An answer to a question of Isaacs on character degree graphs

by

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Abstract. Let N be a normal subgroup of a finite group G. We consider the graph $\Gamma(G|N)$ whose vertices are the prime divisors of the degrees of the irreducible characters of G whose kernel does not contain N and two vertices are joined by an edge if the product of the two primes divides the degree of some of the characters of G whose kernel does not contain N. We prove that if $\Gamma(G|N)$ is disconnected then G/N is solvable. This proves a strong form of a conjecture of Isaacs.

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1 Introduction

Given a finite group G, it is well-known that the set $\mathrm{cd}(G)$ of the degrees of its complex irreducible characters encodes a lot of structural information of the group G. In 1998, M. Isaacs and G. Knutson [13] proposed to study the set

$$\mathrm{cd}(G|N) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G|N)\},$$

where

$$\mathrm{Irr}(G|N) = \{\chi \in \mathrm{Irr}(G) \mid N \not\leq \mathrm{Ker}\,\chi\}$$

and N is a normal subgroup of the group G. In that paper, Isaacs and Knutson used their study of this set to provide a proof of the fact that the derived length of a solvable group with 4 character degrees does not exceed 4 easier that the original proof in S. Garrison's Ph. D. Thesis. The study of this set was also an invaluable tool in the proof of M. Lewis [15] of the result that the derived length of solvable groups with 5 character degrees does not exceed 5.

When studying problems on character degrees it is useful to attach a graph to the set $\mathrm{cd}(G)$. A graph that has been widely studied is the graph $\Gamma(G)$ whose vertices are the prime divisors of the degrees of the irreducible characters of G and two vertices p and q are joined by an edge if the product pq divides the degree of some irreducible character of G. If we know the structure of $\Gamma(G)$ we can often say a lot on the structure of the group G. For instance, the groups G such that $\Gamma(G)$ is disconnected have been completely classified (see [14, 16]).

Following the aim of [13], Isaacs [12] proposed to study the relative character degree graph $\Gamma(G|N)$. The vertices of this graph are the prime divisors of the members of $\mathrm{cd}(G|N)$ and two vertices p and q are joined by an edge if pq divides the degree of some character of $\mathrm{Irr}(G|N)$. More precisely, he studied the graph $\mathcal{G}(G|N)$ whose vertices are the members of $\mathrm{cd}(G|N)$ and two vertices are joined by an edge if they are not coprime. But, as happens with the analog situation in the non-relative case, these two graphs are closely related. In particular, $\Gamma(G|N)$ is disconnected if and only if $\mathcal{G}(G|N)$ has more than one non-trivial connected component.

In this paper, we prove the following theorem, which was conjectured in [12]. (This was also conjectured by M. Lewis in private communication.)

Theorem A. *Let N be a normal subgroup of a finite group G that is contained in G'. Assume that N is solvable and that $\Gamma(G|N)$ is disconnected. Then G is solvable.*

In fact, we can prove the following stronger result.

Theorem B. *Let N be a normal subgroup of a finite group G that is contained in G'. If $\Gamma(G|N)$ is disconnected then G/N is solvable.*

By the work in [12] in order to prove Theorem A it suffices to consider the case when N is a non-abelian p-group. This is the case that we will handle here. We prove the following.

Theorem C. *Let N be a normal subgroup of a finite group G that is contained in G'. Assume that N is a non-abelian p-group and that $\Gamma(G|N)$ is disconnected. Then G/N is the product of (at most) two abelian subgroups of coprime orders. In particular, G/N is metabelian.*

From the results of Isaacs in [12] when N is not a p-group or when N is an abelian p-group and Theorem C it can be seen that the groups G with $\Gamma(G|N)$ disconnected have quite a restricted structure, so it might be possible to classify them. We have not tried to push this further, however.

In our proof we will use the following result, which was conjectured by R. Higgs in [9] and has independent interest.

Theorem D. *Let N be a normal subgroup of a finite group G and let $\varphi \in \mathrm{Irr}(N)$ be G-invariant. Assume that $\chi(1)/\varphi(1)$ is odd for all $\chi \in \mathrm{Irr}(G|\varphi)$. Then G/N is solvable.*

We will use several times along the paper results that depend on the classification of finite simple groups. We also make use of the Deligne-Lusztig theory of characters of groups of Lie type.

The notation is standard. Given a character φ of a normal subgroup of a group G, we write $\mathrm{Irr}(G|\varphi)$ to denote the set of irreducible constituents of φ^G and $\mathrm{cd}(G|\varphi)$ to denote the set of degrees of the members of $\mathrm{Irr}(G|\varphi)$. In Section 2 we prove Theorem D and another result of Higgs on projective characters. In Section 3 we state some preliminary results and in Section 4 we begin to study the structure of groups with disconnected graph. Finally, we prove Theoem C in Section 5 and Theorems A and B in Section 6.

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2 Higgs's conjecture

We begin with the proof of Theorem D.

Proof of Theorem D. By the theory of isomorphic character triples (see Chapter 11 of [11]), we may assume that N is a cyclic central subgroup of G and

φ is faithful. By the results of [9], we may assume that G/N is isomorphic to one of the following groups: $\mathrm{PSL}(n,q)$, $\mathrm{PSU}(n,q)$, $E_6(q)$ or ${}^2E_6(q)$. As pointed out by Higgs, if one of these groups is a counterexample, then it would be a counterexample for Brauer's height-zero conjecture. This conjecture has been proved for the central quotients of the special linear and special unitary groups in [4], so all we need to do is to prove that there are even degree characters lying over φ in the last two cases.

The following argument is due to G. Malle. We may assume that the group is $3 \cdot E_6(q)$ for some q such that 3 divides $q-1$ or $3 \cdot^2 E_6(q)$ for some q such that 3 divides $q+1$. We consider first the case when q is odd. Take s in the adjoint group of $E_6(q)$ of order q^6+q^3+1. Then s does not centralize a Sylow 2-subgroup of the adjoint group of $E_6(q)$ and the corresponding (faithful) semisimple character of $3 \cdot E_6(q)$ has even degree (see Theorem 8.4.8 of [6]). In the other case we may argue in a similar way taking an element of order q^6-q^3+1 of the adjoint group of ${}^2E_6(q)$.

Now, we may assume that q is even. In both cases, it suffices to consider any semisimple element that is not in the derived subgroup of the adjoint group and whose centralizer is not a torus (but bigger). Then the corresponding regular character has even degree (see Theorem 8.4.9 of [6]). $\square$

It is possible to argue in a similar way in the cases of the special linear and special unitary groups.

The following elementary (and probably well-known) lemma will be very useful.

Lemma 2.1. *Let G be a perfect group and N a normal subgroup of G. If $\lambda \in \mathrm{Irr}(N)$ is a G-invariant linear character that has order n, then n divides $\chi(1)$ for every $\chi \in \mathrm{Irr}(G|\lambda)$.*

Proof. We have that $\chi_N = \chi(1)\lambda$. Furthermore, since G is perfect, we have that $\det(\chi) = 1_G$ so

$$1_N = \det(\chi)_N = \det(\chi_N) = \det(\chi(1)\lambda) = \lambda^{\chi(1)}$$

and it follows that $\chi(1)$ is a multiple of n. $\square$

Corollary 2.2. *Let G be a group, $N \trianglelefteq G$ a cyclic p-group and $\lambda \in \mathrm{Irr}(N)$ faithful and G-invariant. If there exists a perfect normal subgroup D of G such that $D \cap N > 1$, then p divides all the members of $\mathrm{cd}(G|\lambda)$.*

Proof. Let $\nu = \lambda_{D\cap N}$. It follows from Clifford theory that it suffices to see that p divides all the members of $\mathrm{cd}(D|\nu)$. Since λ is faithful, p divides the order of ν. Also, it is clear that ν is D-invariant. Now the result follows from Lemma 2.1. $\square$

We will also need the following result, which was proved by Higgs in [8]. For completeness, we sketch a short proof.

Theorem 2.3. *Let N be a normal subgroup of a finite group G and let $\varphi \in \mathrm{Irr}(N)$ be G-invariant. Assume that $\chi(1)/\varphi(1)$ is a power of a fixed prime p for every $\chi \in \mathrm{Irr}(G|\varphi)$. Then G/N is solvable.*

Proof. By Theorem D, we may assume that $p = 2$. Also, assume that G is a minimal counterexample. Let R/N be the largest normal solvable subgroup of G/N. By the minimality of G, we have that G/R is simple non-abelian. Let $\psi \in \mathrm{Irr}(R)$ lying over φ. Put $T = I_G(\psi)$. If $T < G$, then T/R is solvable and $|G : T|$ is a power of 2. Now [7] or [18] imply that G/R is isomorphic to some $\mathrm{PSL}(2, 2^m - 1)$ for some $m > 1$ and T/R is the stabilizer of a line. In particular, T/R is an odd order non-nilpotent group. This contradicts the fact that all the members of $\mathrm{Irr}(G|\psi)$ have 2-power degree. Therefore, we may assume that ψ is G-invariant. By the theory of isomorphic character triples, we may assume that R is cyclic and central and that ψ is faithful. Now we have that G' is perfect. Therefore, G' is quasisimple. By Lemma 2.1, $R \cap G'$ is a 2-group. It is clear that we may assume that $Z(G') = R \cap G' > 1$.

We want to see that the quasisimple group G' has some character lying over $\psi_{Z(G')}$ whose degree is not a 2-power. Assume that all the characters of G' lying over $\psi_{Z(G)}$ have degree 2^a for some fixed $a > 0$. Then $|G'/Z(G')| = 2^{2a}k$, where $k = |\mathrm{Irr}(G'|\psi_{Z(G')})|$ is the number of $\psi_{Z(G')}$-special classes of $G/Z(G')$ (see Problem 11.11 of [11]). It follows that $k(G'/Z(G')) \geq k \geq |G'|_{2'}$. By Theorem 1 of [3], for instance, the elements of $Z(G')$ are commutators. We conclude, using Problem 11.15(b) of [11], that $k(G'/Z(G')) > k$, so $k(G'/Z(G')) > |G'|_{2'}$. But this contradicts Theorem 9 of [17].

Hence, we have that $|\,\mathrm{cd}(G'|\psi_{Z(G')})| \geq 2$. Now, [19] implies that G' is $2 \cdot \mathrm{Sp}(6, 2)$, $2 \cdot O^+(8, 2)$ or a double cover of an alternating group. In the first two cases, it suffices to look at the character tables of these groups available in GAP to see that they have faithful characters whose degree is not a 2-power. Confirming a conjecture in [19], the faithful characters of 2-power degree of the double cover of the alternating group were classified in [2], and it is clear that these groups have faithful characters whose degree is not a 2-power. This completes the proof. □

Alternatively, one could make more extensive use of the classification in [19] to deal with the case considered in the second paragraph. In this way, it would not be necessary to quote the results in [3] and [17]. Also, it is not difficult to prove that there are faithful characters of the double covers of the alternating group whose degree is not a 2-power, without using [2].

3 Preliminary results

We begin by stating two group theoretical facts that we will need.

Lemma 3.1. *Let G be a non-solvable π'-solvable group, where π is a set of primes. If H is a subgroup of G such that $|H|_\pi = |G|_\pi$, then H is not solvable.*

Proof. Let R be the largest normal solvable subgroup of G and let S/R be a chief factor of G. Since S/R is not abelian, we have that it is a π-group. Since $|H|_\pi = |G|_\pi$, we deduce that $S \leq HR$ and it follows that HR is not solvable. Therefore H is not solvable. $\square$

We also need the following deep theorem.

Lemma 3.2. *Let G be a non-abelian simple group with abelian Sylow 2-subgroups. Then G is isomorphic to one of the following groups:* $\mathrm{PSL}(2, 2^f)$ *for some $f \geq 2$,* $\mathrm{PSL}(2, q)$ *for some $q \equiv 3 \pmod 8$ or $q \equiv 5 \pmod 8$, J_1 or a Ree group.*

Proof. This follows from [22] and [5]. See Theorem XI.13.7 of [10]. $\square$

Next, we prove some easy results on character degree graphs. We will use repeatedly, and without further explicit mention, the following fact.

Lemma 3.3. *Let N and M be normal subgroups of a finite group G such that $N \leq M$. If p belongs to $\Gamma(M|N)$ then p belongs to $\Gamma(G|N)$.*

Proof. If p belongs to $\Gamma(M|N)$ then p divides the degree of some character $\varphi \in \mathrm{Irr}(M|N)$. It follows from Clifford theory that if we take $\chi \in \mathrm{Irr}(G|\varphi)$ then p divides $\chi(1)$ and $\chi \in \mathrm{Irr}(G|N)$. The result follows. $\square$

We will also use without further explicit mention the Ito-Michler theorem, which asserts that all the character degrees of a finite group G are coprime to p if and only if G has a normal abelian Sylow p-subgroup (see Theorem 12.33 of [11] and [21]).

We also need the following elementary fact that relates degree graphs.

Lemma 3.4. *Let N be a normal subgroup of a finite group G. Assume that $G/N = A/N \times B/N$, where $A/N > 1$ is abelian. Then*

$$\Gamma(G/N|A/N) = \Gamma(B/N).$$

Proof. It suffices to see that $\mathrm{cd}(G/N|A/N) = \mathrm{cd}(B/N)$. We have that

$$\begin{aligned}\mathrm{cd}(G/N|A/N) &= \{\alpha(1)\beta(1)|\alpha \in \mathrm{Irr}(A/N) - \{1_{A/N}\}, \beta \in \mathrm{Irr}(B/N)\} \\ &= \{\beta(1) \mid \beta \in \mathrm{Irr}(B/N)\} = \mathrm{cd}(B/N),\end{aligned}$$

as desired. $\square$

4 Groups with disconnected graph

Before proceeding to prove our main results on degree graphs, we will obtain some partial results on the structure of groups with disconnected graph $\Gamma(G|N)$. In this section, we will consider the following hypothesis.

Hypothesis 4.1. *The graph $\Gamma(G|N)$ is disconnected and N is a non-abelian normal p-subgroup of the finite group G that is contained in G'. Furthermore, $\Gamma(G|N)$ is the disjoint union of the connected subgraphs $\Gamma(G/N'|N/N')$ and $\Gamma(G|N')$.*

Notice that the last assumption is not really relevant. If $\Gamma(G/N'|N/N')$ is disconnected, then we are in the situation of Theorem C of [11] and in this case we already know the structure of G. Therefore, it is no loss of generality to assume that $\Gamma(G/N'|N/N')$ is connected. Similarly, the degree of all the members of $\mathrm{Irr}(G|N')$ is a multiple of p, so $\Gamma(G|N')$ is connected. Observe also that the degree of all the members of $\mathrm{Irr}(G/N'|N/N')$ is coprime to p.

Lemma 4.2. *Assume Hypothesis 4.1. Put $C/N' = C_{G/N'}(N/N')$. Then C is solvable.*

Proof. Assume that C is not solvable. We will work in the group $\overline{C} = C/N'$ and we will use the bar convention. Since we are assuming that C is not solvable, there exists an integer n such that $\overline{C}^{(n)} > 1$ is perfect. If $\overline{C}^{(n)} \cap \overline{N} > \overline{1}$, then Lemma 2.1 implies that p divides the degree of any character of $\overline{C}^{(n)}$ lying over a non-principal character of $\overline{C}^{(n)} \cap \overline{N}$. This would imply that p belongs to $\Gamma(G/N'|N/N')$, which is a contradiction.

It follows that $\overline{C}^{(n)} \cap \overline{N} = 1$, so by Lemma 3.4

$$\Gamma(C^{(n)}N/N'|N/N') = \Gamma(C^{(n)}N/N).$$

In particular, all prime divisors of the non-abelian simple groups involved in the chief factors of C belong to $\Gamma(G/N'|N/N')$.

Now, take $\mu \in \mathrm{Irr}(N)$ non-linear. By the structure of the graph, we have that the inertia group of μ in C, C_μ contains the whole π-part of $|C|$, where

π is the set of prime divisors of the non-abelian simple groups involved in C. By Lemma 3.1, we have that C_μ is not solvable. Let $N \leq Y \leq X \leq C_\mu$ where X/Y is a non-abelian chief factor of C_μ. Let $\tau \in \mathrm{Irr}(Y)$ lying over μ. We know that all the degrees in $\mathrm{cd}(X|\mu)$ are coprime to $|X/Y|$ and this means that τ extends irreducibly to $\gamma \in \mathrm{Irr}(X)$. But now if $\theta \in \mathrm{Irr}(X/Y)$ is not linear then $\gamma\theta \in \mathrm{Irr}(X|\mu)$ (using Corollary 6.17 of [11]) and this contradicts the fact that the members of $\mathrm{cd}(X|\mu)$ are coprime to $|X/Y|$. We conclude that C is solvable. □

In our next lemma, we study the structure of C.

Lemma 4.3. *Assume Hypothesis 4.1 and put $C/N' = C_{G/N'}(N/N')$. Then C/N' has a normal abelian Sylow p-subgroup P/N'.*

Proof. Since p does not belong to $\Gamma(G/N'|N/N')$, we deduce that any character of N/N' extends irreducibly to L, where $L/N = O_{p,p'}(C/N)$. (This is because if H/N is a Hall p'-subgroup of L/N and P/N is the normal Sylow p-subgroup of L/N then any character of N/N' extends to P by the structure of the graph and also extends to H by Corollary 8.16 of [11]. Now, Corollary 11.31 of [11] implies that any such character extends to L.) Now, let $K/L = O_p(C/L)$ and assume that $K > L$. Then any character of N/N' also extends to K, and it follows from Gallagher's theorem (Corollary 6.17 of [11]), using an appropriate character of K/N, that p belongs to $\Gamma(G/N'|N/N')$. This contradiction implies that $K = L$ and the same argument shows that the Sylow p-subgroup P/N of C/N is abelian. It is also clear that $P' = N'$, so P/N' is abelian. □

Next, we study the behavior of the linear characters of N.

Lemma 4.4. *Assume Hypothesis 4.1 and let $C/N' = C_{G/N'}(N/N')$. Then all the linear characters of N have extensions to C that have p-power order.*

Proof. Write $C = QP$, where Q is a Hall p-complement of C and P is the normal Sylow p-subgroup. Let $\lambda \in \mathrm{Irr}(N/N')$. Since P/N' is abelian, we have that λ extends to P. Also, since Q acts trivially on N/N', we have that λ extends to any Sylow subgroup of C and, by Corollary 11.31 of [11], it also extends to C. Let ν be such an extension. Therefore, ν_P is C-invariant. The canonical extension of ν_P to C extends λ and has p-power order. □

Next, we study the behavior of these extensions in the whole group G.

Lemma 4.5. *Assume Hypothesis 4.1 and put $C/N' = C_{G/N'}(N/N')$. Let $\tau \in \mathrm{Irr}(C/N'|N/N')$ linear of p-power order and let T_1 be the inertia group of τ in G. Put $R/C = O_p(T_1/C)$ and let $\delta \in \mathrm{Irr}(R)$ be an extension of τ*

(such an extension exists because all the members of $\mathrm{Irr}(G/N'|N/N')$ *have* p'*-degree). Let* T_2 *be the inertia group in* T_1 *of* δ*. Then*

(i) R/C *is abelian;*

(ii) $|G:R|$ *is a* p'*-number; and*

(iii) δ *extends to* T_2*.*

Proof. Taking into account that p does not belong to $\Gamma(G/N'|N/N')$ and using Gallagher's theorem, we can see that R/C is abelian. Put $J/R = O_{p'}(T_2/R)$ and let $\hat{\delta}$ be the canonical extension of δ to J. Now, assume that $J < T_2$ and let D/J be a chief factor of T_2/R. We have that p divides $|D/J|$. Assume first that D/J is abelian. If $\hat{\delta}$ extends to D, then using Gallagher's theorem again we would obtain that p belongs to $\Gamma(G/N'|N/N')$, a contradiction. Otherwise, the same contradiction would follow using Corollary 11.29 of [11]. Hence, we may assume that D/J is not abelian. Now, we will work in the group $\overline{D} = D/\operatorname{Ker}\hat{\delta}$. We want to prove that in this group there is some character lying over $\hat{\delta}$ whose degree is divisible by p. We have that $\overline{D}'$ is perfect. If it does not intersect trivially with $\overline{J}$ then we may apply Corollary 2.2 to obtain what we want. If it intersects trivially with $\overline{J}$, then we can extend $\hat{\delta}$ to $D'J$ and we obtain that p divides the degree of some character of D lying over $\hat{\delta}$. This contradiction implies that $J = T_2$. Therefore δ extends to T_2. Also, it follows from Clifford's correspondence that there is some character in $\mathrm{Irr}(G/N'|N/N')$ that is induced from a character of T_2. In particular, $|G:T_2|$, and hence $|G:R|$, is a p'-number. $\square$

The next result is an easy consequence of Lemmas 4.4 and 4.5.

Corollary 4.6. *Assume Hypothesis 4.1. For any* $\lambda \in \mathrm{Irr}(N/N')$*,* λ *extends to its inertia group in* G*.*

Proof. It follows from Lemmas 4.4 and 4.5 that λ extends to a Sylow p-subgroup of G. Also, λ extends to all other Sylow subgroups of its inertia group. We conclude that λ extends to its inertia group. $\square$

The next easy fact will be important in the proof of our main results.

Lemma 4.7. *Assume that* N *is a normal subgroup of a group* G *and that* $\varphi \in \mathrm{Irr}(N)$ *extends to its inertia group* T*. Assume that* $K \leq \operatorname{Ker}\varphi$*. Let* π *be the set of prime divisors* q *of* $|T/N|$ *such that* T/N *has a normal abelian Sylow* q*-subgroup and* q *does not divide* $|G:T|$*. Then all the prime divisors of* $|G/N|$ *except for possibly those primes in* π *belong to* $\Gamma(G/K|N/K)$*.*

Proof. This follows from Clifford's correspondence, Gallagher's theorem and the Ito-Michler theorem. $\square$

5 Proof of Theorem C

Now, we are ready to complete the proof of Theorem C.

Proof of Theorem C. Recall that we are assuming that $\Gamma(G|N)$ is disconnected, where N is a non-abelian p-group. We may assume that $\Gamma(G|N)$ is the disjoint union of $\Gamma(G/N'|N/N')$ and $\Gamma(G|N')$, so Hypothesis 4.1 holds. First, we want to prove that G is solvable.

Let (G, N) be a counterexample with G of minimal order. Put $C/N' = C_{G/N'}(N/N')$ and write $C = QP$ where Q is a Hall p-complement and P is the normal Sylow p-subgroup of C. We may assume that Q acts faithfully on P. (Otherwise, $G/O_{p'}(F(C))$ would be a counterexample of smaller order.) Since Q acts trivially on N, we have that it acts faithfully on P/N.

Let R be the largest normal solvable subgroup of G. Since G is a minimal counterexample, we have that G/R is a non-abelian simple group and that G/N is perfect.

Assume first that 2 does not belong to $\Gamma(G/N'|N/N')$. In particular, it follows from Corollary 4.6 and Lemma 4.7 that G/N has an abelian Sylow 2-subgroup. Also, we know that for any $\lambda \in \mathrm{Irr}(N/N')$ the inertia group in G of λ is contained in $NN_G(S)$ and contains NS, where S is a Sylow 2-subgroup of G. We know that SR/R does not centralize any Sylow r-subgroup of G/R for any $r \neq 2$. (This follows from an analysis of the groups that appear in Lemma 3.2 or from [1]). This implies that all the prime divisors $r \neq 2$ of $|G/R|$ belong to $\Gamma(G/N'|N/N')$. Now, take $\psi \in \mathrm{Irr}(R)$ lying over a non-principal character of N'. We have that $\chi(1)/\psi(1)$ divides $|G : R|$ for any $\chi \in \mathrm{Irr}(G|\psi)$ and also that the prime divisors of $\chi(1)$ belong to $\Gamma(G|N')$. It follows that $\chi(1)/\psi(1)$ is a 2-power. In particular, the inertia group of ψ in G is a 2-power index subgroup. Since G/R does not contain any proper 2-power index subgroup (this can be seen directly or it follows from [7, 18]), we deduce that ψ is G-invariant. Now, it follows from Theorem 2.3 that G/R is solvable. This contradiction implies that we may assume that 2 belongs to $\Gamma(G/N'|N/N')$.

Hence, we may assume that 2 belongs to $\Gamma(G/N'|N/N')$. In particular, 2 does not belong to $\Gamma(G|N')$ and $p > 2$. If we take $\alpha \in \mathrm{Irr}(N')$ and put $T_1 = I_G(\alpha)$, we have by Theorem D that T_1 is solvable. If $\lambda \in \mathrm{Irr}(N/N')$ and T is its inertia group in G, we know that λ extends to T and that all the prime divisors of G/N except for possibly those in the set π defined in Lemma 4.7 belong to $\Gamma(G/N'|N/N')$. Notice that G/N has an abelian Hall π-subgroup. Furthermore, the set of prime divisors of the members of $\mathrm{Irr}(T_1|\alpha)$ is a subset of π and $|G : T_1|$ is a π-number. We conclude using the generalized Gluck-Wolf theorem (see Theorem 12.9 of [20]) that T_1/N

has an abelian Hall π'-subgroup, which is a Hall π'-subgroup of G/N. It follows that G/N can be written as a product of two abelian subgroups. We conclude, using a well-known theorem of Ito that G/N is metabelian. In particular, G is solvable.

Notice that in the last paragraph the assumption that 2 belongs to $\Gamma(G/N'|N/N')$ was only used to guarantee that T_1 is solvable. Once we have proved that G is solvable, we can repeat the same argument to prove in full generality that G/N can be written as the product of two abelian subgroups of coprime orders. □

6 Proof of Theorems A and B

Now it is easy to complete the proof of Theorem A.

Proof of Theorem A. We are assuming that N is solvable and that $\Gamma(G|N)$ is disconnected. We may assume that N is minimal among the normal subgroups with this property. By Theorem D of [12], we are done if N is not a p-group. Hence, we may assume that N is a p-group and now the result follows from Theorem C of [12] and Theorem C above. □

Finally, we prove Theorem B.

Proof of Theorem B. Recall that we are assuming that N is a normal subgroup of a finite group G that is contained in G' and that $\Gamma(G|N)$ is disconnected. We want to prove that G/N is solvable. We may assume that N is minimal among the normal subgroups of G that are contained in G' such that $\Gamma(G|N)$ is disconnected. By Theorem A, we may assume that N is not solvable.

Assume first that N is perfect. If $\Gamma(N)$ is connected then, since all the degrees of the characters in $\mathrm{Irr}(G|N)$ are multiples of the degrees of the non-linear characters of N, we have that $\Gamma(G|N)$ is connected. Hence, we may assume that $\Gamma(N)$ is disconnected. By Lemma 4.3 and Theorem 6.2 of [16], G has a normal subgroup $M \leq N$ such that $N/M \cong \mathrm{PSL}(2,q)$ for some $q \geq 4$. Now, $\frac{G/M}{C_{G/M}(N/M)\times N/M}$ is solvable. It follows that we may assume that $C_{G/M}(N/M)$ is not solvable. Now, for any character ψ of N/M there exists an even degree character of G lying over ψ. It follows that 2 is joined to all the vertices of $\Gamma(G/M|N/M)$, so this graph is connected. Now, Lemma 4.3 and Theorem 6.2 of [16] imply that M is solvable, and we may assume that $M > 1$. Since we may assume that G/N is not solvable, Theorem A implies that $\Gamma(G|M)$ is connected. Therefore $\Gamma(G|N)$ is the disjoint union of

$\Gamma(G/M|N/M)$ and $\Gamma(G|M)$. We have that 2 belongs to $\Gamma(G/M|N/M)$ and we will see that 2 also belongs to $\Gamma(G|M)$. This contradiction will imply that we may assume that N is not perfect. If q is odd, Lemma 4.3 and Theorem 6.2 of [16] imply that there exists $L < M$ such that $N/L \cong \mathrm{SL}(2, q)$ and it follows from Lemma 2.1 that the members of $\mathrm{Irr}(N/L|M/L)$ have even degree. Hence, we may assume that q is a power of 2. Now, the results of Lewis and White imply that M is elementary abelian of order q^2. It follows again from Lemma 2.1 that 2 divides the degree of any member of $\mathrm{Irr}(N|M)$ and, therefore, it also divides the degree of any member of $\mathrm{Irr}(G|M)$, as desired.

Hence, we may assume that N is not perfect. By Theorem A of [12], we have that $\mathcal{G}(G|N)$ has two connected components. The degrees of one of them are all divisible by every prime divisor of $|N : N'|$ and all the degrees of the other component are coprime to $|N : N'|$. By the choice of N, we have that $\Gamma(G|N')$ is connected. Also, by Theorem A, we may assume that $\Gamma(G/N'|N/N')$ is connected. Thus, we have that $\Gamma(G|N)$ is the disjoint union of $\Gamma(G|N')$ and $\Gamma(G/N'|N/N')$. Also, if G is a minimal counterexample, N' is isomorphic to a direct product of a number of copies of a non-abelian simple group S. Assume that the prime divisors of $|N : N'|$ belong to $\Gamma(G/N'|N/N')$. Then $(|N : N'|, |S|) = 1$ and then, the Schur-Zassenhaus theorem together with the fact that all the characters of N' extend to N implies that N is isomorphic to the direct product of two nontrivial normal subgroups of G. It follows from Theorem 2.4 of [12] that $\Gamma(G|N)$ is connected. This contradiction implies that the prime divisors of $|N : N'|$ belong to $\Gamma(G|N')$.

We have that 2 belongs to $\Gamma(G|N')$, so it does not belong to $\Gamma(G/N'|N/N')$. Now, we may argue as in the proof of Theorem C. If G is a minimal counterexample and R/N is the largest normal solvable subgroup of G/N, then G/R is simple and G/N is perfect. Assume that there is some non-principal $\lambda \in \mathrm{Irr}(N/N')$ that extends to T, its inertia group in G. Then G/N has an abelian Sylow 2-subgroup. Also, we know that for any $\lambda \in \mathrm{Irr}(N/N')$ the inertia group in G of λ is contained in $NN_G(Q)$ and contains NQ, where Q is a Sylow 2-subgroup of G. We know that QR/R does not centralize any Sylow r-subgroup of G/R for any $r \neq 2$. This implies that all the prime divisors $r \neq 2$ of $|G/R|$ belong to $\Gamma(G/N'|N/N')$. If $\psi \in \mathrm{Irr}(R)$ lies over a non-principal character of N', then $\chi(1)/\psi(1)$ divides $|G : R|$ for any $\chi \in \mathrm{Irr}(G|\psi)$ and also that the prime divisors of $\chi(1)$ belong to $\Gamma(G|N')$. It follows that $\chi(1)/\psi(1)$ is a 2-power. In particular, the inertia group of ψ in G is a 2-power index subgroup. Since G/R does not contain any proper 2-power index subgroup, we deduce that ψ is G-invariant. Now, it follows from Theorem 2.3 that G/R is solvable. This is a contradiction.

Hence, all we have to prove is that there exists $\lambda \in \mathrm{Irr}(N/N')$ non-principal that extends to T. Let $\lambda \in \mathrm{Irr}(N/N')$ non-principal of p-power order for some prime p. We have that T contains a Sylow p-subgroup of G (because the prime divisors of $|N : N'|$ belong to $\Gamma(G|N')$). Let $P/N \leq T/N$ be a Sylow p-subgroup of G/N. Now, we claim that λ extends to P. Assume not. Then $\mathrm{Irr}(P|\lambda)$ is a set of non-linear characters of p-power degree. If $\chi \in \mathrm{Irr}(G|\lambda)$ then χ_P decomposes as a sum of members of $\mathrm{Irr}(P|\lambda)$ and we conclude that p divides $\chi(1)$. Thus p belongs to $\Gamma(G/N'|N/N')$. This contradiction implies that λ extends to P. Now, if q is a different prime divisor of T/N we have that λ extends to Q, where Q/N is a Sylow q-subgroup of T/N. Thus λ extends to all the Sylow subgroups of T/N and it follows that λ extends to T. This completes the proof of the theorem. $\square$

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