A variation on theorems of Jordan and Gluck

by

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Abstract. Gluck proved that any finite group G has an abelian subgroup A such that $|G : A|$ is bounded by a polynomial function of the largest degree of the complex irreducible characters of G. This improved on a previous bound of Isaacs and Passman. In this paper, we present a variation of this result that looks at the number of prime factors. All these results, in turn, may be seen as variations on the classical theorem of Jordan on linear groups.

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1 Introduction

A classical theorem of Jordan (see Theorem 14.12 of [8]) asserts that a finite group has an abelian subgroup of index bounded in terms of the degree of a faithful complex irreducible character of G. This theorem has motivated a number of results. For instance, the case when the field of complex numbers is replaces by a field of characteristic $p > 0$, was considered in [1]. In a different direction, it seems reasonable to ask whether there is an abelian subgroup of index bounded in terms of the largest degree of the complex irreducible characters. An affirmative answer to this question was provided by M. Isaacs and D. Passman in [9]. Later, D. Gluck proved in [6] that in fact there exists a polynomial bound.

Let G be a complex linear group of degree n. Much research has been devoted to studying complex linear groups and, in particular, to classifying the linear groups of "small" degree. The word small here has two different meaning. One is the obvious one; that the absolute value of n is small. The second meaning involves the prime factorization of n; n is "small" if it has few prime divisors (counting multiplicities). This is the meaning we are interested in here. For instance, the classification of nonsolvable complex linear groups of prime degree has been recently considered in [4, 5].

Given an integer $n = p_1^{a_1} \dots p_t^{a_t}$ as a product of powers of different primes, we define $\omega(n) = a_1 + \cdots + a_n$. It seems reasonable to ask whether every finite linear group G of degree n has an abelian subgroup A such that $\omega(|G : A|)$ is bounded in terms of $\omega(n)$. However, since for every prime p the symmetric group of degree $p + 1$ has an irreducible character of degree p, we have that such a bound does not exist. Therefore, it is perhaps surprising that we can prove the following variation of the theorems of Gluck and Isaacs-Passman.

Theorem A. *There exist (universal) constants K_1 and K_2 such that if G is a non-abelian finite group then G has an abelian subgroup A satisfying*

$$\omega(|G : A|) \leq K_1 \omega(G)^2 \log \omega(G) + K_2$$

where $\omega(G) = \max\{\omega(\chi(1)) \mid \chi \in \mathrm{Irr}(G)\}$.

Our proof of Theorem A shows that if there is not any composition factor that is a non-abelian alternating group, then the bound is quadratic in $\omega(G)$. It is likely that such a quadratic bound exists in complete generality. We will discuss this further in Section 2. However, it is not clear to us whether or not there is a linear bound.

2 Simple groups

In Corollary 2.6 of [13], the following result was proved.

Theorem 2.1. *Let G be a solvable group. Then there exists an abelian subgroup A such that*

$$\omega(|G:A|) \leq 23\omega(G).$$

This result was obtained as a consequence of Theorem A of [13], where it was proved that for G solvable there exist 19 irreducible characters $\chi_1, \dots, \chi_{19}$ such that $|G : F(G)|$ divides $\chi_1(1) \dots \chi_{19}(1)$. The goal of this section is to prove the analogue results for simple groups. We begin with the sporadic groups.

Lemma 2.2. *Let G be a sporadic simple group or the Tits group ${}^2F_4(2)'$. Then there exist $\chi_1, \chi_2, \chi_3, \chi_4 \in \mathrm{Irr}(G)$ such that $|G|$ divides $\chi_1(1)\chi_2(1)\chi_3(1)\chi_4(1)$. In particular,*

$$\omega(|G|) \leq 4\omega(G).$$

Proof. This follows from a routine but tedious inspection of the Atlas [3]. □

A more careful analysis shows that, in fact, 3 characters are enough if $G \ncong M_{22}$. However, for this group 4 irreducible characters are necessary.

Next, we consider the simple groups of Lie type. First, we deal with the classical groups.

Lemma 2.3. *Let G be a classical simple group of Lie type. Then there exist $\chi_1, \chi_2, \chi_3, \chi_4 \in \mathrm{Irr}(G)$ such that $|G|$ divides $\chi_1(1)\chi_2(1)\chi_3(1)\chi_4(1)$. In particular,*

$$\omega(|G|) \leq 4\omega(G).$$

Proof. If G is of type A_1, then G has characters of degree $q-1$, q and $q+1$ and the product of these three integers is a multiple of the order of the group. Then we may assume that G is of one of the types considered in Table 1 of [10]. We use most of the notation in [10]. In particular, $\mathbb{G}$ will be a simple linear algebraic group of adjoint type and σ an endomorphism of $\mathbb{G}$ so that the set $\mathbb{G}_\sigma$ of fixed points is finite and the derived subgroup of $\mathbb{G}_\sigma$ is isomorphic to G. By Section 12.1 of [2], we know that the degrees of the unipotent characters of $\mathbb{G}_\sigma$ and G are the same. By Corollary 11.29 of [8], we know that if $\varphi \in \mathrm{Irr}(G)$ lies under $\psi \in \mathrm{Irr}(\mathbb{G}_\sigma)$, then $\psi(1)/\chi(1)$ divides $d = |\mathbb{G}_\sigma : G|$. Let $\chi \in \mathrm{Irr}(G)$ be a semisimple character lying under $\chi_s \in \mathrm{Irr}(\mathbb{G}_\sigma)$, where χ_s is the same character as in Table 2 of [10] and let $\chi_u \in \mathrm{Irr}(G)$ be a unipotent character of degree given by the unipotent degrees in Table 2 of [10]. Then it is easy to check that $|G|$ divides $\chi(1)^2\chi_u(1)\,\mathrm{St}(1)$, where $\mathrm{St}\in \mathrm{Irr}(G)$ is the Steinberg character. The result follows. □

Now, we consider the exceptional groups of Lie type.

Lemma 2.4. *Let G be an exceptional simple group of Lie type. Then there exist $\chi_1,\chi_2,\chi_3,\chi_4,\chi_5\in \mathrm{Irr}(G)$ such that $|G|$ divides $\chi_1(1)\chi_2(1)\chi_3(1)\chi_4(1)\chi_5(1)$. In particular,*

$$\omega(|G|)\leq 5\omega(G).$$

Proof. The degrees in Table 4 of [10] correspond to degrees of unipotent characters, and are therefore degrees of the simple group G. It is routine to check that taking suitable products of degrees these characters together with $|G|_p$, the degree of the Steinberg character, yields the result. □

Finally, we will consider the alternating groups. The bound that we obtain here is much worse.

Lemma 2.5. *There exists a (universal) constant C such that if G is an alternating group of degree $n\geq 5$, then*

$$\omega(|G|)\leq C\omega(G)\log\omega(G).$$

Proof. Let i be an integer such that $5^i\leq n<5^{i+1}$. We have that

$$5^{(5^i-1)/4}=(5^i!)_5\leq (n!)_5=|G|_5.$$

By [7], we know that G has a 5-block of defect zero. Therefore, $\omega(|G|_5)\leq \omega(G)$, so

$$(5^i-1)/4\leq \omega(G)$$

and we conclude that $i\leq \log_5(4\omega(G)+1)$. Therefore,

$$n\leq 5(4\omega(G)+1)$$

and

$$\omega(|G|)<\omega(n!)\leq \omega((5(4\omega(G)+1))!)\leq \omega((25\omega(G))!).$$

Given an integer k, $\omega(k)\leq \log_2 k$, and we deduce that

$$\omega(|G|)\leq \log_2((25\omega(G))!).$$

By Stirling's formula, we know that for any integer k, $k!\leq Dk^k\exp(-k)\sqrt{2\pi n}$ for some constant D. It follows that

$$\omega(|G|)\leq C\omega(G)\log\omega(G)$$

for some constant C. This completes the proof. □

It would be interesting to decide whether, as happens with the other families of simple groups, there exists a fixed number C of characters of an alternating group such that the order of the group divides the product of the degrees C characters of $\mathrm{Alt}(n)$ is a multiple of $|\mathrm{Alt}(n)|$. In particular, is it possible to take $C = 4$?

We will use the following result.

Lemma 2.6. *Let G be a non-abelian simple group. Then there exists a non-principal irreducible character of G that extends to* $\mathrm{Aut}(G)$.

Proof. This is Lemma 4.2 of [12]. □

We conclude this section with an observation on the orders of the simple groups and their automorphism groups.

Lemma 2.7. *Let G be a simple group. Then*

$$\omega(|\mathrm{Aut}(G)|) \leq 2\omega(|G|).$$

Proof. It suffices to see that $\omega(|\mathrm{Out}(G)|) \leq \omega(|G|)$. This follows easily from [3] (using Zsigmondy's prime theorem (Theorem 6.2 of [11]) for the case of Lie type groups, for instance). □

3 Proof of Theorem A

We begin with an easy lemma.

Lemma 3.1. *The number of non-abelian simple groups in a composition series of a finite group G does not exceed $\omega(G)$.*

Proof. Let G be a minimal counterexample and let N be a maximal normal subgroup of G. Then the number of non-abelian simple groups in a composition series of N does not exceed $\omega(N) \leq \omega(G)$. We deduce that G/N is simple non-abelian. It suffices to chack that $\omega(N) < \omega(G)$.

Let $\varphi \in \mathrm{Irr}(N)$ such that $\omega(\varphi(1)) = \omega(N)$. If φ extends to $\tilde{\varphi} \in \mathrm{Irr}(G)$ then, by Gallagher's theorem (Corollary 6.17 of [8]), we have that $\tilde{\varphi}\chi \in \mathrm{Irr}(G)$ for every $\chi \in \mathrm{Irr}(G/N)$. It suffices to take any non-linear χ to see that $\omega(N) < \omega(G)$. Otherwise, it suffices to take any irreducible constituent of φ^G to see that $\omega(N) < \omega(G)$. This contradiction completes the proof. □

Next, we prove Theorem A (with a better bound) for groups with one non-abelian chief factor that is the socle. Given a group G, we write $S(G)$ to denote the socle of G.

Lemma 3.2. *There exists a universal constant E such that if G is a group with $G/S(G)$ solvable and $S(G)$ a minimal normal subgroup of G, then*

$$\omega(|G|) \leq E\omega(G) \log \omega(G).$$

Proof. We have that $G/S(G)$ is isomorphic to a solvable subgroup of

$$\mathrm{Out}(S(G)) \cong \mathrm{Out}(S) \wr S_t,$$

where S is the non-abelian simple group that appears in a composition series of G and t is the number of times that it appears. By Lemmas 2.2, 2.3, 2.4 and 2.5, we know that there exists a constant C such that $\omega(|S|) \leq C\omega(S) \log \omega(S)$. It follows that $\omega(|S(G)|) \leq C\omega(S(G)) \log \omega(S(G))$.

We may view G as a subgroup of $\mathrm{Aut}(S(G))$. Then $G \cap \mathrm{Aut}(S)^t$ is a normal subgroup of G and $G/(G \cap \mathrm{Aut}(S)^t)$ is a primitive solvable permutation group on t letters. It follows from Lemma 2.7 and the previous paragraph that

$$\omega(|G \cap \mathrm{Aut}(S)^t|) \leq 2C\omega(S(G)) \log \omega(S(G)).$$

By Gluck's theorem of primitive solvable permutation groups, we have that $G/(G \cap \mathrm{Aut}(S)^t)$ has a regular orbit on the power set of $\{1, \ldots, t\}$ except for in the few cases listed in Theorem 5.6 of [11]. In the exceptional cases, the number of prime divisors (counting multiplicites) of the order of the group does not exceed 5. Therefore we may assume, without loss of generality, that there is a regular orbit. Let $\varphi \in \mathrm{Irr}(S)$ be a non-principal character that extends to $\mathrm{Aut}(S)$ (it exists by Lemma 2.6). Thus if we take a character of $S(G)$ that is an appropriate product of copies of the principal character 1_S and φ, we have that the inertia group in G of such character is $G \cap \mathrm{Aut}(S)^t$. By Clifford's correspondence (Theorem 6.11 of [8]), we deduce that $|G/(G \cap \mathrm{Aut}(S)^t)|$ divides the degree of some irreducible character of G. It follows that

$$\omega(|G|) = \omega(|G/(G \cap \mathrm{Aut}(S)^t)|) + \omega(|G \cap \mathrm{Aut}(S)^t|) \leq \omega(G) + 2C\omega(G) \log \omega(G),$$

as desired. □

Now, we are ready to complete the proof of Theorem A.

Proof of Theorem A. Let G be a group such that $\omega(G) \leq n$. We want to see that G has an abelian subgroup A such that $\omega(|G : A|)$ is bounded by a function of the order of $n^2 \log n$.

Let G_1 be a maximal normal subgroup of G such that G/G_1 is not solvable. Then G/G_1 satisfies the hypotheses of Lemma 3.2 and we deduce that $\omega(|G/G_1|) \leq E\omega(G) \log \omega(G)$. We can do the same with G_1 and find a new subnormal subgroup G_2. Iterating this process it follows from Lemma 3.1 that G_n is solvable. We have that

$$\omega(|G : G_n|) \leq En^2 \log n.$$

Now, the result follows from Theorem 2.1. □

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