An elementary method for the calculation of the set of character degrees of some p-groups

by

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Abstract. Let q be a power of a prime number p. In this work, we present an elementary method to find out the set of irreducible character degrees of the Sylow p-subgroups of $Sp(2n, q)$ when q is odd. This simplifies the proof of Previtali [4], where this result was first obtained. This method also works for the groups of unitriangular matrices over finite fields.

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1 Introduction

Let q be a power of a prime number p. A. Previtali [4] has found the set of irreducible (complex) character degrees of the Sylow p-subgroups of $GL(n, q)$ and $Sp(2n, q)$ (q odd or $q = 2$ in the symplectic case). (The linear case had been previously solved by B. Huppert [1].) These are the steps of his proof:

(i) Decompose these groups as a semidirect product (with an elementary abelian kernel).

(ii) Calculate the set of lengths of the orbits of the action of the complement on the kernel. This is a hard step, and the main interest of our method is that we are able to skip it.

(iii) From these lengths, obtain the set of lengths of the orbits of the dual action.

(iv) Using the fact that these groups are q-power-degree groups [3], deduce the result.

The method we present here works for both the linear and the symplectic cases but, since in the linear case an easy proof had already been provided by Huppert [1], we will illustrate it just in the symplectic case. After decomposing a Sylow p-subgroup of $Sp(2n, q)$, where q is a power of the odd prime p, we give a suitable description of the action of the complement on the set of irreducible characters of the kernel. This description will allow us to find orbits of all the desired lengths and we will deduce the set of irreducible character degrees of these groups.

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2 Preliminary results

We begin with the statement of an easy general lemma about the degrees of the irreducible characters of semidirect products. All the groups will be finite. As usual, if G is a group $\mathrm{cd}(G)$ denotes the set of degrees of the irreducible characters of G.

Lemma 2.1. *Let B be a group which acts on an abelian group A and let $R = B[A]$ be the semidirect product of B and A. Then the set of degrees of the irreducible characters of R is the set of numbers $\beta(1)|B : I_B(\lambda)|$ where λ runs over $\mathrm{Irr}(A)$ and β runs over $\mathrm{Irr}(I_B(\lambda))$. In particular, the numbers $|B : I_B(\lambda)|$ where λ runs over $\mathrm{Irr}(A)$ are contained in $\mathrm{cd}(R)$.*

Proof. This is Lemma 2.3 of [4]. □

In the following, q will be a power of an odd prime p. Let G_n be a Sylow p-subgroup $Sp(2n, q)$. Now, we give a well-known decomposition of these groups. In the remaining, $U_n(q)$ will denote the group of lower unitriangular matrices over the finite field with q elements and $S_n(q)$ the additive group of symmetric matrices. Also, for any matrix M, M^{-t} represents the inverse transpose of M.

Proposition 2.2. *The group G_n is isomorphic to the semidirect product $K_n[L_n]$ where*

$$L_n = \left\{ \begin{pmatrix} I_n & 0 \\ A & I_n \end{pmatrix} \mid A \in S_n(q) \right\}$$

and

$$K_n = \left\{ \begin{pmatrix} T & 0 \\ 0 & T^{-t} \end{pmatrix} \mid T \in U_n(q) \right\}.$$

The action is given by

$$\begin{pmatrix} T^{-1} & 0 \\ 0 & T^t \end{pmatrix} \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \begin{pmatrix} T & 0 \\ 0 & T^{-t} \end{pmatrix} = \begin{pmatrix} I & 0 \\ T^tAT & I \end{pmatrix}.$$

This way we have $G_n \cong U_n(q)[S_n(q)]$ with the action given by $A^T = T^tAT$. This is the description of the group G_n that we will use to proof our main result.

3 The action on the irreducible characters of the group $S_n(q)$

In general, if $R = B[A]$, then the action by conjugation of B on A induces an action of B on $\mathrm{Irr}(A)$ given by $\lambda^b(a) = \lambda(a^{b^{-1}})$ for every $a \in A$, $b \in B$ and $\lambda \in \mathrm{Irr}(A)$. The aim of this section is to show that in our case all the q-powers that divide the order of the complement are lengths of some orbit under this action. Since the kernel is an $\mathbb{F}_q$-vector space, the actions of the complement on the irreducible characters of the kernel and the natural action on the dual space are equivalent. We will study this second action. Our first aim is to give a description of this action as an action between matrix groups.

Let $f \in S_n(q)^*$ and $A = (a_{ij}) \in S_n(q)$. Then there exist scalars b_{ij} for $i \leq j$ such that

$$f(A) = \sum_{i \leq j} a_{ij} b_{ij}.$$

Observe that this means that $f(A) = tr(AB)$, where B is the symmetric matrix whose (i,j) entry is b_{ii} if $i = j$ and $b_{ij}/2$ otherwise.

Now let $T \in U_n(q)$ and $A \in S_n(q)$. We have that

$$\begin{aligned} f^T(A) &= f(A^{T^{-1}}) = f(T^{-t}AT^{-1}) \\ &= tr(T^{-t}AT^{-1}B) = tr(AT^{-1}BT^{-t}). \end{aligned}$$

So, we can identify the natural action of $U_n(q)$ on the dual space $S_n(q)^*$ with the action on the group of symmetric matrices given by the formula $B^T = T^{-1}BT^{-t}$. We must admit that this notation is somewhat inconsistent since it coincides with the notation we used for the action of $U_n(q)$ on $S_n(q)$. However, this will create no harm because the unique action that will appear in the remaining is the one of $U_n(q)$ on $S_n(q)^*$.

Proposition 3.1. *For every divisor q^e of $|U_n(q)| = q^{n(n-1)/2}$, there is an orbit of size q^e under this action.*

Proof. It suffices to show that for every integer m with $0 \leq m \leq n(n-1)/2$, there exists a matrix $B \in S_n(q)$ such that the stabilizer of B in $U_n(q)$ has order q^m. Let B be a diagonal matrix with zeros and ones along the diagonal. Then, of course $B \in S_n(q)$ and we compute its stabilizer. A matrix $T \in U_n(q)$ stabilizes B precisely when $TB = BT^{-t}$. But TB is lower triangular and BT^{-t} is upper triangular, and thus these matrices can be equal only if they are diagonal.

If TB is diagonal, we claim that $TB = B$. We prove this by showing that the j th column of TB and B agree for every choice of j with $1 \leq j \leq n$. If the j th column of B is zero, then so is the j th column of TB, and so we can suppose that the j th column of B is nonzero. In this case, since the j th entry of this column is 1, it follows that the j th column of TB equals the j th column of T. But we are assuming that TB is diagonal, and so the j th column of T can be nonzero only in position j, where it must be 1 since T

lies in $U_n(q)$. In other words, the j th column of TB equals the j th column of B in this case too.

We now know that $TB = BT^{-t}$ only when $TB = B$. Conversely, if $TB = B$, then $BT^t = B$, and so $TB = B = BT^{-t}$. Thus T stabilizes B in the action precisely when $TB = B$. We saw in the previous paragraph, however, that the j th columns of TB and B are equal if the j th column of B is zero, but that if the j th column of B is nonzero, then $TB = B$ precisely when the j th column of T agrees with the j th column of B. It follows that the number of matrices T that stabilize B is precisely q^m, where m is the total number of below-diagonal positions in all zero columns of B. As B varies, therefore, we see that m runs over all integers that can be written as a sum of distinct integers chosen from the set $\{1, 2, 3, \ldots, n-1\}$. It is clear that this includes all integers between 0 and $n(n-1)/2$ inclusive, and this completes the proof. □

4 Main result

Now the proof of the main result is straightforward.

Theorem 4.1. *Let G_n be a Sylow p-subgroup of $Sp(2n, q)$, where q is odd. Then* $\mathrm{cd}(G_n) = \{q^j \mid 0 \leq j \leq \binom{n}{2}\}$.

Proof. The inclusion $\supseteq$) follows from the equivalence between the natural action of the complement on the dual space of the kernel of G_n and the action on the irreducible characters of the kernel of G_n, the proposition of the previous section and Lemma 2.1. Now the equality holds by the fact that these groups are q-power-degree groups [3, 5] and by Ito's theorem stating that the degree of an irreducible character divides the index of an abelian normal subgroup. $\square$

Remarks. (i) This result is false over fields of even characteristic, since I.M. Isaacs [3] has constructed characters of degree $q/2$.

(ii) Our method allows us to construct characters of all the degrees. We should start by finding the irreducible characters of the kernel associated to the matrices we have defined. After that it is enough to extend those characters to their inertia subgroups (this is possible by Problem 6.18 of [2]) and induce the extended characters to the whole group G_n.

References

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