Large orbits of p-groups on characters and applications to character degrees

by

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1 Introduction

Let a group A act as automorphisms on a finite group G. There are many papers devoted to studying the orbits of these actions and, in particular, to prove that there are large orbits. As is well-known, these results have applications in character theory. For instance, they are the key to many of the results that have been proved on character degrees.

It may be surprising therefore that, as far as we know, there is not any known result that says that there is a large orbit on the set of irreducible characters of G. As an application of the results of [16], we provide one such result.

Theorem A. *Let A be a p-group that acts faithfully on a solvable p'-group G. Let n be an integer such that $|A : C_A(\chi)| \leq p^n$ for all $\chi \in \mathrm{Irr}(G)$. Then $|A| \leq p^{19n}$.*

It is certainly necessary to assume that $(|A|, |G|) = 1$. Otherwise, we can take G to be any nonabelian p-group and $A = G/Z(G)$. Since $|A|$ can be arbitrarily large and all the A-orbits of characters of G are trivial, we deduce that Theorem A fails in the noncoprime case. It was already proved by W. Burnside [2] that there are p-groups P that have class-preserving outer automorphisms. It follows from a theorem of R. Brauer (Theorem 6.32 of [5]) that these automorphisms fix all the irreducible characters. We can take a direct product of arbitrarily many copies of P and this shows that it is necessary to assume that the action is coprime even it we take A to be a subgroup of outer automorphisms of G. However, we think that it should be possible to remove the solvability assumption on G. As we will see in Section 2, this would allow to solve Conjecture 4 of [12] for p-solvable groups.

In the situation of Theorem A, if we consider the action on the elements of G instead of on the characters, it is possible to prove that $|A| \leq p^{2n}$ and there are even better bounds (see [6]). If seems reasonable to expect that, since the number of characters of a non-abelian group is usually much smaller than the number of elements, the bounds that we will obtain when we consider actions on characters will be worse. However, we know of no example where $|A| > p^{2n}$. It is worth mentioning that for coprime group actions the sizes of the orbits on characters are the same as the sizes of the orbits on the conjugacy classes. Therefore Theorem A can also be applied to actions on conjugacy classes.

As a consequence of Theorem A, we will prove some results that relate ordinary and Brauer character degrees. Fix a prime number p. It follows from the Fong-Swan theorem (Theorem 10.1 of [17]) that for solvable groups G the set of irreducible Brauer characters $\mathrm{IBr}(G)$ may be seen as a subset of $\mathrm{Irr}(G)$. It seems natural to ask the following: What can we say about $\mathrm{Irr}(G)$ if we know $\mathrm{IBr}(G)$? In this note, we focus on character degrees so our question is: What can we say about the degrees of the ordinary irreducible characters if we know the Brauer character degrees? Since $O_p(G)$ is contained in the kernel of all the Brauer characters, it is natural to assume that $O_p(G) = 1$.

First, we look at p-parts of character degrees. It was proved in [7] that if p^2 does not divide the degree of any irreducible Brauer character of a solvable group with $O_p(G) = 1$, then it does not divide the degree of any irreducible ordinary character either. The hypothesis was replaced by the analogue one on class sizes of p-regular elements in Theorem 1.4 of [9], and the same conclusion was obtained. We will extend these results. We define $e_p(G)$ to be the largest integer that occurs as the exponent of the p-part of the degrees of the members of $\mathrm{Irr}(G)$. Similarly, we define $\overline{e}_p(G)$ to be the largest integer that occurs as the exponent of the p-part of the degrees of the members of $\mathrm{IBr}(G)$. We prove that $e_p(G)$ is bounded in terms of $\overline{e}_p(G)$. In fact, we obtain a stronger result.

Theorem B. *Let G be a finite solvable group with $O_p(G) = 1$ and write $\overline{e}_p(G) = n$. Then $|G|_p \leq p^{96n}$.*

As a consequence, we have that for any solvable group such that $\overline{e}_p(G) = n$, $|G : F(G)|_p \leq p^{96n}$. It would be interesting to know what the best possible bounds are. In particular, we know of no example where $|G|_p > p^{2n}$ in the situation of Theorem B.

As shown in [7] it is not possible to obtain any bounds in the situation of Theorem B if we look at q-parts of character degrees for a prime $q \neq p$. However, it we consider all the primes at the same time, we can obtain results of this flavor. Given an integer $n = p_1^{a_1} \ldots p_t^{a_t}$ where $p_i \neq p_j$ if $i \neq j$, we write $\tau(n) = \max_{i=1}^{t} a_i$. Now we define

$$\tau_p(G) = \{\max \tau(\varphi(1)) \mid \varphi \in \mathrm{IBr}(G)\}$$

and

$$\tau(G) = \{\max \tau(\chi(1)) \mid \chi \in \mathrm{Irr}(G)\}.$$

It was proved in Theorem 3.2 of [8] that $\tau(G)$ is bounded by a function that is quadratic in $\tau_p(G)$ and logarithmic in p. After the proof of this theorem, two questions were raised (see 3.3 of [8]). The second of them asks whether or not it is possible to find a bound that does not depend on p. Unfortunately, we have been unable to answer this question. However, we can give an affirmative answer to the first question, which asks whether there is a bound that is linear in $\tau_p(G)$.

Theorem C. *For every solvable group G with $O_p(G) = 1$, we have that*

$$\tau(G) \leq (152 \log_2 p + 97)\tau_p(G).$$

Our proofs are further applications of the results in [11]. The main theorems of [11] have a number of applications to other problems on ordinary character degrees, conjugacy class sizes and zeros of characters (see Section 2 of [16] or [13, 14, 15]). The results of this paper constitute the first applications to results that relate ordinary and Brauer character degrees. For the reader's convenience, we recall Theorem E(i) of [16].

Theorem 1.1. *Let V be a finite completely reducible faithful G-module (possibly of mixed characteristic), where G is a finite solvable group. Then there exists $v \in V$ such that $C_G(v) \leq F_9(G)$, where $F_9(G)$ is the 9th term in the ascending Fitting series of G.*

In order to make the proofs as smooth as possible, the constants that appear in our results are not the best possible ones that could be obtained with our methods. In particular, for odd order groups all the bounds can be considerably improved.

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2 Orbits on characters

In this section, we prove Theorem A. Recall that if G is a finite group, $F_1(G) = F(G)$ is the Fitting subgroup of G and for $i > 1$, $F_i(G)$ is defined by means of $F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G))$. We begin with a lemma.

Lemma 2.1. *Assume that a p-group A acts faithfully on a solvable p'-group G. Let n be an integer such that $|A : C_A(\chi)| \leq p^n$ for all $\chi \in \mathrm{Irr}(G)$. Let $\Gamma = AG$ be the semidirect product. Then $|F_{i+1}(\Gamma) : F_i(\Gamma)|_p \leq p^{2n}$ for all $i \geq 1$.*

Proof. Let $O/F_{i-1}(\Gamma) = O_p(\Gamma/F_{i-1}(\Gamma))$. Replacing Γ by Γ/O we may assume, without loss of generality, that $i = 1$. Let $P/F(\Gamma)$ be the Sylow p-subgroup of $F_2(\Gamma)/F(\Gamma)$. Write $P = QF(\Gamma)$, where $Q \in \mathrm{Syl}_p(P)$. We have to prove that $|Q| \leq p^{2n}$.

We know by Gaschutz's theorem that Q acts faithfully on $\mathrm{Irr}(F(\Gamma)/\Phi(P))$. Replacing A by a conjugate, if necessary, we may assume that $Q \leq A$. Also, it is clear that $Q \trianglelefteq A$. It follows from our hypothesis that $|Q : C_Q(\chi)| \leq p^n$ for all $\chi \in \mathrm{Irr}(G)$.

Now, let $\lambda \in \mathrm{Irr}(F(\Gamma)/\Phi(P))$. By Theorem 13.28 of [5], there exists $\chi \in \mathrm{Irr}(G)$ lying over λ that is $C_Q(\lambda)$-invariant. We claim that $C_Q(\chi) = C_Q(\lambda)$. It is clear that $|Q : C_Q(\lambda)|$ divides the degree of any character of P lying over λ. Therefore, $|Q : C_Q(\lambda)|$ divides the degree of any character of QG lying over λ. Now, Corollary 8.16 of [5] and Clifford's correspondence (Theorem 6.11 of [5]) yield that there exist $\psi \in \mathrm{Irr}(QG)$ lying over χ, whence over λ, such that $\psi(1)_p = |Q : C_Q(\chi)|$. It follows that $C_Q(\chi) = C_Q(\lambda)$, as desired. In particular, $|Q : C_Q(\lambda)| \leq p^n$. We deduce that for all $\lambda \in \mathrm{Irr}(F(\Gamma)/\Phi(P))$, $|Q : C_Q(\lambda)| \leq p^n$. Now, [6], for instance, implies that $|Q| \leq p^{2n}$. This completes the proof of the lemma. □

Now, we are ready to prove Theorem A.

Proof of Theorem A. Let $\Gamma = AG$ be the semidirect product of A and G. By Gaschutz's theorem, $\Gamma/F(\Gamma)$ acts faithfully and completely reducibly on $\mathrm{Irr}(F(\Gamma)/\Phi(\Gamma))$. It follows from Theorem 1.1 that there exists $\lambda \in \mathrm{Irr}(F(\Gamma)/\Phi(\Gamma))$ such that $T = C_\Gamma(\lambda) \leq F_{10}(\Gamma)$. We know that $F(\Gamma)$ is a p'-group and, using Lemma 2.1 that $|F_{i+1}(\Gamma) : F_i(\Gamma)|_p \leq p^{2n}$ for $i = 1, \ldots, 9$. Hence, in order to complete the proof of the theorem, it suffices to see that $|\Gamma : T|_p \leq p^n$.

Let χ be any irreducible character of G lying over λ. Then every irreducible character of Γ that lies over χ also lies over λ and hence has degree divisible by $|\Gamma : T|$. But χ extends to its stabilizer in Γ and thus some irreducible character of Γ lying over χ has degree $\chi(1)|A : C_A(\chi)|$. The p-part of $|\Gamma : T|$, therefore, divides $|A : C_A(\chi)|$, which is at most p^n. This completes the proof of the theorem. □

We conjecture that the solvability assumption is not necessary.

Conjecture 2.2. *Assume that a p-group A acts faithfully on a p'-group G. Let n be an integer such that $|A : C_A(\chi)| \leq p^n$ for all $\chi \in \mathrm{Irr}(G)$. Then*

(i) $|A| \leq p^{f(n)}$ *for some function f.*

(ii) *The function f can be taken to be linear.*

Now, we will see that Conjecture 2.2 holds for simple groups G. In fact, there is a much stronger result in this case.

Theorem 2.3. *Assume that a p-group A acts on a simple p'-group G. Then there exists a regular orbit of A on the irreducible characters of G.*

Proof. It is well-known that if a simple group G has a non-trivial automorphism of coprime order then it is a group of Lie type and the automorphism is, up to conjugation, a field automorphism (see [3]). This implies that A is cyclic and the result follows from the fact that a non-trivial coprime automorphism cannot fix all the irreducible characters. $\square$

The work in [16] was intended to prove Conjecture 4 of [12] for solvable groups. With its help, we have proved that Conjecture 2.2 holds for G solvable. Conversely, we will prove now that Conjecture 2.2 implies Conjecture 4 of [12] for G p-solvable.

Theorem 2.4. *Assume Conjecture 2.2(i). Then there exists a function g such that $|G : O_{p',p}(G)|_p \le p^{g(e_p(G))}$ for any p-solvable group G. Also, the logarithm to the base of p of the largest degree of the irreducible characters of a Sylow p-subgroup of G is bounded by a function of $e_p(G)$.*

Proof. Without loss of generality, we may assume that $O_{p'}(G) = 1$ and $G = O^{p'}(G)$. By Hall-Higman (see [4] and [1]) and Corollary 2.7 of [10] (which asserts that the derived length of a Sylow p-subgroup of G is bounded in terms of $e_p(G)$), the p-length of G is bounded in terms of $e_p(G)$.

Let $R/O_p(G) = O_{p'}(G/O_p(G))$ and $N/R = O_p(G/R)$. By Conjecture 2.2(i), we have that $\log_p |N/R|$ is bounded in terms of $e_p(G)$. We can repeat this argument $l_p(G)$ times to deduce the result.

Now, we prove the second statement. We may also assume that $O_{p'}(G) = 1$. Let P be a Sylow p-subgroup of G. Since $O_p(G) \trianglelefteq P$, we have that

$$e_p(P) \le e_p(O_p(G)) \log_p |G : O_p(G)|_p \le e_p(G) + g(e_p(G)),$$

as wanted. $\square$

3 p-parts of character degrees and conjugacy classes

We begin with the proof of Theorem B.

Proof of Theorem B. Consider the ascending (p', p)-series

$$1 = K_0 < N_0 < K_1 < \cdots < G,$$

so that $N_i/K_i = O_{p'}(G/K_i)$ and $K_{i+1}/N_i = O_p(G/N_i)$ for $i \ge 0$. Observe that $F(G) \le N_0$ and $F_2(G) \le K_1$. Continuing this way, one can see that $F_{10}(G) \le K_5$.

By Gaschutz's theorem, $G/F(G)$ acts faithfully and completely reducibly on $\mathrm{Irr}(F(G)/\Phi(G)) = \mathrm{IBr}(F(G)/\Phi(G))$. It follows from Theorem 1.1 that the inertia group of some Brauer character of $F(G)$ is contained in $F_{10}(G) \leq K_5$. By Clifford's correspondence (Theorem 8.9 of [17]), there is some irreducible Brauer character of G that is induced from some Brauer character of $F_{10}(G)$. Now, our hypothesis implies that $|G : K_5|_p \leq p^n$.

Now, it suffices to see that $|K_{i+1} : K_i|_p \leq p^{19n}$ for $i = 0, \ldots, 4$. By hypothesis, the action of K_{i+1}/N_i on N_i/K_i satisfies the hypothesis of Theorem A. Now, the result follows from Theorem A. $\square$

As an immediate consequence of Theorem B, we have the following.

Corollary 3.1. *Let G be a finite solvable group with $O_p(G) = 1$ and write $\overline{e}_p(G) = n$. Then $|G|_p \leq p^{96n}$. Then*

(i) *$e_p(G) \leq 96n$;*

(ii) *The largest degree of the characters of a Sylow p-subgroup of G is less than p^{48n};*

(iii) *The derived length of a Sylow p-subgroup of G is bounded by a logarithmic function of n.*

It was known that the derived length of a Sylow p-subgroup of $G/O_p(G)$ is bounded by a function of the order of $n \log n$ (see [19] or Corollary 14.15 of [11]). Our result here improves considerably the order of magnitude of the bound.

Now, we prove the analogous result replacing the hypothesis on Brauer character degrees by an hypothesis on the sizes of the conjugacy classes of p-regular elements.

Theorem 3.2. *Let G be a solvable group with $O_p(G) = 1$. Let n be an integer such that $|C|_p \leq p^n$ for every conjugacy class C of p-regular elements. Then $|G|_p \leq p^{683n}$. In particular, $e_p(G) \leq 683n$, the largest degree of the characters of a Sylow p-subgroup of G is less than $p^{683n/2}$ and its derived length is bounded by a logarithmic function of n.*

Proof. We will use the notation of the first paragraph of the proof of Theorem B. By the proof of Theorem C$'$ of [16], there exists a p-regular element $x \in F(G)$ such that $C_G(x) \leq F_{10}(G)$, so $|G : F_{10}(G)|_p \leq p^n$. Now, it suffices to see that $|K_5|_p \leq p^{682n}$.

The p-group K_1/N_0 acts faithfully on N_0. By hypothesis, the orbits of this action have size $\leq p^n$, so by [6], for instance, $|K_1/N_0| \leq p^{2n}$. By Hall-Higman's Lemma 1.2.3 K_{i+1}/N_i acts faithfully and coprimely N_i/K_i for all $i > 0$. Similarly, N_{i+1}/K_{i+1} acts faithfully and coprimely on K_{i+1}/N_i. Now, we can apply Theorem 1 of [18] repeatedly to deduce that $|N_1 : K_1| \leq p^{4n}$, $|K_2 : N_1| \leq p^{8n}$ and, more generally, $|K_{i+1} : N_i| \leq p^{2^{2i+1}n}$. We conclude that

$$|K_5|_p = \prod_{i=0}^{4} |K_{i+1} : N_i| \leq \prod_{i=0}^{4} p^{(2^{2i+1})n} = p^{(\sum_{i=0}^{4} 2^{2i+1})n} = p^{682n},$$

as desired. □

4 Ordinary and Brauer character degrees

In this section we prove Theorem C. The ideas of the proof are those involved in the proofs of the other results of this paper, so we will just sketch it.

Proof of Theorem C. We use the notation of the first paragraph of the proof of Theorem B. As in the proof of Theorem B, there exists an irreducible Brauer character that is induced from $F_{10}(G)$, so

$$\tau(|G/K_5|) \leq \tau(|G/F_{10}(G)|) \leq \tau_p(G).$$

By Theorem A, we know that $|K_{i+1} : N_i| \leq p^{19\tau_p(G)}$ for all i. This implies, using Theorem 1 of [18] again, that $|N_i : K_i| \leq p^{38\tau_p(G)}$ for all $i \geq 1$. In particular,

$$\tau(|K_{i+1}/N_i|) \leq 19\tau_p(G)$$

and

$$\tau(|N_i/K_i|) \leq \log_2 p^{38\tau_p(G)} = 38\tau_p(G) \log_2 p.$$

We conclude that

$$\begin{aligned}\tau(G) &\leq \tau(|G/K_5|) + \tau(|K_5/N_0|) + \tau(|N_0|) \\ &\leq \tau_p(G) + 95\tau_p(G) + 152\tau_p(G) \log_2 p + \tau_p(G) \\ &= (152 \log_2 p + 97)\tau_p(G).\end{aligned}$$

□

As in the case of Theorem B, there is an analogous statement replacing the hypothesis on Brauer character degrees by the corresponding hypothesis of class sizes of p-regular elements. Since the proof is routine, we will omit it.

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