\omega^*(G) = \max\{\omega(|\operatorname{cl}_G(x)|) \mid x \in G\}$ and $\tau^*(G) = \max\{\tau(|\operatorname{cl}_G(x)|) \mid x \in G\}$.

**Theorem 2.2.** *Let $G$ be a solvable group. Then the derived length of $G$ is bounded by a logarithmic function of $\tau^*(G)$ (and therefore it is bounded by the same logarithmic function of $\omega^*(G)$).*

*Proof.* Put $n = \tau^*(G)$. By Theorem C' of [22], we have that $|G : F_{10}(G)|$ divides the size of some conjugacy class of $G$. As in the proof of Theorem B, we deduce that the derived length of $G/F_{10}(G)$ is bounded by a logarithmic function of $n$.

Since for any normal subgroups $M \geq N$ of $G$ and $x \in M$, $|\operatorname{cl}_{M/N}(xN)|$ divides $|\operatorname{cl}_G(x)|$, we have that $\tau(P) \leq \tau(G)$ whenever $P$ is a Sylow subgroup of $F_{i+1}(G)/F_i(G)$ for any $i = 0, \ldots, 9$. Now, using for instance Theorem VIII.9.12 of [7] and Satz III.7.11 of [4], we obtain the desired logarithmic bound. $\qquad \square$

Now, we prove Theorem C. We assume that a solvable group $G$ satisfies the $n$-primes hypothesis and we want to bound the derived length of $G$ by a logarithmic function in $n$.

*Proof of Theorem C.* Applying Theorem C of [22] to $G$ and $G/F_{10}(G)$, we have that there exist $d_1, d_2 \in \operatorname{cd}(G)$ such that $|G : F_{10}(G)|$ divides $d_1$ and $|G : F_{20}(G)| \mid d_2 < |G : F_{10}(G)|$. Since $G$ satisfies the $n$-primes hypothesis, we have that $n \geq \omega(d_1, d_2) \geq \omega(|G : F_{20}(G)|)$. By Lemma 2.1, we have that the derived length of $G/F_{20}(G)$ is bounded by a logarithmic function of $n$.

Now, we will bound the derived length of any Sylow subgroup $P$ of $F_{i+1}(G)/F_i(G)$ by a linear function of $n$ and the result will follow. By Taketa's Theorem (Theorem 5.12 of [10]), $P^{(n+1)}$ is contained in the kernel of any character whose degree is one of the smallest $n + 1$ character degrees of $P$. Therefore, $p^{n+1}$ divides $d$ for any $d \in \operatorname{cd}(P|P^{(n+1)})$. Using Clifford's theory, we have that $p^{n+1}$ divides $d$ for any $d \in \operatorname{cd}(G/F_i(G)||P^{(n+1)})$. By the $n$-primes hypothesis, we deduce that $|\operatorname{cd}(G/F_i(G)|P^{(n+1)})| \leq 1$. Now, Corollary 3.2 of [11] yields

$$\operatorname{dl}(P^{(n+1)}) \leq |\operatorname{cd}(G/F_i(G)|P^{(n+1)})| \leq 1,$$

i.e, the derived length of $P$ does not exceed $n + 2$, as desired. $\qquad \square$

## 3. Sylow $p$-subgroups and $p$-parts of character degrees

Finally, we prove Theorem A. Recall that we assume that a solvable group $G$ has $n$ different $p$-parts of character degrees and we want to bound the derived length of a Sylow $p$-subgroup by a function of $n$ of the order of $n \log n$.

*Proof of Theorem A.* Let $h(G) = 10h + r$ be the Fitting height of $G$, where $h \geq 0$ is some integer and $0 \leq r \leq 9$. Applying Theorem C of [22] to $G/F_{10i}(G)$ for $i = 0, \ldots, h - 1$ we obtain degrees $d_i \in \operatorname{cd}(G)$ such that $|G : F_{10(i+1)}(G)|$ divides $d_i$ and $d_i$ is a proper divisor of $|G : F_{10i}(G)|$. In particular, among that $h$ different character degrees $d_0, \ldots, d_{h-1}$ there are at most $n$ different $p$-parts. It is not difficult

to see that this implies that $p$ divides the order of at most $10n$ of the factor groups $F_{i+1}(G)/F_i(G)$, where $i = 0, \ldots, h(G) - 1$.

Now, we will find a logarithmic bound for the derived length of a Sylow $p$-subgroup of $F_{i+1}(G)/F_i(G)$ for $i > 0$. Let $P$ be a Sylow $p$-subgroup of $F_{i+1}(G)/F_i(G)$. Since $i > 0$, we have that $H = G/F_i(G)$ acts faithfully and completely reducibly on $\frac{F_i(G)/F_{i-1}(G)}{\Phi(G/F_{i-1}(G))}$. We can view this latter group as a direct sum $\oplus_{j=1}^k V_j$ of irreducible modules over finite fields (of possibly different characteristics) for the group $G/F_i(G)$. We also have that $H$ embeds into the direct product of the groups $H/C_H(V_j)$, so $P$ embeds into the direct product of the groups $O_p(H/C_H(V_j))$ (by Proposition 9.5 of [21], for instance). The group $H/C_H(V_j)$ acts faithfully and irreducibly on $V_j$ and hence $p$ does not divide $|V_j|$. Therefore, we can apply Theorem 1.5 of [15] to the action of $H/C_H(V_j)$ on $V_j$ to deduce that the derived length of $O_p(H/C_H(V_j))$ is bounded by a logarithmic function of the number of $p$-parts of the sizes of the orbits of the action of $H/C_H(V_j)$ on $V_j$. It is easy to see using Problem 6.18 of [10] that these orbits sizes correspond to degrees of irreducible characters of $G$. Hence, we deduce that the derived length of $O_p(H/C_H(V_j))$ is bounded by a logarithmic function in $n$. Since $P$ is embedded in the direct product of these groups, the same holds for $P$, as desired.

Finally, we will show that the derived length of $N = O_p(G)$ is at most $n$. This will complete the proof. Clearly, the hypothesis of Theorem 3.1 of [11] holds. Applying that theorem to $N, N', \ldots, N^{(n-1)}$ we deduce that $N^{(n)} = 1$, as we wanted to prove. $\square$

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