Heights of characters in blocks of p-solvable groups

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Abstract

In this paper, it is proved that if B is a Brauer p-block of a p-solvable group, for some odd prime p, then the height of any ordinary character in B is at most $2b$, where p^b is the largest degree of the irreducible characters of the defect group of B. Some other results that relate heights of characters with properties of the defect group are obtained.

1 Introduction

Let p be a prime, let G be a finite group and let B a p-block of G with defect group D. If $\chi \in \mathrm{Irr}(B)$ is an irreducible complex character of B, the height h of χ is the non-negative integer satisfying

$$\chi(1)_p = p^{a-d+h}\,,$$

where $|G|_p = p^a$ and $|D| = p^d$.

One of the main problems in finite group theory is to relate the representation theory invariants of a finite group G with those of certain local subgroups of G. If B is a Brauer p-block of G having defect group D, it is believed that the complexity of D reflects and is reflected in the set $\mathrm{Irr}(B)$ of complex irreducible characters in B. Brauer's famous height conjecture, for instance, asserts that D is abelian if and only all characters in $\mathrm{Irr}(B)$ have height zero. The "only if" part of this conjecture was proved for p-solvable groups by P. Fong (see Theorem 3C of [?] or Theorem 10.21 of [?]), who in fact showed, in this case, that if D is the defect group of a block B and $|D : Z(D)| = p^n$, then the height of any character of B does not exceed n. (The converse of Brauer's conjecture was proved for p-solvable groups by D.

Gluck and T. Wolf in [**?**], but both parts of the conjecture remain open for arbitrary groups.)

For p-solvable groups, we go further. Instead of comparing the heights of the characters in $\mathrm{Irr}(B)$ with the group theoretical structure of D, we compare them against the heights (that is, the exponents of the character degrees) of the group D.

Theorem 1. *Let G be a finite p-solvable group and let B be a Brauer p-block of G with defect group D, where p is an odd prime. Suppose that p^b is the largest degree of the irreducible complex characters of D. If $\chi \in \mathrm{Irr}(B)$ has height h, then $h \leq 2b$.*

Of course, if $\gamma \in \mathrm{Irr}(D)$, then $\gamma(1)^2 \leq |D : Z(D)|$, and from Theorem 1 we recover the bound $p^h \leq |D : Z(D)|$ for odd primes. We have been unable to decide whether or not Theorem 1 holds when $p = 2$. The bound that we obtain in this case is $h \leq 5b$.

In order to prove Theorem 1, we need the following result of independent interest.

Theorem 2. *Suppose that A acts coprimely as automorphisms on a finite group G. If $C_A(G) = 1$, then there exists a nilpotent A-invariant subgroup H of G such that $C_A(H) = 1$.*

The proof of Theorem 2 that we present here, which relies on the classification of finite simple groups, was provided to us, independently, by R. Guralnick and G. R. Robinson.

Now, we come back to discuss heights of characters and we compare them with the order of the defect group. It is clear that if h is the height of an irreducible character of a block B and d is the defect of B, then $h \leq d$. Furthermore, $h = d$ if and only if $d = 0$ (by Theorem 3.18 of [**?**]). When $d > 0$, the ratio h/d can be as close to 1 as we wish. For instance, for $G = SL_n(p)$ in the defining characteristic p, all characters except the Steinberg character lie in the principal block, and there exists a unipotent character with height $(n-1)(n-2)/2$. On the other hand, the defect of the block is $n(n-1)/2$.

Our next result shows that for p-solvable groups this cannot happen.

Theorem 3. *Let B be a p-block of defect d of a p-solvable group G. If $\chi \in \mathrm{Irr}(B)$, then the height h of χ is less than or equal to $3d/4$.*

As we will remark after the proof of this theorem, we can also obtain a sharper result that "almost" yields the bound $h \leq d/2$ for large primes.

The result of Fong that we mentioned before was refined by A. Watanabe in [**?**], who proved that if the height of some character is n (where $p^n = |D : Z(D)|$), then D is abelian. Our next result has, as an immediate consequence, the solvable case of both Fong and Watanabe's theorems.

Theorem 4. *Suppose that G is a solvable group. Let $\chi \in \mathrm{Irr}(B)$ of height h and suppose that B has defect group D. Suppose that $|D : Z(D)| = p^n$. Then*

$$h \leq 7n/8\,.$$

It seems likely that this result can be extended to p-solvable groups and that, at least, the bound $h \leq 3n/4$ holds. In fact, we will prove a strong form of Theorem 4 (see Theorem **??**) that "almost" yields this bound for large primes.

G. R. Robinson ([**?**]) has shown that if Dade's conjectures hold, then the Fong-Watanabe result should hold for arbitrary groups. We will provide examples that show that, unfortunately, there is no hope to extend our results to arbitrary groups.

We thank Guralnick and Robinson for their proof of Theorem 2 and G. Malle for showing us Example 4.1(ii). Some of this work was done while the first author was visiting the University of Wisconsin, Madison. He thanks the Mathematics Department for its hospitality.

2 Coprime action

We begin with the proof of Theorem 2. As usual, we will write $F^*(G)$ to denote the generalized Fitting subgroup of G. Recall that for any group G, we have that $C_G(F^*(G)) \le F^*(G)$. So the following lemma, which is a well-known application of the three subgroups lemma, applies when $N = F^*(G)$.

Lemma 2.1. *Suppose that A acts coprimely on G and suppose that $N \trianglelefteq G$ is A-invariant with $C_G(N) \le N$. If $[N, A] = 1$, then $[G, A] = 1$.*

Proof. We have that $[N, A, G] = [G, N, A] = 1$, and therefore $[G, A, N] = 1$. Hence $[G, A] \le N$ and thus $[G, A] = [G, A, A] = 1$. □

Lemma 2.2. *Suppose that A acts faithfully and coprimely on a finite simple group G. Then A acts faithfully on some A-invariant Sylow subgroup of G.*

Proof. It is well-known that alternating and sporadic groups do not admit coprime automorphisms, so by the classification of finite simple groups we may assume that G is of Lie type. Let p be the characteristic of G. We claim that A acts faithfully on any A-invariant Sylow p-subgroup of G.

Clearly, we may assume that $A = \langle a \rangle$ is cyclic of prime order. Let U be an A-invariant Sylow p-subgroup of G and, by way of contradiction, assume that $[U, A] = 1$. First, we note that G must have rank 1. Let $B = N_G(U)$. Then $U = F^*(B)$, so that $[B, a] = 1$. If G has rank greater than 1, then G is generated by parabolic subgroups of the form $N_G(V)$, where $1 \neq V \trianglelefteq U$. Notice that $N_G(V)$ is A-invariant as $[U, a] = 1$. But for such subgroups, $F^*(N_G(V)) = O_p(N_G(V)) \leq U$ so that a centralizes $F^*(N_G(V))$ and hence centralizes $N_G(V)$ (by Lemma **??**). Thus G must have rank 1 as claimed.

It follows that G is one of the groups $L_2(q), U_3(q)$, ${}^2G_2(q),{}^2B_2(q)$ where q is a power of p (where q is an odd power of p in the last two cases, and $p = 3, 2$, respectively). In all cases, G is a doubly transitive group on the cosets of B and we have $|G : B| = 1 + |U|$. Now $G = B \cup BwB$, where w is an involution. Furthermore, w normalizes $B \cap B^w = T$, and T is a (necessarily A-invariant) Hall p'-subgroup of B. Hence $G = BN_G(T)B$. Furthermore, $|N_G(T) : T| = 2$ in all cases, so it follows that $[N_G(T), a] \leq T$ and $[N_G(T), a, a] \leq [T, a] = 1$. Thus $[N_G(T), a] = 1$ by coprime action, so that $[G, a] = 1$, contrary to assumption. $\square$

Proof of Theorem 2. We argue by induction on $|G|$. Let $Z = Z(G)$. We claim that we may assume that $Z = 1$. Suppose not. Let $D = C_A(G/Z)$. We have that A/D acts faithfully on G/Z. By the inductive hypothesis, there exists an A-invariant nilpotent subgroup V/Z of G/Z on which A/D acts faithfully. Hence, V is A-invariant and nilpotent. Now, $C_A(V)D/D$ centralizes V/Z, and therefore $C_A(V) \leq D$. Since $C_A(V) \leq C_A(Z)$, it follows by coprime action that $C_A(V) \leq C_A(G) = 1$, and we are done in this case.

Next, we claim that G cannot be expressed as the product of two proper

A-invariant normal subgroups $G = MN$ with $[M, N] = 1$. Otherwise, let $B = C_A(M)$, $C = C_A(N)$ and notice that $B \cap C = 1$. Now, A/B acts faithfully on M, and by induction there is a nilpotent A-invariant subgroup H of M with $C_A(H) = B$. Similarly, A/C acts faithfully on N, and again by induction there is a nilpotent A-invariant subgroup K of M with $C_A(K) = C$. Then HK is nilpotent, A-invariant and $C_A(HK) = 1$.

By Lemma **??**, we may assume that $G = F^*(G)$. Since we may write $F^*(G) = E(G)F(G)$, where $E(G)$ is the layer of G, by the second paragraph we may assume that $G = E(G)$. Now, by using Theorem X.13.18 of [**?**], we may write

$$G = K_1 \times \cdots \times K_r\,,$$

where K_i is simple non-abelian.

Now, A acts on $\{K_1, \ldots, K_r\}$ (by Theorem X.13.16 of [**?**]). Let $\Delta_1, \ldots, \Delta_s$ be the distinct A-orbits, and $\Lambda_i = \prod_{T \in \Delta_i} T$. We have $G = \Lambda_1 \times \cdots \times \Lambda_s$. If $s > 1$, then the theorem is again proved by using the claim in the second paragraph. Therefore, we may assume that A acts transitively on the set $\{K_1, \ldots, K_r\}$. Write $K_i = K^{a_i}$ for some $a_i \in A$, where $K = K_1$ and $a_1 = 1$.

Now, let $B = N_A(K)/C_A(K)$. We have that B acts faithfully on the nonabelian simple group K. By Lemma **??**, there exists a B-invariant Sylow q-subgroup $1 \neq Q$ of K such that B acts faithfully on Q. We claim that $C_A(Q) \leq C_A(K)$. Let $a \in C_A(Q)$. Suppose that $K^a = K_2$, for instance. Then $Q = Q^a \leq K_1 \cap K_2 = 1$, and this is not possible. Therefore, $a \in N_A(K)$. Hence, $aC_A(K) \in B$ centralizes Q, and the claim follows. Now, let $Q_i = Q^{a_i}$ and let $U = Q_1 \cdots Q_r$, which is again nilpotent. Notice that $C_A(Q_i) \leq C_A(K_i)$. Now, let $a \in C_A(U)$. Hence, $a \in C_A(Q_i)$. Then a centralizes G. It remains to show that U is A-invariant. Let $x \in A$. If

$K_i^x = K_j$, it suffices to show that $Q_i^a = Q_j$. Now, we have that $K^{a_i x} = K^{a_j}$. Hence, $a_i x a_j^{-1} \in N_A(K)$. Thus $Q^{a_i x a_j^{-1}} = Q$, and $Q_i^x = Q_j$, as desired. $\square$

3 Heights of characters

Is this section, we prove Theorems 1, 3 and 4. In order to prove Theorem 1, we need the following regular orbit theorem.

Theorem 3.1. *Suppose that G is a p-solvable group with $O_p(G) = 1$, p odd, and let $P \in \mathrm{Syl}_p(G)$. Suppose that V is a faithful FG-module, where* $\mathrm{char}F = p$. *Then there exists $v \in V$ such that $C_P(v) = 1$.*

Proof. We argue by induction on $|G|$. If G is solvable, then this is the main result of [**?**]. (An alternative proof is available in [**?**].) Now, let $N = O_{p'}(G)$, $H = NP$ and notice that $C_P(N) = 1$, since $C_G(N) \leq N$. By Theorem 2, there exists a nilpotent P-invariant subgroup K of N such that $C_P(K) = 1$. In this case, $O_p(KP) = 1$, and since V_{KP} is faithful, we may assume that $G = KP$. Then G is solvable, and we are done. $\square$

If G is a finite group, we write $b(G)$ for the biggest irreducible complex character degree of G.

Theorem 3.2. *Suppose that G is a p-solvable group and let $P \in \mathrm{Syl}_p(G)$.*

(i) *If p is odd, then $|G : O_{p'p}(G)|_p \leq b(P)$.*

(ii) *If $p = 2$, then $|G : O_{p'p}(G)|_p \leq b(P)^4$.*

Proof. For the first part, argue as in Proposition 2.2 of [**?**], using Theorem (2.4) instead of the main result of [**?**]. For the second part, use Proposition 2.3 of [**?**] and Theorem 12.26 of [**?**] $\square$

Next is Theorem 1.

Theorem 3.3. *Let G be a finite p-solvable group and let B be a Brauer p-block of G with defect group D. Suppose that p^b is the largest degree of the irreducible complex characters of D. If $\chi \in \mathrm{Irr}(B)$ has height h, and p is odd then $h \leq 2b$. If $p = 2$, then $h \leq 5b$.*

Proof. Put $N = O_{p'}(G)$. Let $\theta \in \mathrm{Irr}(N)$ be lying under χ. Since N is a p'-group, we have that $\{\theta\}$ is a block and it is covered by B (by Theorem 9.2 of [**?**]). Let T be the inertia group of θ, which coincides with the inertia group of the block $\{\theta\}$, and let $\varphi \in \mathrm{Irr}(T|\theta)$ be the Clifford correspondent of χ (see Theorem 6.11 of [**?**]). By the Fong-Reynolds theorem (Theorem 9.14 of [**?**]) the defect of the block of φ coincides (up to conjugacy) with the defect of B and $\mathrm{height}(\varphi) = \mathrm{height}(\chi)$. Thus, we may assume that θ is G-invariant.

By Fong's theorem (Theorem 10.20 of [**?**]) $\mathrm{Irr}(B) = \mathrm{Irr}(G|\theta)$ and D is a Sylow p-subgroup of G. Also, the height h of $\chi \in \mathrm{Irr}(B)$ is $\chi(1)_p = p^h$. Write $b(P) = p^b$. Now, let $M = O_{p'p}(G)$ and let $\eta \in \mathrm{Irr}(M)$ be under χ. Hence, we have $\chi(1)_p \leq \eta(1)_p|G : M|_p$ by Corollary 11.29 of [**?**]. Now, since θ is M-invariant, it follows that θ has an extension $\hat{\theta}$ to M (by Corollary 8.16 of [**?**]). Hence, Gallagher's Theorem (Theorem 6.17 of [**?**]) yields that $\eta = \hat{\theta}\gamma$ for some $\gamma \in \mathrm{Irr}(M/N)$. Since $M/N \leq PN/N \cong P$, we have that $\eta(1)_p = \gamma(1) \leq b(P)$. Now, if p is odd, we can use the first part of Theorem **??**, we conclude that $\chi(1)_p \leq b(P)^2$, as desired. Similarly, the theorem follows for $p = 2$ by using the second part of Theorem **??** $\square$

Now, we prove Theorem 3.

Proof of Theorem 3. Let $\theta \in \mathrm{Irr}(O_{p'}(G))$ lie under χ. Arguing as in the beginning of the proof of Theorem 1, we may assume that θ is G-invariant. We may also assume that $\mathrm{Irr}(B) = \mathrm{Irr}(G|\theta)$ and that the defect groups of B are the Sylow p-subgroups of G. Let $P \in \mathrm{Syl}_p(G)$.

Write $|G|_p = p^a, |O_{p'p}(G)|_p = p^n$ and $|G : O_{p'p}(G)|_p = p^m$ so that $n + m = a = d$. Also, put $\overline{G} = G/O_{p'}(G)$ and use the bar convention. Note that $F(\overline{G}) = O_p(\overline{G})$. By Gaschutz's theorem (Satz III.4.2 and III.4.5 of [**?**]), $\overline{G}/F(\overline{G})$ acts faithfully and completely reducibly on the elementary abelian p-group $F(\overline{G})/\Phi(\overline{G})$. Since $|F(\overline{G})/\Phi(\overline{G})| \leq p^n$ we deduce that $\overline{G}/F(\overline{G})$ is a completely reducible subgroup of $\mathrm{GL}(n, p)$. It follows from a theorem of Wolf (see [**?**]) that $m < n$. Thus $m < (n + m)/2 = d/2$.

We have that θ extends to $O_{p'p}(G)$. Let $\hat{\theta}$ be an extension of θ, so that all the characters of $O_{p'p}(G)$ that lie over θ are of the form $\hat{\theta}\mu$, where $\mu \in \mathrm{Irr}(O_p(\overline{G}))$. Of course, $(\hat{\theta}\mu(1))_p < p^{n/2}$ and since χ lies over some of these characters, we deduce using Clifford theory that $\chi(1)_p < p^{m+n/2} = p^{m/2+d/2} = p^{3d/4}$, as desired. $\square$

We note that, in order to keep the proof less technical, we have used a weak form of Wolf's theorem. Using the full strength of his result, we would obtain that $h \leq f(p)d$ where f is a function of p whose value at any prime is less than $3/4$ and such that $f(p) \to 1/2$ when p goes to infinity.

Now, we begin work toward a proof of a slightly strengthened version of Theorem 4. We need the following lemma.

Lemma 3.4. *Suppose that V is a finite dimensional faithful completely reducible FG-module, where F is the field with p elements and G is solvable. Let $P \in \mathrm{Syl}_p(G)$ of order p^n and suppose that $\dim(V) - \dim C_V(P) = t$. Then $n < t(p + 1)/(p - 1)$.*

Proof. By [**?**], we know that

$$\dim C_V(P) \leq \frac{2}{p+1}\dim(V)\,.$$

(Actually, this is stated for irreducible modules, but works for completely reducible. Furthermore, this is the only place where solvability is required.) By [**?**], we also know that $|P| < |V|$. Now, we have that

$$\dim(V) \leq t + \frac{2}{p+1}\dim(V)$$

so $\dim(V) \leq t(p+1)/(p-1)$ and the result follows. $\square$

We remark that in his review of [**?**] D. Gluck (MR 2000e:20023) suggested that a similar bound for $\dim C_V(P)$ could possibly be extended to arbitrary groups G. If this were the case, then the remaining of our proof of Theorem 4 would work unchanged for p-solvable groups.

Finally, we can prove the strong form of Theorem 4.

Theorem 3.5. *Suppose that G is a solvable group. Let $\chi \in \mathrm{Irr}(B)$ having height h and suppose that B has defect group D. Suppose that $|D : Z(D)| = p^n$. Then*

$$h \leq (3/4 + 1/4p)n\,.$$

Proof. Let $\theta \in \mathrm{Irr}(O_{p'}(G))$ be a character lying under χ. Arguing as in the beginning of the proof of Theorem 1, we may assume that θ is G-invariant. We may also assume that $\mathrm{Irr}(B) = \mathrm{Irr}(G|\theta)$ and that the defect groups of B are the Sylow p-subgroups of G. Let $P \in \mathrm{Syl}_p(G)$.

As in the proof of Theorem 3, put $\overline{G} = G/O_{p'}(G)$ and use the bar convention. Note that $\overline{P} \cong P$. Observe also that $F(\overline{G}) = O_p(\overline{G})$ and by Gaschutz's theorem , $H = \overline{G}/F(\overline{G})$ acts faithfully and completely reducibly on the elementary abelian p-group $V = F(\overline{G})/\Phi(\overline{G})$. Put $Q = \overline{P}/F(\overline{G})$ so that Q is a Sylow p-subgroup of H. Note that $Z(\overline{P}/\Phi(\overline{G})) \leq F(\overline{G})$. In particular, $C_V(Q) \geq Z(\overline{P})$.

Now, we have that θ extends to $O_{p'p}(G)$, so all the characters of $O_{p'p}(G)$ that lie over θ are of the form $\hat{\theta}\mu$ for some $\mu \in \mathrm{Irr}(F(\overline{G}))$. Notice that $\hat{\theta}$ has p'-degree and that

$$\mu(1) \leq |F(\overline{G}) : Z(F(\overline{G}))|^{1/2} \leq |F(\overline{G}) : Z(\overline{P})|^{1/2} = p^{t/2},$$

where the second inequality follows from the fact that $Z(F(\overline{G})) \geq Z(\overline{P})$ and we have defined t by means of $p^t = |F(\overline{G}) : Z(\overline{P})|$. Thus, the p-part of the degree of any irreducible character of $O_{p'p}(G)$ lying over θ does not exceed $p^{t/2}$.

Let $|Q| = p^m$ and observe that $n = m + t$. Therefore, using Clifford theory, we have that the height of our character χ is at most $m + t/2$. Now, all we need to show is that

$$m + t/2 \leq (3/4 + 1/4p)(m + t).$$

This is true if and only if $(1 - 1/p)m \leq (1 + 1/p)t$, i.e, if and only if

$$m \leq \frac{1 + 1/p}{1 - 1/p} t = t(p + 1)/(p - 1).$$

Applying Lemma **??** to the action of H on V, we have that $m \leq e(p + 1)/(p - 1)$, where

$$e = \log_p |V : C_V(Q)| \leq \log_p |\overline{F} : Z(\overline{P})| = t.$$

The result follows. $\square$

4 Examples

First, we see that it is not possible to obtain any analog of Theorem 1 if we remove the p-solvability hypothesis.

Example 4.1. (i) If $G = L_3(q)$ where $q \equiv -1 \pmod 4$ then the Sylow 2-subgroup of G is semidihedral and it follows from work of Olsson [**?**] on the structure of these blocks that there are characters in the principal block of height $d-2$, where $|G|_2 = 2^d$. This means that it is not possible obtain any bound in the situation of Theorem 1 and that it is not possible to improve on the bound conjectured by Robinson for the height in terms of the index of the center of the defect group.

(ii) The following example is the analog to (i) for odd primes and has been communicated to us by G. Malle. If $G = GL_r(q)$, where q is a power of a prime p, $r \neq p$ is prime and r^a is the exact power of r that divides $q-1$, then the Sylow r-subgroup of G is $C_{p^a} \wr C_p$. On the other hand, there are non-unipotent characters in the principal r-block whose height is $a(r-1)$.

It is perhaps remarkable, however, that the conclusion of Theorem 1 remains true for the symmetric groups, the general linear groups in the defining characteristic and the sporadic groups.

In view of the bound obtained in Theorem **??** and of Theorem 4, it seems natural to hope that the coefficient $1/4p$ can be erased from Theorem **??**. Still, the question of what the best possible bound is would remain open. Our next example shows that the coefficients in the bounds of Theorems C and D cannot be smaller than $1/2$, so we are not too far from these best possible bounds.

Example 4.2. (i) Let E be an extraspecial p-group of order p^{2n+1}. By [**?**], we have that the group of outer automorphisms of E that fix all the elements of $Z(E)$, contains as a subgroup either the symplectic group $\mathrm{Sp}_{2n-2}(p)$ or an orthogonal group $O_{2n}(2)$. Choosing n large enough, we may assume that $\mathrm{Out}(E)$ has a subgroup that is a solvable Frobenius group $F = PH$ whose complement P is a cyclic p-group of order p^m with $m \geq 2$. Let G be this extension of E by F. First, note that $O_{p'}(G) = 1$, so by Corollary 15.40 of [**?**] we have that G has a unique p-block. Now, let $\theta \in \mathrm{Irr}(E)$ be non-linear. Since E is an extraspecial group, θ vanishes outside $Z(E)$. By the choice of F, we have that θ is G-invariant. Since P is cyclic, θ extends to PE and by coprimeness, θ also extends to HE. Now, Corollary 11.31 of [**?**] yields that θ extends to G. Let $\hat{\theta}$ be an extension of θ to G and let $\mu \in \mathrm{Irr}(F)$ non-linear. Then $\hat{\theta}\mu \in \mathrm{Irr}(G)$ and

$$\hat{\theta}\mu(1) = \hat{\theta}(1)\mu(1) = p^n p^m = p^{n+m},$$

so the height of this character is $n + m$ which is bigger than one-half of the defect of the block. This is also an example of a group with $h > b$, where $p^b = b(D)$.

(ii) The direct product of n copies of $\mathrm{GL}(2, 3)$ shows that if one could prove that $h \leq 2b$ when $p = 2$, then this bound would be best possible. However, it seems likely that our bound in Theorem 1 can be improved for large primes.

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