Nonsolvable groups with few character degrees

by

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1 Introduction

Given a group G, let $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ be the set of degrees of the ordinary complex irreducible characters of G. There are several papers devoted to studying groups with few character degrees. For instance, groups with 2 character degrees were studied in [11]. One year later, M. Isaacs proved that groups with 3 character degrees are solvable of derived length ≤ 3 (see [8] or Theorem 12.15 of [9]). Several decades later, the structure of (solvable) groups with 3 character degrees was analyzed in detail in [22] (see also [18]). As the alternating group $\mathfrak{A}_5$ shows, it is not possible to prove that G is solvable when $|\,\mathrm{cd}(G)| = 4$. Solvable groups with 4 character degrees have also been considered. It was proved in Garrison's Ph. D. thesis [6] that they have derived length ≤ 4. (A new proof of this theorem was provided in [10]). In view of the complexity of these results and of the analysis of the structure of groups with 3 character degrees, it seems clear that it is hopeless to try to classify the solvable groups with 4 character degrees. However, solvable groups with 5 character degrees have also been studied (see [16, 17, 25]). On the other hand, there appears to be no result on the structure of nonsolvable groups with few character degrees. The aim of this note is to provide some such results.

Problem 45 of [1], which is attributed to Isaacs, asks for a classification of nonsolvable groups with 4 character degrees. The following result solves this problem. Recall that M_{10} is the stabilizer of a point in M_{11} in its natural permutation representation.

Theorem A. *Let G be a nonsolvable group with $|\,\mathrm{cd}(G)| = 4$. Then one of the following holds*

(i) *$G \cong \mathrm{L}_2(2^n) \times A$ for some $n \geq 2$ and some abelian A;*

(ii) *G has a normal subgroup U such that $U \cong \mathrm{L}_2(q)$ or $\mathrm{SL}_2(q)$ for some odd $q \geq 5$ and if $C = C_G(U)$ then $C \leq Z(G)$ and $G/C \cong \mathrm{PGL}_2(q)$; or*

(iii) *the group G has a normal subgroup of index 2 that is a direct product of $\mathrm{L}_2(9)$ and a central subgroup C. Furthermore, $G/C \cong M_{10}$.*

Conversely, if (i), (ii) or (iii) holds, then $|\,\mathrm{cd}(G)| = 4$.

In Theorem 1.2 of [24] it was proved that if $\mathrm{cd}(G) = \{1, n, m, nm\}$ for nonnegative integers n and m, then G is solvable. As a by-product of Theorem A, we have the following complementary result.

Corollary B. *Let G be a nonsolvable group with $|\,\mathrm{cd}(G)| = 4$. Then $\mathrm{cd}(G) = \{1, q-1, q, q+1\}$ for some prime power $q > 3$ or $\mathrm{cd}(G) = \{1, 9, 10, 16\}$.*

In a series of papers, N. Ito [12, 13, 14] classified the finite simple groups with 4 and 5 conjugacy class sizes and he obtained some partial results on simple groups with 6 conjugacy class sizes. Using the classification of finite simple groups and Deligne-Lusztig theory, we can describe the simple groups with few character degrees.

Theorem C. *Let G be a non-abelian finite simple group. Then $|\operatorname{cd}(G)| \geq 8$, or one of:*

(a) $|\operatorname{cd}(G)| = 4$ *and* $G = \mathrm{L}_2(2^f)$, $f \geq 2$, *or*

(b) $|\operatorname{cd}(G)| = 5$ *and* $G = \mathrm{L}_2(p^f)$, $p \neq 2$, $p^f > 5$, *or*

(c) $|\operatorname{cd}(G)| = 6$ *and* $G = {}^2B_2(2^{2f+1})$, $f \geq 1$, *or* $G = \mathrm{L}_3(4)$, *or*

(d) $|\operatorname{cd}(G)| = 7$ *and* $G = \mathrm{L}_3(3), \mathfrak{A}_7, M_{11}$ *or* J_1.

2 Simple groups with few character degrees

We start with the proof of Theorem C.

Proof of Theorem C. We use the classification of finite simple groups. For the sporadic groups and the Tits group ${}^2F_4(2)'$, the statement follows directly from the character tables printed in the Atlas [2]: only M_{11} and J_1 have less than 8 different character degrees. For the alternating groups we argue as follows: the irreducible characters of $\mathfrak{S}_n$ corresponding to hook-partitions are the exterior powers of the reflection representation of degree $n-1$ (see [7, Prop. 5.4.12]). Thus for $n \geq 16$ we obtain at least 8 different degrees for $\mathfrak{S}_n$. Since the corresponding hooks are not self-dual, each such representation restricts irreducibly to the alternating group $\mathfrak{A}_n$, and we find the same number of different degrees for $\mathfrak{A}_n$. For smaller n it is easy to check the assertion from the well-known hook-formula for character degrees of the symmetric group.

For the groups of Lie type we make use of Lusztig's classification of character degrees and in particular of the formulae for degrees of unipotent characters. For G of exceptional type G_2, ${}^3D_4$, ${}^2F_4$, F_4, E_6, ${}^2E_6$, E_7 or E_8 the tables in [3, 13.9] show that G has at least eight unipotent characters of different degrees. For the Suzuki groups ${}^2B_2(2^{2f+1})$, the character degrees were worked out by Suzuki [29], and it turns out that each such group has exactly six different degrees if $f \geq 1$. The case $f = 0$ leads to the solvable Frobenius group of order 20. The character tables of the Ree groups ${}^2G_2(3^{2f+1})$ were worked out by Ward [30]; here all groups have at least eight different degrees if $f \geq 1$. The case $f = 0$ belongs to the non-simple group $\mathrm{Aut}(\mathrm{L}_2(8))$.

It remains to treat the families of classical groups of Lie type. Here the unipotent characters are parametrized by certain combinatorial objects, either partitions (for types A_n and ${}^2A_n$) or so-called symbols, see for example [3, 13.8]. Let first G be of type B_n or C_n, $n \geq 2$, that is, G is a projective symplectic group $\mathrm{S}_{2n}(q)$ or an odd-dimensional orthogonal group $\mathrm{O}_{2n+1}(q)$. By [3, 13.8] G has unipotent characters parametrized by the symbols

$$\begin{pmatrix} 1 & n \\ 0 & \end{pmatrix}, \begin{pmatrix} 0 & n \\ 1 & \end{pmatrix}, \begin{pmatrix} 0 & 1 & n \\ & & \end{pmatrix},$$

of degrees

$$\frac{q}{2}\frac{(q^n+1)(q^{n-1}-1)}{q-1}, \frac{q}{2}\frac{(q^n-1)(q^{n-1}+1)}{q-1}, \frac{q}{2}\frac{(q^n-1)(q^{n-1}-1)}{q+1}.$$

Together with the trivial character and the Steinberg character of degree q^{n^2} this already accounts for five different character degrees. If $n > 2$ then the Alvis-Curtis duals (see [3, 8.2]) of the first three unipotent characters yield another three new character degrees (they differ from the first ones by a non-trivial power of q). Now let $n = 2$. Here the irreducible characters were determined by Srinivasan [27] for odd q, and by Enomoto [5] for even q. The tables show that there always exist at least 8 different degrees (exactly five of which belong to unipotent characters) if $q \neq 2$. The case $q = 2$ leads to $\mathrm{S}_4(2) \cong \mathfrak{S}_6$.

For G an even-dimensional orthogonal group $\mathrm{O}_{2n}^+(q)$, of type D_n, $n \geq 4$, we consider the unipotent characters parametrized by the symbols

$$\begin{pmatrix} 0 & n-1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & n-1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & n-1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & n-1 \\ & - & & \end{pmatrix},$$

of degrees

$$\frac{q^3}{2}\frac{(q^n-1)(q^{n-1}-1)(q^{n-2}+1)(q^{n-3}+1)}{(q^2-1)^2},$$
$$\frac{q^3}{2}\frac{(q^n-1)(q^{n-1}+1)(q^{n-2}-1)(q^{n-3}+1)}{(q^2+1)(q-1)^2},$$
$$\frac{q^3}{2}\frac{(q^n-1)(q^{n-1}+1)(q^{n-2}+1)(q^{n-3}-1)}{(q^2-1)^2},$$
$$\frac{q^3}{2}\frac{(q^n-1)(q^{n-1}-1)(q^{n-2}-1)(q^{n-3}-1)}{(q^2+1)(q+1)^2}.$$

Together with the trivial and the Steinberg character, this yields six distinct character degrees. For $n > 4$, the Alvis-Curtis duals of the above four characters add another four degrees, while for $n = 4$ the character labelled by

$$\binom{3}{1}$$

of degree $q(q^2+1)^2$ and its Alvis-Curtis dual of degree $q^7(q^2+1)^2$ provide two further unipotent character degrees.

For G a non-split even-dimensional orthogonal group $\mathrm{O}_{2n}^-(q)$, of type ${}^2D_n$, $n \geq 5$, we consider the unipotent characters parametrized by

$$\begin{pmatrix} 1 & 2 & n-1 \\ & 0 & \end{pmatrix}, \begin{pmatrix} 0 & 2 & n-1 \\ & 1 & \end{pmatrix}, \begin{pmatrix} 0 & 1 & n-1 \\ & 2 & \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ & n-1 & \end{pmatrix},$$

of degrees

$$\begin{aligned}
&\frac{q^3}{2}\frac{(q^n+1)(q^{n-1}+1)(q^{n-2}-1)(q^{n-3}-1)}{(q^2-1)^2}, \\
&\frac{q^3}{2}\frac{(q^n+1)(q^{n-1}-1)(q^{n-2}+1)(q^{n-3}-1)}{(q^2+1)(q-1)^2}, \\
&\frac{q^3}{2}\frac{(q^n+1)(q^{n-1}-1)(q^{n-2}-1)(q^{n-3}+1)}{(q^2-1)^2}, \\
&\frac{q^3}{2}\frac{(q^n+1)(q^{n-1}+1)(q^{n-2}+1)(q^{n-3}+1)}{(q^2+1)(q+1)^2}.
\end{aligned}$$

Together with their Alvis-Curtis duals, the trivial and the Steinberg character this gives ten distinct character degrees. For $n = 4$ the formulae in [3, 13.8] show that there exist unipotent characters of degrees

$$\begin{aligned}
&1,\ q(q^4+1),\ q^2(q^4+q^2+1),\ \frac{1}{2}q^3(q^2-q+1)(q^4+1), \\
&\frac{1}{2}q^3(q^2+q+1)(q^4+1),\ q^6(q^4+q^2+1),\ q^7(q^4+1),\ q^{12}.
\end{aligned}$$

For $G = \mathrm{L}_2(q)$, $q \geq 4$, the 2-dimensional projective special linear group, the well-known character table shows that the occurring character degrees are $1, q, q-1$, $q+1$ (if $q \neq 5$), $(q-1)/2$ (if $q \equiv 3 \pmod 4$) and $(q+1)/2$ (if $q \equiv 1 \pmod 4$). This leads to cases (a) and (b) and deals with the groups $\mathrm{L}_2(q)$.

The characters of $\mathrm{L}_3(q)$ and $\mathrm{U}_3(q)$ were worked out by Simpson and Frame [26]. It turns out that $\mathrm{L}_3(q)$ has at least 8 different degrees unless $q \leq 4$, $\mathrm{U}_3(q)$ has at least 8 different degrees unless $q = 2$. The tables of $\mathrm{L}_3(2) \cong \mathrm{L}_2(7)$, $\mathrm{L}_3(3)$ and $\mathrm{L}_3(4)$ are to be found in the Atlas, and $\mathrm{U}_3(2)$ is solvable. By the results of Steinberg [28] and Nozawa [23] there are no examples for $\mathrm{L}_4(q)$ or $\mathrm{U}_4(q)$. The groups $\mathrm{L}_5(q)$ and $\mathrm{U}_5(q)$ already have 7 unipotent characters of different degrees, all but the trivial one divisible by

q. But both $\mathrm{L}_5(q)$ and $\mathrm{U}_5(q)$ are reductive, so they contain at least one non-trivial semisimple element. The corresponding semisimple character (see [3, 8.4]) is non-linear and has degree prime to q, hence is different from the unipotent degrees.

The groups $\mathrm{L}_n(q)$, $\mathrm{U}_n(q)$, for $n \geq 6$, have at least 11 unipotent characters, and it is easy to find 8 among them with different degrees. Indeed, given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r)$ of n the q-part of the degree of the corresponding unipotent character indexed by λ is given by $q^{a(\lambda)}$ where

$$a(\lambda) = \sum_{1 \leq j \leq r} (j-1)\lambda_j$$

(see [3, 13.8]). Taking hook-partitions for $n \geq 8$ and further suitable partitions for $n = 6, 7$ we find that there exist degrees with different q-parts already. □

Note that the alternating groups $\mathfrak{A}_5$ and $\mathfrak{A}_6$ occur as the linear groups $\mathrm{L}_2(4)$ respectively $\mathrm{L}_2(9)$ with 4 respectively 5 character degrees, as does $\mathrm{L}_3(2)$ as $\mathrm{L}_2(7)$.

It would be similarly easy to extend the above result to slightly larger bounds on $|\operatorname{cd}(G)|$.

We begin work toward a proof of Theorem A, which classifies the non-solvable groups with 4 character degrees. The next result considers the almost simple groups with 4 character degrees.

Theorem 1. *Let G be a finite almost simple group, that is $S \leq G \leq \operatorname{Aut}(S)$ for some non-abelian simple group S. Then $|\operatorname{cd}(G)| \geq 5$, or $|\operatorname{cd}(G)| = 4$ and $G = \mathrm{PGL}_2(q)$, $q > 3$, $G = \mathrm{L}_2(2^f)$, $f > 1$, or $G = M_{10}$.*

Proof. Let G be almost simple with generalized Fitting subgroup $F^*(G) = S$. We treat the different possibilities for S according to the classification. Again, sporadic groups and ${}^2F_4(2)'$ can be ruled out from the Atlas. For small alternating groups, the same remark applies. For $n \geq 7$ the full automorphism group of $S = \mathfrak{A}_n$ is the symmetric group $\mathfrak{S}_n$, and it follows from the argument given in the previous proof that no case (other than $\mathfrak{A}_5 \cong \mathrm{L}_2(4)$ and $\mathfrak{S}_5 \cong \mathrm{PGL}_2(5)$) arises.

For G of type ${}^2B_2$ or ${}^2G_2$ the explicit results of Suzuki [29] and Ward [30] allow to conclude.

For the remaining groups of Lie type we use some facts on unipotent characters. First assume that S is not of type D_n for n even. Let H be the extension of S by the group of diagonal automorphisms. Thus H is the group of fixed points under a Frobenius map of a connected reductive algebraic group of adjoint type. Then restriction defines a natural bijection between unipotent characters of H and of S, by [20]. In particular, any unipotent character of S has a canonical extension to a unipotent character of H. Let K be the extension of H by the group of graph automorphisms. Then by [4] restriction defines a natural bijection between unipotent characters of K and of H. In particular, any unipotent character of H has a canonical extension to a unipotent character of K. Finally, all unipotent characters of K are invariant under field automorphisms, thus we find an extension to $\mathrm{Aut}(S)$ of any unipotent character of S. Now if S is of type D_n with $n > 4$ even, then the above statements remain true for all unipotent characters not parametrized by a degenerate symbol. (The latter are not invariant under graph automorphisms.) For $n = 4$ there are two more unipotent characters not invariant under triality.

Now for groups of exceptional Lie type, the eight different unipotent character degrees exhibited in the previous proof do the job. More generally, this argument applies to all families of classical groups of Lie type where five unipotent characters with distinct degrees and not parametrized by degenerate symbols were found. It hence remains to consider $\mathrm{L}_2(q)$, $\mathrm{L}_3(q)$, $\mathrm{U}_3(q)$.

We stated already that $|\,\mathrm{cd}(\mathrm{L}_2(q))| = 4$ when q is even. The only outer automorphisms are field automorphisms, and these leave the characters of degree $1, q$ invariant, as well as at least one character of degree $q-1$. Moreover, if $q > 4$ (so $q \geq 8$) there exist characters of degree $q-1$ which are not invariant under field automorphisms. Hence we get at least five degrees unless $q = 4$. But $\mathrm{Aut}(\mathrm{L}_2(4)) \cong \mathrm{PGL}_2(5)$. For q odd we have $|\,\mathrm{cd}(\mathrm{L}_2(q))| = 5$, with degrees $1, q, (q+\epsilon)/2, q-1, q+1$, where $q \equiv \epsilon \pmod 4$. The diagonal automorphism of order 2 only fuses the two characters of degree $(q+\epsilon)/2$, while the field automorphisms leave these and the characters of degrees $1, q$ invariant. For $q \geq 25$, for any non-trivial field automorphism γ there exist characters of degrees $q \pm 1$ fixed by γ as well as characters not fixed by γ. Hence $|\,\mathrm{cd}(\mathrm{PGL}_2(q))| = 4$, while all extensions by field automorphisms have at least five character degrees (for $q = 9$ this can be checked from the Atlas). Also, products of diagonal and field automorphisms lead to at least five character degrees for $q \geq 25$. For $q = 9$ the corresponding extension is the group M_{10}, the stabilizer of a point in M_{11} in its natural permutation representation. Here the Atlas shows that $|\,\mathrm{cd}(M_{10})| = 4$.

For the groups $\mathrm{L}_3(q)$ and $\mathrm{U}_3(q)$ the tables in [26] show that we always get at least five character degrees. $\square$

3 Nonsolvable groups with four character degrees

We will use several times the following result.

Lemma 2. *Let S be a nonabelian finite simple group. Then there exists $1_S \neq \varphi \in \mathrm{Irr}(S)$ that extends to* $\mathrm{Aut}(S)$.

Proof. This is Lemma 4.2 of [21] $\square$

In the next lemma, we consider the case when G has a unique minimal normal subgroup which is nonsolvable. As usual, given a normal subgroup N of a group G, we write

$$\mathrm{cd}(G|N) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G|N)\},$$

where

$$\mathrm{Irr}(G|N) = \{\chi \in \mathrm{Irr}(G) \mid N \not\leq \mathrm{Ker}\,\chi\}.$$

Lemma 3. *Let G be a finite group with a unique minimal normal subgroup N, which is nonabelian. If $|\,\mathrm{cd}(G)| = 4$, then G is almost simple.*

Proof. We know that $N = S_1 \times \cdots \times S_m$ is a direct product of m copies of a nonabelian simple group S, for some integer m. We have to prove that $m = 1$.

By way of contradiction, assume that $m > 1$. Since N is the unique minimal normal subgroup of G, we have that G/N is isomorphic to a subgroup of $\mathrm{Out}(N) \cong \mathrm{Out}(S) \wr \mathfrak{S}_m$. We may view G as a subgroup of $\mathrm{Aut}(N) \cong \mathrm{Aut}(S) \wr \mathfrak{S}_m = \Gamma$. Let $B = \mathrm{Aut}(S)^m \cap G$. By the definition of N, G/B is a permutation group on $\Omega = \{1, \ldots, m\}$.

Assume first that $m > 2$. Consider the character

$$\psi_1 = \varphi \times 1_S \times \cdots \times 1_S \in \mathrm{Irr}(N),$$

where $\varphi \in \mathrm{Irr}(S)$ extends to $\mathrm{Aut}(S)$ (such a character exists by Lemma 2). It follows from the character theory of wreath products (see Theorem 4.3.34 of [15], for instance) that ψ_1 extends to its inertia group in Γ. Similarly, the character

$$\psi_2 = 1_S \times \varphi \times \cdots \times \varphi \in \mathrm{Irr}(N)$$

also extends to its inertia group in Γ and it is clear that both characters have the same inertia group T. Then $T \cap G$ is the inertia group of both ψ_1 and ψ_2 in G. It follows from Clifford's correspondence (Theorem 6.11 of [9]) that

$$|G : T \cap G|\varphi(1) < |G : T \cap G|\varphi(1)^{m-1}$$

belong to the set of character degrees of G.

Now, we will find a third member of $\mathrm{cd}(G)$ that is a multiple of $\varphi(1)$. Consider the character

$$\psi_3 = \psi \times \varphi \times \cdots \times \varphi \in \mathrm{Irr}(N),$$

where $\psi \in \mathrm{Irr}(S)$ and $\varphi(1) \neq \psi(1) > 1$. If we take $\theta \in \mathrm{Irr}(B)$ lying over ψ_3, it is clear that $I_G(\theta) \leq T \cap G$, so for any $\chi \in \mathrm{Irr}(G)$ lying over ψ_3, $\chi(1)$ is a multiple of $|G : T \cap G|\psi(1)\varphi(1)^{m-1}$.

Since $|\mathrm{cd}(G)| = 4$, we conclude that $\varphi(1)$ divides the degree of any non-linear character of G. Now, if we take a prime divisor p of $\varphi(1)$, we have that G has a normal p-complement (by Thompson's theorem [9, Corollary 12.2]). In particular, S has a normal p-complement. This is a contradiction.

Now, we have that $m = 2$ and $|G/B| = 2$. By Theorem B of [10] (or the proof of Theorem 1), there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathrm{Irr}(B/S_1|N/S_1)$ of pairwise different degrees. These three characters induce irreducibly to G and in this way we obtain 3 different character degrees of G. Therefore, $\mathrm{cd}(G) = \{1, 2\gamma_1(1), 2\gamma_2(1), 2\gamma_3(1)\}$. In particular, 2 divides $\chi(1)$ for every non-linear $\chi \in \mathrm{Irr}(G)$. It follows from Thompson's theorem again that G has a normal 2-complement and, in particular, G is solvable. This is the final contradiction. $\square$

In the proof of Theorem A, we will use the classification of nonsolvable groups G with disconnected character degree graph. Recall that the vertices of this graph are the prime divisors of the character degrees and two vertices p and q are joined by an edge if pq divides the degree of some irreducible character of G. We will write $\Delta(G)$ to denote this graph. The nonsolvable groups for which $\Delta(G)$ is disconnected were classified in [19]. In our classification of nonsolvable groups with 4 character degrees, we will use the following.

Theorem 4. *Let G be a nonsolvable group with $\Delta(G)$ disconnected. Then either $G \cong \mathrm{L}_2(2^n) \times A$ for some abelian A and some integer $n \geq 2$ (in this case $\Delta(G)$ has 3 connected components) or $\Delta(G)$ has two connected components and G has normal subgroups $V \leq U$ such that*

(i) *$U/V \cong \mathrm{L}_2(q)$ where $q \geq 4$ is a power of a prime p.*

(ii) *If $C/V = C_{G/V}(U/V)$ then $C/V \leq Z(G/V)$ and G/U is abelian.*

(iii) *If $V > 1$, then either $U \cong \mathrm{SL}_2(q)$ or there is a normal subgroup L of G so that $U/L \cong \mathrm{SL}_2(q)$, L is elementary abelian of order q^2 and U/L acts transitively on the nonprincipal characters of* $\mathrm{Irr}(L)$.

Proof. This follows from [19]. $\square$

Finally, we are ready to complete the proof of Theorem A and Corollary B.

Proof of Theorem A. We begin with the proof of the last assertion. If (i) holds, then $\mathrm{cd}(G) = \{1, 2^n - 1, 2^n, 2^n + 1\}$. If (iii) holds, then it is well-known that $\mathrm{cd}(G) = \{1, 9, 10, 16\}$. If (ii) holds, then we have that $\mathrm{cd}(UC) = \{1, q-1, q, q+1\} \cup \mathcal{S}$, where $\mathcal{S} \subseteq \{(q-1)/2, (q+1)/2\}$ is not empty. We have that $|G/UC| = 2$. Furthermore, it is well-known that the characters of UC whose degree lies in $\mathcal{S}$ are not G-invariant, while all the other characters are G-invariant. We conclude that $\mathrm{cd}(G) = \{1, q-1, q, q+1\}$.

Assume that $|\mathrm{cd}(G)| = 4$ for some nonsolvable G. We want to see that (i), (ii) or (iii) holds. Let M be a normal subgroup of G maximal such that G/M is not solvable. Let N/M be a chief factor of G/M. Then N/M is the unique minimal normal subgroup of G/M. Also, N/M is not solvable. By Lemma 3, G/M is an almost simple group. By Theorem 1 and the hypothesis $|\mathrm{cd}(G)| = 4$, we know that G/M is isomorphic to $\mathrm{PGL}_2(q)$ for $q > 3$ odd, $\mathrm{L}_2(2^f)$ for some $f \geq 2$ or to M_{10}. By inspection we have that $\Delta(G/M) = \Delta(G)$ is disconnected. If $G/M = \mathrm{L}_2(2^f)$ then $\Delta(G)$ has 3 connected components and it follows from Theorem 4 that $G \cong \mathrm{L}_2(2^f) \times A$, for an abelian group A.

Finally, we may assume that $G/M \cong \mathrm{PGL}_2(q)$ for some odd $q \geq 5$ or M_{10}. It follows from Theorem 4 and the hypothesis $|\mathrm{cd}(G)| = 4$ that G has a normal subgroup U such that $U \cong \mathrm{SL}_2(q)$ or $\mathrm{L}_2(q)$. We have that $V = Z(U)$, where V is the group in the statement of Theorem 4. Since $|V| \leq 2$, V is central in G. Write $C/V = C_{G/V}(U/V)$. Since $\mathrm{Aut}(\mathrm{SL}_2(q)) = \mathrm{Aut}(\mathrm{L}_2(q))$ and G/U is abelian (by Theorem 4) we have that $C = C_G(U)$ is central in G. It is clear that G/C is an almost simple group. Now, it follows from Theorem 1 that $G/C \cong M_{10}$ or $\mathrm{PGL}_2(q)$ for some $q > 3$. In the second case, we have that (ii) holds, so we may assume that $G/C \cong M_{10}$.

In this case, $U/V \cong UC/C \cong \mathrm{L}_2(9)$. If $V = 1$, then we have that G has a normal subgroup of index 2 isomorphic to the direct product of $\mathrm{L}_2(9)$ and a central subgroup C such that $G/C \cong M_{10}$, so (iii) holds. Now, we have to see that if $|V| = 2$, then G has at least five character degrees. This is because in this case, UC has an irreducible character of degree 10 that is not invariant, so the set of character degrees of G contains 1, 9, 10, 16 (these are the degrees of M_{10}) and 20. This completes the proof of the theorem. $\square$

Proof of Corollary B. This follows immediately from Theorem A (and its proof). $\square$

References

[1] Ya. G. Berkovich, E. M. Zhmud', "Characters of Finite Groups". Part 2, Translations of Mathematical Monographs **181**, Providence, 1999.

[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, "Atlas of Finite Groups", Clarendon Press, Oxford, 1985.

[3] R.W. Carter, "Finite Groups of Lie Type", Wiley-Interscience, New York, 1985.

[4] F. Digne and J. Michel, Groupes réductifs non connexes. *Ann. Sci. École Norm. Sup.* **27** (1994), 345–406.

[5] H. Enomoto, The characters of the finite symplectic group Sp(4, q), $q = 2^f$, *Osaka J. Math.* **9** (1972), 75–94.

[6] S. Garrison, "On Groups with a Small Number of Character Degrees" Ph. D. Thesis, University of Wisconsin, Madison, 1973.

[7] M. Geck, G. Pfeiffer, "Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras", Clarendon Press, Oxford 2000.

[8] I. M. Isaacs, Groups having at most three irreducible character degrees, *Proc. Amer. Math. Soc.* **21** (1969), 185–188.

[9] I. M. Isaacs, "Character Theory of Finite Groups", Dover, New York, 1994.

[10] I. M. Isaacs, G. Knutson, Irreducible character degrees and normal subgroups, *J. Algebra* **199** (1998), 302–326.

[11] I. M. Isaacs, D. S. Passman, A characterization of groups in terms of the degrees of their characters II, *Pacific J. Math.* **24** (1968), 467–510.

[12] N. Ito, On finite groups with given conjugate types III, *Math. Z.* **117** (1970), 267–271.

[13] N. Ito, Simple groups of conjugate type rank 4. *J. Algebra* **20** (1972) 226–249.

[14] N. Ito, Simple groups of conjugate type rank 5. *J. Math. Kyoto Univ.* **13** (1973), 171–190.

[15] G. James, A. Kerber, "The Representation Theory of the Symmetric Group", Encyclopaedia of Mathematics and its Applications, Addison-Wesley, Reading MA, 1981.

[16] M. L. Lewis, Derived lengths of solvable groups having five irreducible character degrees I, *Algeb. Represent. Theory* **4** (2001), 469-489.

[17] M. L. Lewis, Derived lengths of solvable groups having five irreducible character degrees II, *Algeb. Represent. Theory* **5** (2002), 277-304.

[18] M. L. Lewis, J. M. Riedl, Affine semi-linear groups with three irreducible character degrees, *J. Algebra* **246** (2001), 708-720.

[19] M. L. Lewis, D. L. White, Connectedness of degree graphs of nonsolvable groups, *J. Algebra* **266** (2003), 51–76.

[20] G. Lusztig, On the representations of reductive groups with disconnected centre. Pp. 157–166 in: *Orbites unipotentes et représentations, I.* Astérisque **168**, (1988)

[21] A. Moretó, Complex group algebras of finite groups: Brauer's Problem 1, preprint.

[22] T. Noritzsch, Groups having three complex irreducible character degrees, *J. Algebra* **175** (1995), 767–798.

[23] S. Nozawa, On the characters of the finite general unitary group U(4, q^2), *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **19** (1972), 257–293.

[24] G. Qian, W. Shi, A note on character degrees of finite groups, *J. Group Theory* **7** (2004), 187–196.

[25] J. M. Riedl, Fitting heights of odd order groups with few character degrees, *J. Algebra* **267** (2003), 421–442.

[26] W. A. Simpson, J. S. Frame, The character tables for $\mathrm{SL}(3,q)$, $\mathrm{SU}(3,q^2)$, $\mathrm{PSL}(3,q)$, $\mathrm{PSU}(3,q^2)$, *Canad. J. Math.* **25** (1973), 486–494.

[27] B. Srinivasan, The characters of the finite symplectic group Sp(4, q), *Trans. Amer. Math. Soc.* **131** (1968), 488–525.

[28] R. Steinberg, The representations of $\mathrm{GL}(3,q)$, $\mathrm{GL}(4,q)$, $\mathrm{PGL}(3,q)$, and $\mathrm{PGL}(4,q)$, *Canadian J. Math.* **3** (1951), 225–235.

[29] M. Suzuki, On a class of doubly transitive groups, *Ann. of Math.* **75** (1962), 105–145.

[30] H. N. Ward, On Ree's series of simple groups, *Trans. Amer. Math. Soc.* **121** (1966), 62–89.