Heights of characters and defect groups

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1. Introduction

An important result in ordinary character theory is the Ito-Michler theorem, which asserts that a prime p does not divide the degree of any irreducible character of a finite group G if and only if G has a normal abelian Sylow p-subgroup. The famous Brauer's height zero conjecture can be thought as the block version of this result. Given a block B with defect d the height of a character $\chi \in \mathrm{Irr}(B)$ is the integer h such that $\chi(1)_p = p^{a-d+h}$, where $|G|_p = p^a$. The height zero conjecture asserts that the height of any character $\chi \in \mathrm{Irr}(B)$ is zero if and only if the defect group is abelian. The goal of this note is to discuss some possible extensions of the Ito-Michler theorem and its "block version", the height zero conjecture.

P. Fong [3] proved that all characters in a block with abelian defect group have height zero in a p-solvable group. The converse was proved by D. Gluck and T. Wolf in [4]. (However, both parts of the height zero conjecture remain open for arbitrary finite groups.) In their work in [4] Gluck and Wolf prove in fact a stronger result, namely that if e is the largest height of the characters in B, then the derived length of the defect group D cannot exceed $2e + 1$ (we will keep this notation thorought this note). This generalized a result of Isaacs [8] which asserts that the derived length of a Sylow p-subgroup of a p-solvable group does not exceed $2f + 1$, where p^f is the largest p-part of the degrees of the irreducible characters of G (we will also maintain this notation in the remaining of this paper). The following conjecture asserts that these results should hold for arbitrary finite groups.

Conjecture A *Let G be a finite group. Then the derived length of a Sylow p-subgroup is bounded in terms of f, where p^f is the largest p-part of the degrees of the irreducible characters of G. More precisely, if B is a p-block of G with defect group D, then the derived length of D is bounded in terms of the largest height of the irreducible characters in B.*

It is clear that the first statement of Conjecture A follows from the second statement for the principal block. Isaacs bound was improved to a logarithmic bound in the case of solvable groups in [15] and it seems reasonable to hope that a logarithmic bound should hold in both statements for arbitrary groups.

The work in [15] began as an attempt to relate p-parts of character degrees not with the derived length of a Sylow p-subgroup P but with the largest degree of the irreducible characters of P. It was conjectured in [12] that if we write $b(P)$ to denote the largest degree of the irreducible characters of P then $\log_p b(P)$ is bounded by a function of f, where p^f is the largest p-part of the degrees of the irreducible characters of G. Now, we restate this conjecture and present its block version.

Conjecture B *Let P be a Sylow p-subgroup of a finite group G and $b(P) = p^b$. Then b is bounded in terms of f. More precisely, if B is a p-block of G with defect group D then $\log_p b(D)$ is bounded in terms of e*

Again, the first statement of the conjecture follows from the second one for the principal block. It is tempting to conjecture that the bound $\log_p b(D) \leq 2e$ holds. For $p \leq 3$, there are examples due to Isaacs in [8] and [12] that show that this bound would be best possible. It is clear that Conjecture B implies Conjecture A and that this bound would imply the desired logarithmic bound in Conjecture A (using Theorem D of [13], for instance).

Our final conjecture relates the derived length of a Sylow p-subgroup (or of a defect group) and the number of p-parts of character degrees (or the number of heights of the characters in the block).

Conjecture C *The derived length of a Sylow p-subgroup of a finite group is bounded in terms of the number of different p-parts of the degrees of the irreducible characters of the group. More precisely, the derived length of the defect group of a block B is bounded in terms of the number of different heights of the characters in B.*

As before, the second statement for the principal block implies the first statement and it is clear that this conjecture implies Conjecture A. It also seems tempting to conjecture that a logarithmic bound should hold but, among other things, one would need to solve the hard problem that says that the derived length of a p-group is bounded by a logarithmic function of the number of character degrees.

In this note we discuss these conjectures for several important classes of finite groups: in Section 2 we consider p-solvable groups, in Section 3 the general linear groups in the defining characteristic, in Section 4 the symmetric groups and in Section 5 the sporadic groups. We prove that most of them hold for these groups.

Before proceeding to do this, we consider the possible existence of reversed inequalities. There are p-groups of derived length 2 with arbitrarily many character degrees, so there is no hope that we can find any class of reversed bound in the situation of Conjectures A and C. Also, as Frobenius groups whose complement is an abelian p-group show, there is no possible reversed inequality for the first part of Conjecture B. Hence, it is perhaps surprising that something can be said about the second part of Conjecture B. It has recently been proved in [14] that if G is p-solvable, then the height of any character in a p-block with defect group D does not exceed $2\log_p b(D)$ if p is odd and $5\log_p b(D)$ if $p = 2$. However, as shown by an example due to G. Malle (see Section 4 of [14]) there is no hope to obtain any bound for arbitrary groups.

2. p-solvable groups

As commented in the introduction, Conjecture A was proved for p-solvable groups in [8] and [4]. A logarithmic bound for the first statement for solvable groups was obtained in [15]

For solvable groups, the first statement of Conjecture B was recently proved in Corollary B of [15] and that of Conjecture C in Theorem A of [13]. In the remainder of this section we discuss the block forms of these conjectures. Our first result reduces the p-solvable case of the modular form of Conjecture B to a problem in ordinary character theory.

Theorem 2.1. *The second statement of Conjecture B for p-solvable groups holds if the following is true: if Z is a cyclic central p'-subgroup of a p-solvable group G, $P \in \mathrm{Syl}_p(G)$, $\lambda \in \mathrm{Irr}(Z)$ and for any $\chi \in \mathrm{Irr}(G|\lambda)$, $\chi(1)_p \leq p^n$, then $\log_p b(P)$ is bounded in terms of n.*

At first sight this statement looks very similar to that of the first part of Conjecture B, which was proved for solvable groups in [15]. However, it doesn't seem possible to prove it using the methods of [15]

Proof of Theorem 2.1. We begin working toward a proof of the second statement of Conjecture B for p-solvable groups. Using an argument due to Fong (see Theorems 9.14 and 10.20 of [16] or the first paragraph of the proof of Theorem A of [14]) we may assume that the defect group of B is a Sylow p-subgroup of G and that $\mathrm{Irr}(B) = \mathrm{Irr}(G|\theta)$ where $\theta \in \mathrm{Irr}(O_{p'}(G))$ is G-invariant. Put $N = O_{p'}(G)$. Let (G^*, N^*, θ^*) be a character triple isomorphic to (G, N, θ) (see Definition 11.23 of [10]). By Theorem 5.2 of [9] and Theorem 11.28 of [10], we may assume that $N^* \leq Z(G^*)$ is a cyclic p'-group. Furthermore, by Lemma 11.24 of [10] we know that the sets of p-parts of the degrees of the characters of G^* that lie over θ^* and the set of p-parts of the degrees of G that lie over θ coincide. Also, by the definition of character triple, the Sylow p-subgroups of G/N and G^*/N^* are isomorphic, and the result follows.

$\square$

We remark that the same proof would allow to obtain the corresponding restatement of the block form of Conjecture C for p-solvable groups. Using the results of [15] it is possible to reduce the block form of Conjecture B to a somewhat more restricted situation, but we do not think that it is worth including it here.

3. General linear groups

The goal of this section is to prove Conjectures A, B and C for $\mathrm{GL}(n,q)$ where $q=p^e$. We prove the following result.

Lemma 3.1. *Suppose that $G=GL(n,q)$, where $q=p^u$ and p is a prime. Then the set of p-heights of the ordinary irreducible characters of G lying in any of the p-blocks of defect bigger than 0 of G contains $\{ut(t-1)/2 \mid t=1,\dots,n-1\}$.*

Proof. By [2], G has $q-1$ blocks of defect zero and $q-1$ blocks of full defect. Let B be a block of G of full defect whose characters lie over a character $\rho\in\mathrm{Irr}(Z(G))$. By Proposition 3.1 of [18], the number of characters in B whose p-part of the degree is p^{ui} is bigger than 0 whenever there is a partition μ of n such that $n'(\mu)=i$, where if $\mu=(a_1^{l_1},\dots,a_\delta^{l_\delta})$ then

$$n'(\mu)=\sum_{j=1}^{\delta}\binom{a_j}{2}l_j.$$

Now, it suffices to consider the partitions $\mu_t=(t,1^{n-t})$ for $t=1,\dots,n-1$. The result follows. $\square$

Now, we can prove Conjectures A, B and C for the general linear group. Note that the bounds we obtain are good. By the structure of the blocks of $\mathrm{GL}(n,q)$ we may assume that B has full defect.

Theorem 3.2. *Let $G=\mathrm{GL}(n,q)$, where q is a power of p. Suppose that B is a block of full defect of G. Then the derived length of a Sylow p-subgroup P of G is bounded logarithmically in terms of the number of heights in B and also in terms of the largest height of the characters in B. Furthermore, $\log_p b(P)\le 2e$.*

Proof. By Satz III.16.3 of [6], the derived length of P is of the order of $\log n$. We have just proved that the number of heights is at least $n-1$ and that the largest height is at least $(n-1)(n-2)/2$, so the first claim follows. In order to prove the second claim it suffices to use, for instance, the information on the character degrees of P that appears in [7]. $\square$

4. Symmetric Groups

In this section we will find the set of heights of the characters in the blocks of the symmetric groups when $p\ge 5$. As a consequence, we will see that Conjectures A, B and C hold for these groups. We need to recall some terminology. It is well-known that the irreducible characters of the symmetric group S_n are labelled by the partitions of n. If λ is a partition of n and $[\lambda]$ is the Young diagram associated to λ a **p-hook** of $[\lambda]$ is the part of the diagram associated to a hook of length p.

The diagram (or the partition) obtained by succesively removing all the rim p-hooks successively is called the **p-core** of λ (it is known that this does not depend on the order). The **p-weight** of a partition λ is the number of rim p-hooks that have to be removed before obtaining the p-core. By Nakayama's conjecture (which was proved independently by Brauer [1] and de Robinson [19]), we know that two characters of S_n belong to the same block if and only if the associated partitions have the same p-core. Hence, it makes sense to define the weight of a block as the weight of the partition associated to any character of the block. For more details on all these concepts see [11].

Next, we need to recall some notation from [17]. We write $c(p,n)$ to denote the number of p-core partitions of n and $F_p(x)$ is the formal power series

$$F_p(x) = \sum_{n=0}^{\infty} c(p,n)x^n.$$

We also put

$$F_p(x)^s = \sum_{n=0}^{\infty} C_p(s,n)x^n.$$

Now, we can prove the key result of this section.

Theorem 4.1. *If $p \geq 5$, then the set of heights of the characters in any p-block B of S_n is the set of integers $\{0,1,2,\ldots,(w-a_0-a_1-\cdots-a_r)/(p-1)\}$, where w is the weight of B and $w = a_0 + a_1p + \cdots + a_rp^r$ is the p-adic decomposition of w.*

Proof. It was proved in Corollary 3.8 of [17] that the maximal possible height of characters in B is $(w-a_0-a_1-\cdots-a_r)/(p-1)$ and that the number of characters of such height is $C_p(p,w)$. It was proved in [5] that for every integer n there exists a p-core partition on n when $p \geq 5$. Hence $c(p,n) > 0$. This means that for any s all the coefficients of $F_p(x)^s$ are non-zero positive integers. Hence it follows from Proposition 3.5 of [17] that the set of heights is the one asserted in the statement of the theorem (here we are using the fact that the set $E_a(p,w)$ defined in [17] is not empty for all a with $0 \leq a \leq (w-a_0-a_1-\cdots-a_r)/(p-1)$). □

Corollary 4.2. *Let B be a p-block of S_n for $p \geq 5$ and D the defect group of B. Then $\log_p b(D)$ is the maximum height of the characters in B.*

Proof. Write $b(D) = p^b$. We have to show that $b = (w-a_0-a_1-\cdots-a_r)/(p-1)$, where w is the weight of B and $w = a_0+a_1p+\cdots+a_rp^r$ is the p-adic decomposition of w. It is well-known that D is isomorphic to a Sylow p-subgroup of S_{pw}. We have that $pw = a_0p + a_1p^2 + \cdots + a_rp^{r+1}$, so D is isomorphic to the direct product of a_0 copies of a Sylow p-subgroup of S_p, a_1 copies of a Sylow p-subgroup of S_{p^2},$\ldots$, a_r copies of a Sylow p-subgroup of $S_{p^{r+1}}$. It was proved in [7] that the largest character degree of a Sylow p-subgroup of S_{p^n} (for $n \geq 2$) is $p^{1+p+\cdots+p^{n-2}}$. We deduce that

$$\begin{aligned} b &= a_1 + a_2(1+p) + a_3(1+p+p^2) + \cdots + a_r(1+p+\cdots+p^{r-1}) \\ &= \sum_{i=1}^{r} a_i(p^i - 1)/(p-1) = (w - a_0 - a_1 - \cdots - a_r)/(p-1), \end{aligned}$$

as desired. □

Hence, we have proved that Conjecture B holds for symmetric groups when $p \geq 5$. It is also clear from Theorem 4.1 that Conjectures A and C also hold in this case. Similarly, one could prove that all these conjectures also hold when $p \leq 3$ for symmetric groups, but this would require some more tedious calculations using the structure of the 2-cores and 3-cores and the results of [17].

5. Sporadic Groups

Using the information provided by the GAP character table library, we have checked that the bound $\log_p b(D) \leq 2e$ holds for the sporadic groups. Actually, in order to check this bound we do not need to know the structure of the defect groups; it suffices to know their order.

Acknowledgements. I thank G. Navarro for helpful conversations. This research was partially supported by the Spanish Ministerio de Ciencia y Tecnología, grant BFM2001-0180, the FEDER and the Basque Government.

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