Character degree graphs, normal subgroups and blocks

by

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1 Introduction

There are several graphs attached to the set of character degrees of a finite groups that have been studied. Results on these graphs are often useful to prove results that provide structural information of the group from some property of the set of character degrees. The graph that has been most commonly studied is the graph $\Gamma(G)$ whose vertices are the prime divisors of the character degrees of the group G and two vertices are joined by an edge if the product of the primes divides some character degree. This graph was introduced in [7]. However, it is often interesting in character theory to study only certain subsets of the set of character degrees of a group. For instance, the sets of degrees of the members of $\mathrm{Irr}(G|N) = \{\chi \in \mathrm{Irr}(G) \mid N \not\leq \mathrm{Ker}\,\chi\}$ or $\mathrm{Irr}_{\pi}(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi(1) \text{ is a } \pi\text{-number}\}$ have been widely studied. The graphs associated to these sets of degrees have also been studied. See [2, 3, 8].

The goal of this paper is to introduce two new graphs associated to certain subsets of character degrees and to prove that they share some of the properties of the previously studied graphs.

A situation that is often of interest in character theory is the following. We have a normal subgroup N of a finite group G, $\theta \in \mathrm{Irr}(N)$ and we want to study the characters of G lying over θ. As usual, we write $\mathrm{Irr}(G|\theta)$ to denote this set of characters. Our first graph considers these characters. We define the graph $\Gamma(G|\theta)$ as follows. The vertices of this graph are the prime divisors of the numbers $\chi(1)/\theta(1)$, where $\chi \in \mathrm{Irr}(G|\theta)$. We join two vertices by an edge if the product of the two different primes is divisible by some member of $\{\chi(1)/\theta(1) \mid \chi \in \mathrm{Irr}(G|\theta)\}$.

Our main result shows that the number of connected components of this graph behaves as in the previously studied situations.

Theorem A. *Let N be a normal subgroup of a finite group G and $\theta \in \mathrm{Irr}(N)$. Then $\Gamma(G|\theta)$ has at most three connected components. Furthermore, if G/N is solvable then $\Gamma(G|\theta)$ has at most two connected components.*

In particular, if $N = 1$ and θ is the trivial character, we have that $\Gamma(G)$ has at most three connected components and that if G is solvable then it has at most two connected components. In this way, we recover the main result of [7] and Proposition 2 of [6].

As a consequence of the proof of Theorem A, we obtain the following.

Corollary B. *Let N be a normal subgroup of a finite group G and $\theta \in \mathrm{Irr}(N)$. If G/N is solvable then the diameter does not exceed* 4.

It has been proved that in the case when $N = 1$ and $\theta = 1$ the diameter of this graph is at most three when G is solvable (see [10]) and at most 4 for arbitrary finite groups (see [5]). It seems likely that these two results can be extended to the graph $\Gamma(G|\theta)$, but we have been unable to do it.

Another subset of the set of character degrees of a group that is often important is the set of degrees of the characters in a Brauer p-block B. We define the graph $\Gamma(B)$ whose vertices are the set of prime divisors of the degrees of the members of $\mathrm{Irr}(B)$ and we join two vertices by an edge if the product of the two different primes divides the degree of some character in $\mathrm{Irr}(B)$. An easy consequence of Theorem A and Corollary B is the following.

Corollary C. *Let B be a p-block of a p-solvable group G. Then $\Gamma(B)$ has at most three connected components. Furthermore, if G is solvable, then it has at most two connected components and the diameter of $\Gamma(B)$ does not exceed 4.*

We have been unable to obtain any bound for the number of connected components of $\Gamma(B)$ when G is an arbitrary finite group. It would be interesting to do this.

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2 Proofs

We begin work toward a proof of Theorem A. We need the following elementary lemma. For the reader's convenience, we include a proof.

Lemma 2.1. *Let G be a perfect group and N a normal subgroup of G. If $\lambda \in \mathrm{Irr}(N)$ is a G-invariant linear character that has order n, then n divides $\chi(1)$ for every $\chi \in \mathrm{Irr}(G|\lambda)$.*

Proof. We have that $\chi_N = \chi(1)\lambda$. Furthermore, since G is perfect, we have that $\det(\chi) = 1_G$ so

$$1_N = \det(\chi)_N = \det(\chi_N) = \det(\chi(1)\lambda) = \lambda^{\chi(1)}$$

and it follows that $\chi(1)$ is a multiple of n. □

Proof of Theorem A. It follows from Clifford theory that we may assume that θ is G-invariant. (Otherwise, we would have that $\Gamma(G|\theta)$ is connected.) Now, by the theory of isomorphic character triples (see Chapter 11 of [1]), we may assume that N is a cyclic central subgroup of G and that θ is faithful.

Pick a chief series $N = N_0 < N_1 < \cdots < N_j = G$ in the following way. Put $N_0 = N$. If G/N_0 has an elementary abelian minimal normal subgroup, then N_1/N_0 is abelian. Otherwise, let N_1/N_0 be any minimal normal subgroup of G/N_0. Proceed in this way for every i. Let R be the member of that series maximal such that θ extends to R and let T be the subgroup of that series such that T/R is a minimal normal subgroup of G/R. Since θ extends to R it follows from Gallagher's theorem (Corollary 6.17 of [1]) that $\Gamma(R|\theta) = \Gamma(R/N)$. Therefore this graph has at most three connected components (by the main result of [7]).

Now, assume that $\Gamma(R|\theta)$ has three connected components. By Theorem 4.1 of [4] we have that $R/N \cong \mathrm{PSL}_2(2^n) \times A$, for some integer $n \geq 2$ and some abelian subgroup A. Let S_1/N be the normal subgroup of R/N such that $S_1/N \cong \mathrm{PSL}_2(2^n)$. We know that θ extends to S_1, so all the characters of N extend to S_1 (recall that N is cyclic) and it follows that S_1 has a normal subgroup S isomorphic to $\mathrm{PSL}_2(2^n)$. Notice that S_1/N is the derived subgroup of R/N and that $S = S_1'$. Observe that S is also normal in G. Put $B = F(R)$, which is a normal abelian subgroup of G. (The group B is abelian because B/N is abelian and θ is a faithful irreducible character of the cyclic group N that extends irreducibly to B.) Notice that $R = S \times B$. Let $\hat{\theta}$ be an extension of θ to B.

We have that $C_T(S) \times S$ is a normal subgroup of G that contains R. If $C_T(S) \times S = R$, then T/R is a subgroup of $\mathrm{Out}(S)$, and it follows that T/R is a cyclic group of prime order p. We know that $1_S \times \hat{\theta}$ does not extend to T (because $\hat{\theta}$ is not T-invariant) and the same happens for $\varphi \times \hat{\theta}$ for any $\varphi \in \mathrm{Irr}(S)$. We conclude that p divides the degree of any irreducible character of T lying over θ, so $\Gamma(G|\theta)$ is connected.

Thus, we may assume that $C_T(S) \times S = T$. Then $C_T(S)/B$ is an abelian chief factor of G. This contradicts our choice of the chief series.

Hence, we may assume that $\Gamma(R/N)$ has at most two connected components. It follows that the characters of $\mathrm{Irr}(G|\theta)$ lying over non-linear characters of $\mathrm{Irr}(R|\theta)$ belong to at most two different connected components of the graph $\Gamma(G|\theta)$. Now, it suffices to see that the characters of $\Gamma(G|\theta)$ lying over linear characters of R belong to the same connected component. In order to see this, we may assume that R is abelian.

If T/R is an elementary abelian p-group, then we would have that p divides the degree of any such character and the result would follow. Thus, we may assume that T/R is a direct product of isomorphic non-abelian simple groups. Then T' is perfect. Assume that $T' \cap N > 1$ has order m. By Lemma 2.1 we have that m divides the degree of any character of T' lying over $\theta_{T' \cap N}$ and hence the same happens with the degrees of the characters of G lying over θ. Thus, we may assume that $T' \cap N = 1$. We conclude that

$T' \times N$ is a normal subgroup of T and that θ has an extension to $T'N$ whose kernel contains T'. Since T/T' is abelian, we conclude that θ also extends to T, and this is a contradiction. This completes the proof of the first part of the theorem.

Now, we assume that G/N is solvable and we want to prove that our graph has at most two connected components.

If G/N is abelian, then G is nilpotent and $\Gamma(G|\theta)$ is connected. Hence, we may assume that G/N is not abelian. Let M/N be a maximal normal subgroup of G/N such that G/M is not abelian. By Lemma 12.3 of [1], we have that either G/M is a non-abelian q-group for some prime q or a Frobenius group whose kernel is an elementary abelian r-group K/M for some prime r and G/K is cyclic.

Assume first that G/M is a q-group. Let $\rho(G|\theta)$ be those primes p that divide the degree $\chi(1)$ of some irreducible character $\chi \in \mathrm{Irr}(G|\theta)$. It is clear that

$$\rho(M|\theta) \subseteq \rho(G|\theta) \subseteq \rho(M|\theta) \cup \{q\}.$$

It follows from Gallagher's theorem and Corollary 11.29 of [1] that for any $\tau \in \mathrm{Irr}(M)$ there exists $\chi \in \mathrm{Irr}(G|\tau)$ whose degree is divisible by q. Therefore, $\Gamma(G|\theta)$ is connected.

Now, we may assume that G/M is a Frobenius group whose kernel is an elementary abelian r-group K/M for some prime r and G/K is cyclic. We have that $K/M = F(G/M)$. We claim that for every $v \in \Gamma(G|\theta)$, $d(v,r) \leq 1$ or $d(v,q) \leq 1$ for all $q \in \pi(G/K)$. (Here d is the distance function. The distance between two vertices of the graph is the length of the shortest path that joins them.)

Let $\chi \in \mathrm{Irr}(G|\theta)$ such that v divides $\chi(1)$. Assume that the claim does not hold. Let π be the set of prime divisors of $|G/K|$ that are coprime to $\chi(1)$. Let H/K be a Hall π-subgroup of G/K and let $\psi \in \mathrm{Irr}(H)$ lying under χ. Since $(\psi(1), |H/M|) = 1$, we have that $\psi_M \in \mathrm{Irr}(M)$. Let $\alpha \in \mathrm{Irr}(H/M)$ with $\alpha(1) = |H : K|$. By Gallagher's theorem $\psi\alpha \in \mathrm{Irr}(H)$ and if we choose any $\varphi \in \mathrm{Irr}(G|\alpha\psi)$ we can see that the prime divisors of $|H/K|$ and v are joined in the graph. This proves the claim.

Hence, $\Gamma(G|\theta)$ has at most two connected components. $\square$

As a consequence of this proof, we have Corollary B.

Proof of Corollary B. Using the notation of the proof of Theorem A, we may assume that G/M is a Frobenius group. (Otherwise, either we have a complete graph or the diameter is at most 2.) We have seen that for every

$v \in \Gamma(G|\theta)$, $d(v,r) \leq 1$ or $d(v,q) \leq 1$ for all $q \in \pi(G/K)$. If $\Gamma(G|\theta)$ has two connected components, we have that $\mathrm{diam}(\Gamma(G|\theta) \leq 2$.

Suppose that $\Gamma(G|\theta)$ is connected. Let $q \in \pi(G/K)$. We have that $d(q,r) \leq 2$. (Otherwise, there is a vertex s in a shortest path between q and r that is not joined to q and is not joined to r, so $d(s,q) > 1$ and $d(s,r) > 1$ and this is a contradiction.) Now, it is clear that the diameter of the graph does not exceed 4. □

Proof of Corollary C. Let $b = \{\theta\}$ be a block of $O_{p'}(G)$ covered by B and let T be the inertia group of θ in G. By the Fong-Reynolds theorem (Theorem 9.14 of [9]), $\mathrm{Irr}(B) = \{\psi^G \mid \psi \in \mathrm{Irr}(B_1)\}$ for some $B_1 \in \mathrm{Bl}(T|b)$ and by Fong's theorem (Theorem 10.20 of [9]) $\mathrm{Irr}(B_1) = \mathrm{Irr}(T|\theta)$. Now, the result follows from Theorem A and Corollary B. □

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