Fitting height and character degree graphs

by

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Abstract. Given a group G, $\Gamma(G)$ is the graph whose vertices are the primes that divide the degree of some irreducible character and two vertices p and q are joined by an edge if pq divides the degree of some irreducible character of G. By a definition of M. Lewis, a graph Γ has bounded Fitting height if the Fitting height of any solvable group G with $\Gamma(G) = \Gamma$ is bounded (in terms of Γ). In this note, we prove that there exists a universal constant C such that if Γ has bounded Fitting height and $\Gamma(G) = \Gamma$ then $h(G) \leq C$. This solves a problem raised by Lewis.

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1 Introduction

There are many results that show that properties of the degrees of the complex irreducible characters of finite groups give structural information of the group G. A useful tool in these problems is the graph associated to the character degrees. This graph, which we will denote $\Gamma(G)$, is defined as follows. Its vertices are the prime integers that divide the degree of some irreducible character and two vertices p and q are joined by an edge if pq divides the degree of some irreducible character. This graph has been widely studied. For the most recent account of results on this graph see [4].

This graph tends to have many edges. For instance, it is known that it has at most 3 connected components. Moreover, if G is solvable then it has at most two connected components and in the case that it has two connected components the Fitting height of the group G is at most 4. (See Theorem 19.6 of [6], for instance, for a proof of this result of P. Palfy.)

Motivated by this result, M. Lewis considered in [2] the following question: which are the graphs Γ that have *bounded Fitting height*? Here the phrase bounded Fitting height means, as one could expect, that there is a bound for the Fitting height of the solvable groups with character degree graph Γ. In that paper Lewis obtained a nice characterization of these graphs. He proved that if Γ has n vertices for some integer n, then Γ has bounded Fitting height if and only if it has at most one vertex of degree $n-1$. (Recall that the degree of a vertex is the number of edges that touch the vertex.)

The bound that Lewis obtained for the Fitting height of the groups with bounded Fitting height graph is linear in the number of vertices of the graph. More precisely he proved that $h(G) \leq 4(n-1)+2$. He suggested that it is likely that there is a constant bound and that, in fact, he was not aware of any solvable group G with bounded Fitting height graph and $h(G) > 4$. (This fact has now been formally stated as a conjecture in [4].)

Further evidence for these facts was given in the subsequent paper [3], where Lewis proved that this happens in a special case (when the vertices of Γ are the primes in $\pi_1 \cup \pi_2 \cup \{p\}$, where this is a disjoint union, π_i for $i = 1, 2$ are nonempty sets of primes and no vertex in π_1 is adjacent to any vertex in π_2). The goal of this note is to prove Lewis' suggestion in full generality. We obtain a universal constant bound for the Fitting height of groups whose degree graphs have bounded Fitting height.

Theorem A. *Let G be a solvable group with bounded Fitting height graph. Then the Fitting height of G is $h(G) \leq 49$. Furthermore, if $|G|$ is odd then $h(G) \leq 7$.*

Certainly, the bounds given in this theorem are not best possible. Thus the question of whether or not 4 is the "right" bound remains open.

Our proof of Theorem A is another application of the main results in [8]. (See [7] and the references in there for other applications of these results.) The work in [8], in the case of even order solvable groups, gives useful information only for groups of Fitting height larger than 10. This is the reason why it is not possible to obtain sharp bounds with our techniques.

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2 Preliminary results

We begin by recalling the results from [8] that we will use. Recall that given a group G, $F(G)$ is the Fitting subgroup of G and we define the Fitting series of G by means of $F_1(G) = F(G)$ and $F_{i+1}(G)/F_i(G) = F(G/F_i(G))$ for $i \geq 1$. The Fitting height of the solvable group G, which we denote $h(G)$, is the smallest integer i such that $F_i(G) = G$.

Theorem 2.1. *Let G be a solvable group. Then there exists $\chi \in \mathrm{Irr}(G)$ such that $|G : F_{10}(G)|$ divides $\chi(1)$. Furthermore, if $|G|$ is odd, then we may take $\chi \in \mathrm{Irr}(G)$ such that $|G : F_3(G)||F_2(G) : F(G)|$ divides $\chi(1)$.*

Proof. This is part of Theorems C and D of [8]. □

We will also use the following deep result. As usual, if N is a normal subgroup of a group G and $\theta \in \mathrm{Irr}(N)$, we write $\mathrm{Irr}(G|\theta)$ to denote the set of irreducible constituents of θ^G.

Theorem 2.2 (Generalized Gluck-Wolf). *Let G be a solvable group, N be a normal subgroup of G and π a set of primes. Let $\theta \in \mathrm{Irr}(N)$ and assume that $\chi(1)/\theta(1)$ is a π'-number for all $\chi \in \mathrm{Irr}(G|\theta)$. Then G/N has an abelian Hall π-subgroup. In particular, G/N has π-length at most one.*

Proof. See Theorem 12.9 of [6] or [5]. □

We notice that this theorem is also implicitly used in the proof of Theorem 3 of [2]. Lewis's proof depends on a result of J. Pense [9], which in turn relies on the Gluck-Wolf Theorem.

We will need Lewis's characterization of graphs with bounded Fitting height.

Theorem 2.3. *Let Γ be a graph with n vertices. Then Γ has bounded Fitting height if and only if it has at most one vertex of degree $n-1$.*

Proof. This is the Main Theorem of [2]. □

In fact, we will just use the "only if" part of this theorem which is quite straightforward: it suffices to observe that there are $\{p,q\}$-groups of arbitrarily large Fitting height and that if Γ has two primes p and q of degree $n-1$ then for any group G with $\Gamma(G)=\Gamma$, the direct product $G\times R$, for any $\{p,q\}$-group R, has the same degree graph.

Finally, we need the following variations of Theorem 1 and Corollary 2 of [2]. Given an integer n, we write $\pi(n)$ to denote the set of prime divisors of n. For brevity, if G is a group, we will write $\pi(G)=\pi(|G|)$. Given a set of primes π, we write $l_\pi(G)$ to denote the π-length of G. If π is the empty-set, we will put $l_\pi(G)=0$ for any group G.

Lemma 2.4. *Let $m\geq 1$ be an integer. Let G be a solvable group and assume that $\pi(G)=\pi_1\cup\cdots\cup\pi_m$ and that G has nilpotent Hall π_i-subgroups for every i. Then*

$$h(G)\leq\sum_{i=1}^{m} l_{\pi_i}(G).$$

Proof. We argue by induction on $|G|$. Since $h(G)=h(G/\Phi(G))$ (by Theorem 1.12 of [6], for instance), we may assume that $\Phi(G)=1$, i.e., $F=F(G)$ is the direct product of the minimal normal subgroups of G. Assume first that there are two different minimal normal subgroups N_1 and N_2. Since G embeds into $G/N_1\times G/N_2$, there exists $i\in\{1,2\}$ such that $h(G)=h(G/N_i)$. Now the result follows from the inductive hypothesis. Hence, we may assume that F is the unique minimal normal subgroup of G. Assume that F is a p-group for some prime $p\in\pi_1$. Since G has a nilpotent Hall π_1-subgroup, we have $F=O_{\pi_1}(G)$. We also know that $O_{\pi_1'}(G)=1$, so $l_{\pi_1}(G)=l_{\pi_1}(G/F)+1$. Also, $l_{\pi_i}(G/F)\leq l_{\pi_i}(G)$ for $i\neq 1$. By the inductive hypothesis,

$$h(G)=h(G/F)+1\leq\sum_{i=1}^{m} l_{\pi_i}(G/F)+1\leq\sum_{i=1}^{m} l_{\pi_i}(G),$$

as desired. □

Lemma 2.5. *Let G be a solvable group and assume that $\pi(G)=\pi_1\cup\cdots\cup\pi_n$. Then*

$$l_{\pi_n}(G)\leq\sum_{i=1}^{n-1} l_{\pi_i}(G)+1.$$

Moreover, this bound can be improved by one if $O_{\pi_n}(G)=1$.

Proof. We argue by induction on $|G|$. If G is a π_n-group, then the result is obvious. We assume that G is not a π_n-group. Assume first that $O_{\pi_n}(G)=$

1. Suppose that G has at least two different minimal normal subgroups N_1 and N_2. For $i = 1, 2$, let $M_i/N_i = O_{\pi_n}(G/N_i)$. We claim that $M_1 \cap M_2 = 1$. Otherwise, there exists a minimal normal subgroup L of G that is contained in $M_1 \cap M_2$. But notice that for $i = 1, 2$, N_i is the unique minimal normal subgroup of G contained in M_i. This implies that L is both N_1 and N_2. This contradiction proves the claim. Hence G embeds in $G/M_1 \times G/M_2$. Using the inductive hypothesis, we deduce that

$$\begin{aligned} l_{\pi_n}(G) &\leq \max\{l_{\pi_n}(G/M_1), l_{\pi_n}(G/M_2)\} \\ &\leq \max\{\sum_{i=1}^{n-1} l_{\pi_i}(G/M_1), \sum_{i=1}^{n-1} l_{\pi_i}(G/M_2)\} \\ &\leq \sum_{i=1}^{n-1} l_{\pi_i}(G), \end{aligned}$$

as desired.

Hence, we may assume that G has a unique minimal normal subgroup M. Thus there is a prime $p \notin \pi_n$ such that $F(G)$ is a p-group. In particular, $F = F(G)$ is a π_j-group for some $j \neq n$. This implies that $O = O_{\pi_j}(G) > 1$ and $O_{\pi_j'}(G) = 1$. By the inductive hypothesis,

$$l_{\pi_n}(G) = l_{\pi_n}(G/O) \leq \sum_{i=1}^{n-1} l_{\pi_i}(G/O) + 1.$$

It is easy to see that $l_{\pi_j}(G) = l_{\pi_j}(G/O) + 1$. For every $i \neq j$, we have $l_{\pi_i}(G) \geq l_{\pi_i}(G/O)$. Combining these formulas, the result follows.

Now, assume that $O_{\pi_n}(G) > 1$. In this case, it suffices to apply the inductive hypothesis to $G/O_{\pi_n}(G)$ to deduce the result. □

Corollary 2.6. *Let $m \geq 1$ be an integer. Let G be a solvable group and assume that $\pi(G) = \pi_1 \cup \cdots \cup \pi_m \cup \pi_{m+1}$ and that $|\pi_{m+1}| \leq 1$. Assume also that G has abelian Hall π_i-subgroups for $1 \leq i \leq m$. Then*

$$h(G) \leq 2m + 1.$$

Proof. Since G has abelian Hall π_i-subgroups for $i \leq m$, we have $l_{\pi_i}(G) \leq 1$ for $i \leq m$. Hence using Lemmas 2.4 and 2.5, we have

$$h(G) \leq m + l_{\pi_{m+1}}(G) \leq m + \sum_{i=1}^{m} l_{\pi_i}(G) + 1 \leq 2m + 1,$$

as desired. □

3 Proof of Theorem A

Now, we are ready to complete the proof of Theorem A. We will write $\rho(G)$ to denote the set of prime integers that divide the degree of some irreducible character of G, i.e., $\rho(G)$ is the set of vertices of $\Gamma(G)$. Notice that by Ito's theorem (Corollary 12.34 of [1]), if G is solvable then $\rho(G)$ is the set of prime divisors of $|G|$ for which G does not have a normal abelian Sylow subgroup. This implies that $\rho(G) = \pi(G/F(G)) \cup \rho(F(G))$.

Proof of Theorem A. Put $F_0 = 1$, $F = F(G)$ and $F_i = F_i(G)$ for $i \geq 1$. Since F is nilpotent, there exists $\beta_1 \in \mathrm{Irr}(F)$ such that $\pi(\beta_1(1)) = \rho(F)$. By page 254 of [6], there exist $\beta_i \in \mathrm{Irr}(F_i)$ such that

$$\pi(|F_i : F_{i-1}|) = \pi(\beta_i(1))$$

for all $2 \leq i \leq 10$. This implies that

$$\rho(F_{10}) = \pi(\beta_1(1) \dots \beta_{10}(1)).$$

On the other hand, by Theorem 2.1, there exists $\psi \in \mathrm{Irr}(G)$ such that $|G : F_{10}|$ divides $\psi(1)$. In particular,

$$\rho(G) = \pi(\psi(1)) \cup \pi(\beta_1(1)) \cup \dots \cup \pi(\beta_{10}(1)).$$

For $i = 1, \dots, 10$, we define

$$\pi_i = \{p \mid |G : F_{10}| \mid p \text{ divides } \chi(1)/\beta_i(1) \text{ for some } \chi \in \mathrm{Irr}(G|\beta_i)\}.$$

By the Generalized Gluck-Wolf Theorem, G/F_{10} has an abelian Hall π_i'-subgroup for every $i = 1, \dots, 10$.

Assume that p belongs to π_i for all $i = 1, \dots, 10$. Then there exist $\chi_i \in \mathrm{Irr}(G|\beta_i)$ such that p divides $\chi_i(1)/\beta_i(1)$ for all i. Also, p divides $\psi(1)$. This means that p is joined to all the other primes in $\rho(G)$, i.e., p is a vertex of $\Gamma(G)$ of degree $|\rho(G)| - 1$. By Theorem 2.3, p is the unique such vertex. This means that

$$|\bigcap_{i=1}^{10} \pi_i| \leq 1,$$

so $\bigcup_{i=1}^{10} \pi_i'$ contains all the primes in $\pi(G/F_{10})$ except for at most one of them. If there is one prime missing, let this prime be q. Otherwise, choose a prime q that does not divide $|G/F_{10}|$. Set $Q \in \mathrm{Syl}_q(G/F_{10})$ and put $\tau_i = \pi_i' \cap \pi(G/F_{10})$ for every i. We can write $G/F_{10} = H_1 \dots H_{10} Q$, where H_i is an abelian Hall τ_i-subgroup of G/F_{10} for $i = 1, \dots, 10$. In particular, G/F_{10} has τ_i-length at most one for every i. By Corollary 2.6, we have

$$h(G) = h(G/F_{10}) + 10 \leq (2 \cdot 10 + 1) + 10 = 31.$$

This concludes the proof of the first part of the theorem.

For the second part, we argue in a similar way using the more precise results obtained for odd order groups in [8].

Now we have that $|G|$ is odd. As before, there exists $\beta \in \mathrm{Irr}(F)$ such that $\pi(\beta(1)) = \rho(F)$. By Theorem 2.1, there exists $\psi_1 \in \mathrm{Irr}(G)$ such that $|G : F_3||F_2 : F|$ divides $\psi_1(1)$. Applying Theorem 2.1 to the group G/F, we get that there exist $\psi_2 \in \mathrm{Irr}(G)$ such that $|G : F_4||F_3 : F_2|$ divides $\psi_2(1)$. In particular,

$$\rho(G) = \pi(\psi_1(1)) \cup \pi(\psi_2(1)) \cup \pi(\beta(1))$$

and the prime divisors of $|G : F_4|$ are joined to all the vertices in $\Gamma(G)$ except for possibly those that divide $\beta(1)$. Put

$$\pi = \{p \mid |G : F_4| \mid p \text{ divides } \chi(1)/\beta(1) \text{ for some } \chi \in \mathrm{Irr}(G|\beta)\}.$$

By the Generalized Gluck-Wolf Theorem, G has an abelian Hall π'-subgroup.

It is clear that if p belongs to π, then p is joined in $\Gamma(G)$ to all the prime divisors of $\beta(1)$. In particular, $|\pi| \leq 1$. Since G has an abelian Hall π'-subgroup, we may assume that π consists of one prime, say q. Thus $G/F_4 = QA$, where Q is a Sylow q-subgroup of G/F_4 and A is an abelian Hall q'-subgroup of G/F_4.

By Corollary 2.6, we have

$$h(G) = h(G/F_4) + 4 \leq 2 + 1 + 4 = 7.$$

This concludes the proof of the theorem. □

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