Finite groups whose conjugacy class graphs have few vertices

by

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Dedicated to Prof. Otto Kegel on his 70th birthday

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1 Introduction

For a finite group G, the following conjugacy class graph $\Gamma(G)$ is defined in [3]: its vertex set is the set of non-central classes of G, and two distinct vertices C and D are connected by an edge if and only if their class lengths have a non-trivial common divisor, i.e, $(|C|, |D|) > 1$. This graph has been widely studied. See, for instance, [8] for a survey on this graph and the analogue graph for character degrees. Some authors have studied the structure of a finite group G when its class graph have *"small number"* of edges. For example, finite groups whose class graphs have no subgraph K_3 have been classified in [4] (Recall that the graph K_n is the complete graph with n vertices). Now a general question is posed:

Can one say something about the structure of a finite group G if its class graph has no subgraph K_n? especially for $n = 4$ or 5?

For the case when G is a non-abelian simple group, we know that $\Gamma(G)$ is complete (note that this result depends on the classification theorem of finite simple groups) by [5]. Therefore, the above question is somewhat trivial for simple groups. In particular, $\Gamma(G)$ must contain a subgraph K_4 for any non-abelian simple group G.

Observe that if $\Gamma(G)$ has no subgraph K_n then for any n distinct vertices, their class lengths are setwise relatively prime. Consequently the above question can be modified as follows. We say that a group G satisfies property P_n if for every prime integer p, G has at most $n-1$ conjugacy classes whose size is a multiple of p. It is clear that if $\Gamma(G)$ does not have a subgraph K_n, then G satisfies property P_n.

The main goal of this note is to classify the finite groups that satisfy property P_5. From this classification, it is easy to classify the groups G whose class graph does not contain a subgraph K_4 or K_5, extending the result of Fang and Zhang.

Theorem A. *Let G be a finite group that satisfies P_5. Then G is isomorphic to one of the following groups:*

(i) *an abelian group;*

(ii) D_8, Q_8*;*

(iii) *The Frobenius group $Q_8(C_3 \times C_3)$;*

(iv) *The Frobenius group with complement of order 5 and kernel C_{11} or $(C_2)^4$;*

(v) *The Frobenius group with complement of order 4 and kernel $C_5, (C_3)^2, C_{13}$ or C_{17};*

(vi) *The Frobenius group with complement of order 3 and kernel* $(C_2)^2, C_7$ *or* C_{13}*;*

(vii) *The direct product of* C_2 *and* A_4 *or the non-abelian group of order 21;*

(viii) *The Frobenius group with complement of order 2 and kernel* $(C_3)^2$*;*

(ix) *The dihedral group* D_{14}*;*

(x) *The dihedral group* D_{10}*,* $D_{10} \times C_2$ *or* $\langle x, y \mid y^5 = x^4 = 1, y^x = y^{-1} \rangle$*;*

(xi) *The symmetric group* S_3*,* $S_3 \times C_2$*,* $\langle x, y \mid y^3 = x^4 = 1, y^x = y^{-1} \rangle$*,* $S_3 \times C_3$*,* $S_3 \times C_4$*,* $S_3 \times C_2 \times C_2$*,* $\langle x, y \mid y^3 = x^4 = 1, y^x = y^{-1} \rangle \times C_2$*, or* $\langle x, y \mid y^3 = x^8 = 1, y^x = y^{-1} \rangle$*;*

(xii) *The dihedral group* D_{18}*;*

(xiii) S_4*; or*

(xiv) A_5 *or* $L_2(7)$*.*

Conversely, all these groups satisfy property P_5*.*

The proof of this theorem makes heavy use of the classification of finite groups with few conjugacy classes given in the papers [11]. We suggest the reader to have a copy of [11] at hand. It would be possible to give an independent proof, but this would make the paper considerably longer. Using the methods of this paper, we think that it would be possible to extend Theorem A further and classify the groups that satisfy, for instance, property P_6.

In Section 2, we will present some preliminary results that we will need to prove Theorem A. We complete its proof in Section 3 and we will conclude with some remarks in Section 4.

2 Preliminary results

We will use repeatedly, without further explicit mention, the following easy fact.

Lemma 1. *Let* N *be a normal subgroup of a finite group* G *and let* p *be a prime. If* p *divides* $|\operatorname{cl}_{G/N}(xN)|$ *then* p *divides* $|\operatorname{cl}_G(x)|$*. In particular, if* $x_1N, \ldots, x_nN$ *belong to different classes of* G/N *all of whose sizes are multiples of* p*, then* $x_1, \ldots, x_n$ *belong to different conjugacy classes of* G *all of whose sizes are multiples of* p*. It follows that if* G *satisfies* P_n*, then* G/N *also satisfies* P_n*.*

Proof. Since p divides $|\operatorname{cl}_{G/N}(xN)|$, we have that $C_{G/N}(xN)$ does not contain a Sylow p-subgroup of G/N. It follows that $C_G(x)$ does not contain a Sylow p-subgroup of G, and therefore p divides $|\operatorname{cl}_G(x)|$. $\square$

We will also use the following result, whose proof requires character theory.

Lemma 2. *Let $S \leq G \leq \operatorname{Aut}(S)$, where S is a non-abelian simple group. For every $g \in G - \{1\}$, $|S : C_S(g)|$ is not a prime power.*

Proof. This follows from [7]. $\square$

Next, we classify the almost simple groups that satisfy property P_5. In its proof we make use of the classification of finite simple groups in a week way: we use the Conjecture in [1], which was proved using the classification. In fact, we will only use the following weaker result: if p is an odd prime divisor of $|G|$ and G has a nilpotent Hall $\{2,p\}$-subgroup, then G is not simple. This is the only place where we make use of the classification of simple groups. Given a group G, we write $k(G)$ to denote the number of conjugacy classes of G. Also, given a subset A of G, $k_G(A)$ stands for the number of conjugacy classes of G that intersect A nontrivially.

Lemma 3. *Let $S \leq G \leq \operatorname{Aut}(S)$ where S is simple non-abelian. If G satisfies P_5, then G is isomorphic to A_5 or $L_2(7)$.*

Proof. If $k(G) \leq 12$, it follows from [11] that G is A_5 or $L_2(7)$. Now, we have to show that there is not any almost simple group G satisfying P_5 with $k(G) > 12$. By Lemma 2, we know that the sizes of the G-classes contained in $S - \{1\}$ are not prime powers. Assume that the number of different prime divisors of S is not bigger than 5. Then

$$2(k(G) - 1)/|\pi(|S|)| \geq 2 \cdot 12/5 > 4$$

and by the pigeonhole principle, we conclude that G does not satisfy P_5. Thus, we may assume that $|\pi(|S|)| > 5$. Let $p_1, \ldots, p_5$ be 5 different odd prime divisors of $|S|$. Let $P \in \operatorname{Syl}_2(S)$. By [1], we have that for $i = 1, \ldots, 5$, $C_S(P)$ does not contain a Sylow p_i-subgroup of S. Therefore, for every $i = 1, \ldots, 5$ there exists a p_i-element that is not centralized by a Sylow 2-subgroup of G. We conclude that G has at least 5 conjugacy classes of even size. This is the final contradiction. $\square$

3 Proof of Theorem A

Now, we are ready to complete the proof of Theorem A. It is easy to check that the groups listed in Theorem A satisfy property P_5.

Now, we split the classification into two parts. In Theorem 4 we classify the non-solvable groups that satisfy P_5 and in Theorem 5 we classify the solvable ones.

Theorem 4. *Let G be a non-solvable group that satisfies property P_5. Then $G \cong A_5$ or $L_2(7)$.*

Proof. Let $N \trianglelefteq G$ be maximal such that G/N is not solvable. Then G/N possesses a unique minimal normal subgroup M/N. Also, M/N is not solvable and G/M is solvable. Since G has property P_5, it is easy to see that M/N is simple. Therefore, G/N is almost simple and it follows from Lemma 3 that $G/N \cong A_5$ or $L_2(7)$.

Now, we want to prove that $N = 1$. We may assume that N is a minimal normal subgroup of G. Since G satisfies P_5, we can see that N is an elementary abelian group and G/N is isomorphic to a subgroup of $\mathrm{Aut}(N)$ (one can easily check that a non-trivial central extension of A_5 or $L_2(7)$ has at least 5 conjugacy classes of even size). This implies that the size of all the G-classes of elements in $G - N$ is a multiple of the prime divisor of $|N|$, so $k_G(G-N) \leq 4$. Since $|G|$ is divisible by at most 4 primes, we conclude that $k(G) \leq 17$ and the result follows from [11]. □

Theorem 5. *Let G be a solvable group that satisfies property P_5. Then G is one of the groups listed in (i)-(xiii) in Theorem A.*

Proof. We may assume that G is a non-abelian solvable group. Let $N \trianglelefteq G$ maximal such that G/N is not abelian. Assume that there exists such N with G/N a p-group, for some prime p. Write $Z/N = Z(G/N)$. Then G/Z is abelian and $|G : Z| = p^{2a}$ for some integer a (by Lemma 12.3 of [6], for instance). Since p divides the size of any class of elements in $G-Z$, we have that $|G : Z| = 4$ and $k_G(G - Z) \leq 4$. On the other hand, $|C|$ divides $|G|/4$ for every G-class C of elements of $G - Z$. This implies that $k_G(G - Z) \geq 3$ and if $k_G(G - Z) = 3$ then all these 3 classes have size $|G|/4$. First, assume that $k_G(G - Z) = 3$. Then the Sylow 2-subgroup of G is isomorphic to D_8 or Q_8 (otherwise, the centralizer of any 2-element in $G - Z$ would have order bigger than 4). In particular, $G = PN$ where P is isomorphic to D_8 or Q_8 and N has odd order. It is easy to see that if G is nilpotent then G is isomorphic to D_8 or Q_8. Now, we want to prove that if G is not nilpotent then G is the Frobenius group $Q_8(C_3 \times C_3)$.

Let N/K be a chief factor of G. Since G/K satisfies P_5, and $C_{G/K}(xK) \leq Z/K$ for all $x \in N$, we have that the non-trivial elements of N/K constitute a unique conjugacy class of G/K. In particular, if $y \in Z - N$, we deduce that y does not commute with any non-identity element of N/K, i.e., the action of G/N on N/K is Frobenius. We deduce that $G/K \cong Q_8(C_3 \times C_3)$, so all we have to show is that $K = 1$. This follows from the fact that G/K has 4 conjugacy classes of elements whose size is a multiple of 2. This implies that all class sizes of elements in K are odd and if $K > 1$ then this would contradict the fact that the elements of $P - Z$ do not commute with non-trivial elements of N.

Now, we may assume that there are 4 G-classes in $G - Z$, two of them of size $|G|/4$ and two of them of size $|G|/8$. We deduce that $Z(G) = 1$ and all the G-classes contained in $Z - \{1\}$ have size a power of 2. But if $P \in \mathrm{Syl}_2(G)$, we have that $Z(P) \cap Z > 1$ and this is a contradiction.

Hence, by Lemma 12.3 of [6], we may assume that for any normal subgroup N of G maximal such that G/N is not abelian, we have that G/N is a Frobenius group with kernel K/N and cyclic complement isomorphic to G/K. Since for any coset xK, we have a conjugacy class of G whose size is divisible by the prime divisors of K/N, we deduce that $|G/K| \leq 5$.

Assume first that $|G/K| = 5$. Then $k_G(G-K) = 4$ so all the classes of G contained in $G - K$ have size $|G|/5$. In particular, we deduce that $|G|_5 = 5$ and $G = PK$, where P is a Sylow 5-subgroup of G. Also, the action of P on K is Frobenius. Since there are 4 classes in $G - K$ of size $|K|$, we deduce that K is abelian. Also, $k_G(K - \{1\}) \leq 4$. We deduce that G is one of the groups in (iv).

Now, assume that $|G/K| = 4$. As before, either there are 3 classes in $G - K$ all of them of size $|G|/4$ or there are 4 G-classes, two of them of size $|G|/4$ and two of them of size $|G|/8$. In the second case, we obtain a contradiction as in the third paragraph of the proof. Hence, we may assume that there are 3 G-classes of size $|G|/4$ in $G - K$. It follows that $G = PK$, where $P \in \mathrm{Syl}_2(G)$ has order 4 and K has odd order. Also, the action of P on K is Frobenius. Since P has even order, K is abelian and thus $k_G(K - \{1\}) \leq 4$. It follows that G is one of the groups in (v).

Now, assume that $|G/K| = 3$. Since the size of the classes in $G - K$ is a multiple of the primes that divide $|K/N|$, we deduce that $k_G(G - K) \leq 4$. Assume that $|G|_3 > 3$. Then there exists a 3-element $x \in G - K$ such that $|C_G(x)| \geq 9$. The same happens with x^2, which belongs to a different G-class. If $y \in G - K$ is not a 3-element, then its order is at least 6, and the same happens with the order of its centralizer. Since

$$1/9 + 1/9 + 1/6 + 1/6 = 5/9 < 2/3,$$

we conclude that $k_G(G-K) > 4$. Therefore, $|G|_3 = 3$ and $G = PK$, where $P \in \mathrm{Syl}_3(G)$. Using a similar argument it follows from the fact that $k_G(G-K) \leq 4$ that the action of P on K is Frobenius or that $|C_G(x)| = 6$ for $x \in P - \{1\}$. In the first case it is not difficult to see that then G is one of the groups in (vi).

In the second case, we know that there are 4 G-classes in $G-K$ of size $|G|/6$. Then K has a central Hall $2'$-subgroup. Also, it is easy to see that K has an abelian Sylow 2-subgroup, so we deduce that K is abelian. By Fitting's lemma $K = [K,P] \times C_K(P)$, and we deduce that $C_K(P) = Z(G)$ has order 2. The action of P on $[K,P]$ is Frobenius so by the previous paragraph we conclude that G is one of the groups in (vii).

Finally, we may assume that $|G/K| = 2$. It follows that $|K/N| = 3, 5, 7$ or 9. It is not possible to have $|K/N| = 9$ because a Frobenius group of order 18 is supersolvable and this contradicts the maximality of N.

Assume now that $|K/N| = 7$. Again, we want to see that $N = 1$. We may assume that N is a minimal normal subgroup of G. Since G satisfies P_5, it is easy to see that $C_G(N) = N$. Also, $Z(G) = 1$. Since $|G|$ has at most 3 different prime divisors, we conclude that $k(G) \leq 13$. Furthermore, it is easy to see that $k_G(N) > 2$. Now it follows from [11] that G is the group in (ix).

Now, assume that $|K/N| = 5$. Let N/L be a chief factor of G. Assume that $|N/L| > 2$. Then $C_{G/L}(N/L) = N/L$, so the action of K/N on N/L is Frobenius. As before, it follows from [11] that there does not exist such a group satisfying P_5. This implies that we may assume that $|N/L| = 2$, so $G/L \cong D_{10} \times C_2$ or $\langle x, y \mid y^5 = x^4 = 1, y^x = y^{-1} \rangle$. Arguing in a similar way, again, it follows from [11] that $L = 1$. We conclude in this case that G is one of the groups in (x).

Now, assume that $|K/N| = 3$, so that $G/N \cong S_3$. Let N/L be a chief factor of G. If N/L is central in G/L, then $|N/L| \leq 4$. If $|N/L| = 3$ or 4, then it follows from [11] that $L = 1$. Similarly, if $|N/L| = 2$ it follows from [11] that $|L| \leq 2$. We can conclude in in this case that G is one of the groups in (xi).

Therefore, we may assume that N/L is not central in G/L. If $C_{G/L}(N/L) = K/L$, then G/L is a Frobenius group of order 18. It follows from [11] that $L = 1$ and G is the group in (xii) or (viii).

Finally, we may assume that $C_{G/L}(N/L) = N/L$. Assume that $G/L \ncong S_4$. Since $k(G/L) \leq 13$ and $k_{G/L}(N/L) > 2$, again it follows from [11] that this is not possible. Therefore, $G/L \cong S_4$. Now it is easy to deduce, using [11], for instance, that $L = 1$ and G is the group in (xiii). $\square$

4 Further remarks

In this paper, we have considered groups that satisfy property P_n. We may also say that a group G satisfies property P'_n if any prime p divides at most $n-1$ conjugacy class sizes. It would also be interesting to consider the groups that satisfy P'_n. For instance, is it possible to find a bound that depends only on n for the number of conjugacy class sizes of the groups that satisfy property P'_n? The analogue question for character degrees of solvable groups was considered in [2], where a quadratic bound was obtained. It was conjectured in [9] that if any prime integer p divides at most m character degrees of a solvable group G, then G has at most $3m$ character degrees.

We have not been able to obtain a linear bound for the number of non-central conjugacy classes of a group that satisfies property P_n, but we are quite close. We need a lemma.

Lemma 6. *Suppose that G is a finite group with property P_n. If there exists an element $x \in G - Z(G)$ whose order is a product of m distinct primes, then $2^{m-1} + 1 \leq n$.*

Proof. Let $x_1 \in \langle x \rangle - Z(G)$ be of order a prime q_1. Then there are 2^{m-1} elements $x_1 y_j$ of distinct orders where $y_j \in \langle x^{q_1} \rangle$. For those 2^{m-1} distinct classes $\mathrm{cl}_G(x_1 y_j)$, their class lengths have the common divisor $|x_1^G| > 1$. This implies that $n \geq 1 + 2^{m-1}$. □

Theorem 7. *There exists a constant C such that any finite group G that satisfies property P_n has order divisible by at most $C(\log n)^4 \log\log n$ different primes. In particular, the number of non-central conjugacy classes of G is at most $Cn(\log n)^4 \log\log n$.*

Proof. By Lemma 6, we have that the number of different prime divisors of the order of an element of G is bounded by a logarithmic function of n. The result follows from [10], where it is proved that the number of prime divisors of the order of a group is bounded by $C_1 m^4 \log m$, where m is the maximum number of different prime divisors of the element orders and C_1 is an absolute constant. □

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