A proof of Huppert's ρ-σ conjecture for nonsolvable groups

by

Alexander Moretó
Departament d'Àlgebra
Universitat de València
46100 Burjassot. València. SPAIN
E-mail: Alexander.Moreto@uv.es

Abstract. In this note, we prove Huppert's ρ-σ conjecture for nonsolvable groups. This conjecture asserts that the cardinality of the set of primes that divide the degree of some irreducible character of a finite group is bounded in terms of the maximum number of different prime divisors of the irreducible characters of the group.

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1 Introduction

A lot of research has been made on character degrees of finite groups since the eighties. This is in large part due to the interest of B. Huppert and his school in this type of problems. One of the main problems that Huppert raised in the eighties was his well-known ρ-σ conjectures. Given a finite group G, $\rho(G)$ is the set of distinct primes that divide the degree of some irreducible character of G and

$$\sigma(G) = \max\{\sigma(\chi(1)) \mid \chi \in \mathrm{Irr}(G)\},$$

where given an integer n, $\sigma(n)$ is the number of different prime divisors of n. Huppert's ρ-σ conjectures say the following:

(i) There is a real-valued function f such that $|\rho(G)| \leq f(\sigma(G))$ for every finite group G.

(ii) If G is solvable, then $|\rho(G)| \leq 2\sigma(G)$.

Until now, both parts of this conjecture remained open. The first positive answer to question (i) in the case of solvable groups was provided by M. Isaacs in [5]. Isaacs obtained an exponential bound, which was subsequently improved by D. Gluck, O. Manz and T. Wolf [4, 6, 8]. (See also [7] for a detailed discussion of the known results on the ρ-σ conjectures.) The best known bound to date in the case of solvable groups is $|\rho(G)| \leq 3\sigma(G) + 2$ (see [8]). The goal of this paper is to settle part (i) of the conjecture. We obtain a quadratic bound which is by no means best possible. We have not tried to obtain the best possible bound attainable with our methods because it would be far from the best possible bound.

Theorem A. *For any finite group G,*

$$|\rho(G)| \leq \sigma(G)^2 + 5\sigma(G) + 13.$$

2 Proof of Theorem A

We begin with several lemmas that will be necessary in the proof of Theorem A. The first one gives a very crude bound for the number of different prime divisors of the order of the automorphism group of a nonabelian simple group that do not divide the order of the simple group.

Lemma 2.1. *Let S be a nonabelian simple group. Then the number of different prime divisors of $|\mathrm{Out}(S)|$ that do not divide $|S|$ does not exceed one-third of the number of different prime divisors of $|S|$.*

Proof. As it is well-known, any coprime automorphism of s non-abelian simple group is, up to conjugation, a field automorphism of a simple group of Lie type over a finite field with, say, $q = p^e$ elements. The order of a Hall subgroup of $\mathrm{Out}(S)$ for the primes that do not divide $|S|$, divides e. The result follows using, for instance, Zsigmondy's prime divisor theorem and the order formulas of the simple groups (see Theorem 6.2 of [7] and [**?**]). □

Next, we prove a surely well-known number theoretic fact. As usual given a real number x, $\pi(x)$ stands for the number of prime integers $\leq x$.

Lemma 2.2. *For any $x > 41$,*

$$\pi(x/2) \leq 2.5(\pi(x) - \pi(x/2)).$$

Proof. Using Corollary 1 and Corollary 3 of [9], we have

$$\pi(x/2) \leq 1.25506(x/\log x) < 2.5(\pi(x) - \pi(x/2)),$$

as wanted. □

Now, we state an elementary group theoretic fact.

Lemma 2.3. *For $1 \leq i \leq t$, let N_i be the direct product of some number of copies of a nonabelian simple group S_i. Assume that the groups S_i are pairwise nonisomorphic. Then*

$$\mathrm{Aut}(N_1 \times \cdots \times N_t) \cong \mathrm{Aut}(N_1) \times \cdots \times \mathrm{Aut}(N_t).$$

and

$$\mathrm{Out}(N_1 \times \cdots \times N_t) \cong \mathrm{Out}(N_1) \times \cdots \times \mathrm{Out}(N_t).$$

Finally, we will need the following easy consequence of the results in [2] and [10]. Following [2], recall that if G is a permutation group on Ω and μ is a set of prime divisors of $|G|$, we say that $\Lambda \subseteq \Omega$ lies in a μ-semiregular orbit if the intersection of μ and the set of prime divisors of $|G|$ is contained in the set of prime divisors of $|G : G_\Lambda|$. We refer the reader to [2] for further terminology.

Lemma 2.4. *Let G be a permutation group on a set Ω. Let n be the degree of the largest alternating or symmetric group that appears as a primitive constituent of G and let μ be the set of primes p such that $p \leq \max\{32, (n+1)/2\}$. Then G has a μ'-semiregular orbit on the power set of Ω.*

Proof. Let (H, Δ) be a primitive constituent of (G, Ω). If H does not contain $\mathrm{Alt}(\Delta)$, then it follows from [10] that H has a μ'-semiregular orbit on the power set of Δ. If H contains $\mathrm{Alt}(\Delta)$, then we can take Λ to be any subset of Δ of cardinality $[|\Delta|/2]$. It is easy to see that the orbit of Λ is μ'-semiregular. Now the result follows from Corollary 1 of [2]. □

Now, we are ready to complete the proof of the theorem. We will use repeatedly, and without further explicit notice, that if N is a normal subgroup of G, then $\sigma(N) \leq \sigma(G)$ and $\sigma(G/N) \leq \sigma(G)$.

Proof of Theorem A. Let $\sigma = \sigma(G)$ be the maximum number of different prime divisors of an irreducible character of G. Let A be the product of the normal abelian Sylow subgroups of G. By the Ito-Michler theorem (see Theorem 19.10 and Remark 19.11 of [3]), we know that $\rho(G)$ is the number of different prime divisors of $|G/A|$.

Let R be the largest normal solvable subgroup of G. By Theorem 1.4 of [8], we have that the number of different prime divisors of R does not exceed $3\sigma + 2$. Now, let T/R be the socle of G/R. We know that T/R is a direct product of nonabelian simple groups. Now, we have that G/T is isomorphic to a subgroup of $\mathrm{Out}(T/R)$, and we view it as a subgroup of this group. Let U/T be the intersection of G/T and the direct product of the outer automorphism groups of the simple groups involved in T/R (note that by Lemma 2.3 this is a subgroup of $\mathrm{Out}(T/R)$). Therefore, U/R is a direct product of almost simple groups. Assume that we can find σ almost simple groups $S_1, \ldots, S_\sigma$ and σ pairwise different primes $p_1, \ldots, p_\sigma$ such that p_i divides $|S_i|$ for $i = 1, \ldots, \sigma$. If a new prime p divides the order of some other almost simple group S that appears as a direct factor, then we can take a suitable product of characters of $S, S_1, \ldots, S_\sigma$ to deduce that $\sigma(U/R) > \sigma$. This contradiction implies that we cannot find such a new prime that divides the order of a new direct factor.

Now, we take a prime q_1 that divides the order of $|U/R|$. Then q_1 divides the order of some (almost simple) direct factor U_1/R. If there is some prime that divides $|U/R|$ but does not divide $|U_1/R|$, we take such a prime, say q_2, and a direct factor U_2/R of U/R. We repeat this process until we have covered all the prime divisors of U/R. The previous paragraph guarantees that this will happen in at most σ steps.

Using Lemma 2.1 and Theorem A of [1], we have that

$$|\rho(U_j/R)| \leq 4\sigma(U_j/R) \leq 4\sigma$$

for every j. It follows that the number of different prime divisors of U/R does not exceed $4\sigma^2$.

Finally, we have that G/U is a permutation group on the set Ω of direct factors of T/R. With the notation of Lemma 2.4, we have that G/U has a μ'-semiregular orbit on the power set of this set. This means that if Λ lies in such an orbit, and we take a character φ of T/R that is a product of copies of the principal character in the positions corresponding to indices in Λ and non-principal characters elsewhere, then the degree of any character in $\mathrm{Irr}(G)$ lying over φ will be a multiple of all the primes in μ'. This means that the number of prime divisors of $|G/U|$ bigger than $\max\{32, (n+1)/2\}$ does not exceed σ. If $n \leq 63$, then the number of prime divisors of $|G/U|$ is either $\leq \sigma + 11$ (note that $\pi(32) = 11$). In other case,

$$\pi(n-1) - \pi((n+1)/2) \leq \sigma$$

and we deduce using Lemma 2.2 that

$$\pi(n/2) \leq 2.5(\pi(n) - \pi(n/2)) \leq 2.5(\pi(n-1) - \pi((n+1)/2) + 2) \leq \sigma + 5.$$

We see that, in any case, the number of prime divisors of $|G/U|$ does not exceed $2\sigma + 11$.

Putting everything together, we have

$$|\rho(G)| \leq (3\sigma + 2) + 4\sigma^2 + (2\sigma + 11) = \sigma^2 + 5\sigma + 13,$$

as desired. This concludes the proof. $\square$

From the proof above, it seems clear that it should be easy to obtain a linear bound. This linear bound is known in the case of almost simple groups and, using this proof, one only needs to consider the case of a direct product of almost simple groups. However, we have not pursued this further because the linear bound that one can likely obtain with these methods is of the form $|\rho(G)| \leq 9\sigma(G) + C$, where C is a universal constant. However, it seems likely that a bound of the form $|\rho(G)| \leq 3\sigma(G) + C$ holds and it is even possible that the multiplicative constant can be lowered to 2 (see [7, 8]).

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