A dual version of Huppert's ρ-σ conjecture

by

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Abstract

Huppert's ρ-σ conjecture asserts that any finite group has some character degree that is divisible by "many" primes. In this note, we consider a dual version of this problem, and we prove that for any finite group there is some prime that divides "many" character degrees.

1 Introduction

One of the main problems on character degrees of finite groups is Huppert's ρ-σ conjecture, which roughly asserts that any finite group G has some (complex) irreducible character whose degree is divisible by "many", say one-third, of the primes that divide some character degree of G. More precisely, let $\rho(G)$ be the set of primes that divide some character degree of G and let $\sigma(G)$ be the maximum number of different primes that divide any given character degree of G. The weak ρ-σ conjecture asserts that there exists an integer valued function g such that for any finite group G, $|\rho(G)| \leq g(\sigma(G))$. The strong ρ-σ conjecture asserts that $|\rho(G)| \leq 3\sigma(G)$, or perhaps even there exists some additive constant C such that for any finite group G, $|\rho(G)| \leq 2\sigma(G) + C$ and C can be taken to be 0 is G is solvable (see p. 220 and Remarks 17.9(a) of [11]). The strong ρ-σ conjecture remains open. The weak ρ-σ conjecture was finally settled in [14].

In this note, we consider a sort of converse to Huppert's ρ-σ conjectures: is it true that for any finite group G there is some prime that divides "many" of the character degrees of G? Again, one can state two different versions of this problem. We say that a finite group G satisfies *property* P_k if for any prime p the number of different character degrees of G that are divisible by p is at most k. The weak form asserts that there exists an integer valued function f such that for any finite group that satisfies property P_k, the number of character degrees of G is at most $f(k)$. The strong form could assert that the number of character degrees of a nonabelian group that satisfies property P_k is at most $4k$ or even $3k+C$ for some universal constant C, that can be taken to be 0 if G is solvable. In other words, given any nonabelian finite group there is one prime that divides at least one-fourth of the character degrees. A family of examples showing that the bound $|\operatorname{cd}(G)| \leq 3k$ would be sharp for infinitely many values of k and solvable groups was constructed in [1]. (As usual, we write $\operatorname{cd}(G)$ to denote the set of degrees of the irreducible characters of G.) This bound was suggested in [1] and conjectured in [13]. It was proved in [1] for $k \leq 3$ and in [13] for $k = 4$. As the groups $\mathrm{L}_2(2^f)$ show, this bound does not hold for arbitrary finite groups.

Regarding the weak form of the problem it was proved in [1] that for solvable groups there is a quadratic bound. This was later improved in McVey's Ph.D. Thesis [12] but his bound is still quadratic. In this article, we consider the problem for arbitrary finite groups for the first time, and settle the weak form of the problem.

Theorem A. *There exists an integer-valued function f such that if G is a finite group with the property that for any prime p, there are at most k members of $\mathrm{cd}(G)$ that are divisible by p, then $|\mathrm{cd}(G)| \leq f(k)$.*

Our methods would allow us to give an explicit bound. However, this would probably be far from best possible so, for the sake of simplicity, we have refrained from doing so. As usual, we will say that a quantity is k-*bounded* if it is bounded by some real valued function that depends only on k.

Now, we explain the structure of this paper. In Section 2 we prove several elementary results that will be necessary in the proof of Theorem A. In Section 3 we prove Theorem A and several related results for simple groups and in Section 4 we complete the proof of Theorem A.

2 Preliminary results

We recall the following from the introduction:

Definition 2.1. We say that a finite group G satisfies *property P_k* if for any prime p the number of different character degrees of G that are divisible by p is at most k.

Our first result will be used many times in the remainder of this paper without further explicit mention.

Lemma 2.2. *Let N be a normal subgroup of a finite group G. If G satisfies property P_k, then G/N satisfies property P_k.*

Proof. This is an immediate consequence of the fact that $\mathrm{cd}(G/N) \subseteq \mathrm{cd}(G)$. □

The next three lemmas relate character degrees of a group and a normal subgroup.

Lemma 2.3. *Let G be a finite group with normal subgroup N. Then $|\mathrm{cd}(G)| \leq |\mathrm{cd}(N)|\Omega(|G/N|)$, where $\Omega(n)$ is the number of divisors of the integer n.*

Proof. This follows from Corollary 11.29 of [4], for instance. □

Given a normal subgroup N of a finite group G and $n \in \mathrm{cd}(N)$ we write $\mathrm{cd}(G|N, n)$ to denote the set of character degrees of G that lie over characters of N of degree n.

Lemma 2.4. *Let G be a finite group satisfying property P_k. Let N be a normal subgroup of G and $1 < n \in \mathrm{cd}(N)$. Then $|\mathrm{cd}(G|N, n)| \leq k$.*

Proof. This is a straightforward consequence of the fact that all the members of $\mathrm{cd}(G|N, n)$ are multiples of n. □

Property P_k is not inherited by normal subgroups, but we have the following.

Lemma 2.5. *Let G be a finite group satisfying property P_k. If N is a normal subgroup of G and the number of divisors of $|G : N|$ is l, then N satisfies property P_{kl}*

Proof. Assume that a prime p divides $kl + 1$ character degrees of N. It follows from property P_k that the number of character degrees of G lying over these $kl + 1$ character degrees of N is at most k. This means that for some of these at most k degrees of G, the number of character degrees of N lying under it is bigger than l. But this is impossible. □

The next lemma will allow us to "forget" the character degrees that are multiples of a given finite number of primes.

Lemma 2.6. *Let G be a finite group satisfying property P_k. If π is a finite set of primes and G has m character degrees of π'-degree, then $|\mathrm{cd}(G)| \leq m + |\pi|k$.*

Proof. For any prime $p \in \pi$, G has at most k character degrees that are divisible by p. If $d \in \mathrm{cd}(G)$ is not divisible by any prime in π, then it has π'-degree and the result follows. □

Our last result in this section is probably known. Since we have not been able to find a reference in the literature, we include a short proof, which is based on the ideas of Lemma 2.5 of [14]. If a group G acts on a set Ω, we say that $X \subseteq \Omega$ lies in a regular orbit if the size of its orbit is $|G/\mathrm{Ker}_G\,\Omega|$.

Lemma 2.7. *Assume that a finite abelian group G acts on a set Ω. Then G has a regular orbit on $\mathcal{P}(\Omega)$.*

Proof. We argue by induction on $|\Omega|$. Assume first that the action is not transitive and let $\Omega = \Omega_1 \cup \Omega_2$ be a decomposition of Ω as the union of two proper disjoint G-invariant subsets. By the inductive hypothesis, there exist $X_1 \subseteq \Omega_1$ and $X_2 \subseteq \Omega_2$ in regular orbits under the action of $G/\operatorname{Ker}_G \Omega_1$ and $G/\operatorname{Ker}_G \Omega_2$, respectively. Let $X = X_1 \cup X_2$. Hence $X \subseteq \Omega$ lies in a regular orbit under the action of G. Therefore we may assume that the action of G on Ω is transitive. Also, we may assume that this action is faithful.

If G is primitive on Ω, then it is of prime order and the statement is trivial. Now, assume that G is imprimitive and let $\mathcal{B} = \{B_1, \ldots, B_t\}$ be a nontrivial system of blocks. For $1 \leq i \leq t$, let G_{B_i} be the stabilizer of the block B_i. Then $G/\cap_i G_{B_i}$ is a permutation group on $\mathcal{B}$. By the inductive hypothesis, it has a regular orbit on $\mathcal{P}(\mathcal{B})$, say the orbit of $\{B_1, \ldots, B_s\}$, where $s < t$. Also, for every i, there exists $X_i \subseteq B_i$ such that X_i lies in a regular orbit under the action of G_{B_i}. Hence, $X_1 \cup \cdots \cup X_s$ lies in a regular orbit under the action of G, as desired. □

3 Simple groups

In this section we prove Theorem A and some related results for simple groups. All results except for Proposition 3.7, Corollary 3.8 and Lemma 3.9 are of an asymptotic nature. Hence, in their proofs, we may (and will) ignore any finite number of simple groups, like the sporadic groups, the Tits group, groups of Lie type with exceptional Schur multiplier, and alternating groups $\mathfrak{A}_n$ for all $n \leq n_0$, for some bound n_0.

Proposition 3.1. *For all $n \geq 3$ we have:*

$$n/2 \leq \min\{k \mid \mathfrak{A}_n \textit{ satisfies } P_k\} \leq |\operatorname{cd}(\mathfrak{A}_n)| \leq n!$$
$$n/2 \leq \min\{k \mid \mathfrak{S}_n \textit{ satisfies } P_k\} \leq |\operatorname{cd}(\mathfrak{S}_n)| \leq n!$$

Proof. The irreducible characters of the symmetric group $\mathfrak{S}_n$ are parameterized by partitions λ of n, and their degrees are given by the so-called hook formula. In particular, for the hook partitions $\lambda_k = (n-k, 1^k)$ of n, where $0 \leq k < n/2$, the degree of χ_λ is equal to $\binom{n}{k}$. It is obvious that $\binom{n}{k}$ are all different for $0 \leq k < n/2$. An irreducible character χ_λ of $\mathfrak{S}_n$ remains irreducible upon restriction to the alternating group $\mathfrak{A}_n$ if λ is not self-dual. Again, it is obvious that the hook partitions are not self-dual, for $0 \leq k < n/2$. Thus, $\min\{k \mid \mathfrak{A}_n \text{ satisfies } P_k\} \geq n/2$ as claimed. The remaining inequalities are obvious. □

The bounds obtained in this proof are very weak, and much better bounds could be established.

We next prove a similar statement for the character degrees of groups of Lie type. The last inequality has been obtained in Theorem 1.5 of [7], but for completeness we include a proof.

Proposition 3.2. *There exists a monotonous function $f_2 : \mathbb{N} \longrightarrow \mathbb{N}$ such that whenever G is a finite group of Lie type of rank n then*

$$f_1(n) := n/2 - 5 \leq \min\{k \mid G \text{ satisfies } P_k\} \leq |\operatorname{cd}(G)| \leq f_2(n).$$

Proof. We consider the following setup. Let $\mathbf{G}$ be a simple algebraic group defined over a finite field $\mathbb{F}_q$ and $F : \mathbf{G} \to \mathbf{G}$ the corresponding Frobenius endomorphism. Then the group of fixed points $G := \mathbf{G}^F$ is, by definition, a finite group of Lie type.

The fundamental results of Lusztig [8] on the classification of the complex irreducible characters of finite groups of Lie type show the following. The set $\operatorname{Irr}(G)$ is partitioned into so-called Lusztig series $\mathcal{E}(G, s)$, where s runs over a system of representatives of semisimple elements in the dual group G^* of G. Furthermore, the degrees of characters $\chi \in \mathcal{E}(G, s)$ only depend on the structure of $C_{G^*}(s)$. In particular, $\operatorname{cd}(G)$ is controlled by the various $C_{G^*}(s)$. Fix $s \in G^*$ and let $C := C_{G^*}(s)$. The connected component C° of C is a connected reductive subgroup of G^*, hence completely determined by its root system. Now C/C° is a subgroup of the relative Weyl group in G^* of the root system of C°, which is a section of the Weyl group of G^*. In particular, for a given C° there is only a finite number of possible C's.

Now, for a given root system of G (and hence of G^*) there only exist a finite number of sub-root systems which can give rise to a connected centralizer C°. Hence, the number of different connected centralizers is bounded by a function in the root system of G. Our previous argument then shows that the number of different centralizers $C_{G^*}(s)$, up to conjugation, is bounded by a function in the root system of G. Finally, the number of characters in $\mathcal{E}(G, s)$ only depends on the structure of $C_{G^*}(s)$, and thus $|\operatorname{cd}(G)|$ is bounded in terms of a function in the root system of G, independent of the order of the underlying field. In turn, for a fixed rank n there only exist finitely many different root systems of algebraic groups of rank n, hence we have established the existence of the upper bound f_2.

The relation between $\min\{k \mid G \text{ satisfies } P_k\}$ and $|\operatorname{cd}(G)|$ is trivial. For the lower bound we consider the elements of $\mathcal{E}(G, 1)$, the so-called unipotent characters. By Lusztig's description of $\mathcal{E}(G, 1)$, the cardinality $|\mathcal{E}(G, 1)|$ only depends on the root system of G, and it tends to infinity with the rank of G. Indeed, there is a subset E of $\mathcal{E}(G, 1)$, the so-called principal series unipotent characters, which is in bijection with $\operatorname{Irr}(W)$, such that that the degree polynomials of characters in E specialize to the character degrees in $\operatorname{Irr}(W)$. In particular, there are at least as many different degrees of unipotent characters as there are different degrees in $\mathrm{Irr}(W)$. For the finitely many exceptional types, the lower bound is now easily proved by inspection.

So now assume that G is of classical type. Then, if G has rank n, $\mathrm{Irr}(W)$ has the symmetric group $\mathfrak{S}_n$ as a quotient, so indeed we find at least $n/2$ different character degrees by Proposition 3.1. Moreover, all except possibly five of the degrees of unipotent characters are divisible by the defining characteristic of G (see [10, Lemma 2.12], for example). So we have $n/2 - 5 \leq \min\{k \mid G \text{ satisfies } P_k\}$. This proves the validity of the lower bound. $\square$

Lemma 3.3. *If G is a quasisimple group of Lie type, then $|\operatorname{cd}(G)|$ is bounded by some function of $|\operatorname{cd}(G/Z(G))|$.*

Proof. We keep the setup from the proof of Proposition 3.2. If $\mathbf{G}$ is chosen to be simply-connected, then the group of fixed points $G := \mathbf{G}^F$ is the universal covering group of the simple group $S := G/Z(G)$, except for a finite number of cases, and all finite simple groups of Lie type occur among as such an S, except for the Tits group. Clearly, $\mathrm{Irr}(S) \subset \mathrm{Irr}(G)$ in a natural way, and in order to prove the assertion, it is enough to establish a function bounding $|\operatorname{cd}(G)|$ in terms of $|\operatorname{cd}(S)|$. Now the unipotent characters $\mathcal{E}(G, 1)$ have $Z(G)$ in their kernel, so descend to characters of S. Thus, if n denotes the rank of $\mathbf{G}$, then by Proposition 3.2 we have $|\operatorname{cd}(S)| \geq n/2 - 5$, which tends to infinity with n, while $|\operatorname{cd}(G)| \leq f_2(n)$. So our claim follows. $\square$

Lemma 3.4. *There exists an integer valued function f_3 such that if an almost simple group G with socle S satisfies property P_k, then S satisfies property $P_{f_3(k)}$.*

Proof. First consider groups of Lie type. By Proposition 3.2 the minimal k such that S has property P_k is bounded above by a function only depending on the rank of G. Hence, we may assume that the rank is larger than 8, so that G is of classical type. Now the unipotent characters of S are invariant under diagonal and field automorphisms of S by Lusztig's results [8]. Furthermore, the subgroup of $\mathrm{Aut}(S)$ generated by inner, diagonal and field automorphisms has index at most 6 in $\mathrm{Aut}(S)$. (The factor group consists of the graph automorphisms, of which there are at most six.)

Since all but at most five unipotent characters have degree divisible by p, the defining characteristic of S (see [10, Lemma 2.12]), and the number of different unipotent character degrees goes to infinity with the rank, this proves the claim for groups of Lie type.

For alternating groups, the claim follows directly from Proposition 3.1 $\square$

Lemma 3.5. *There exists an integer valued function f_4 such that whenever $S \le G \le \mathrm{Aut}(S)$, where S is a nonabelian simple group, and G satisfies property P_k, then $|\operatorname{cd}(G)| \le f_4(k)$.*

Proof. This is just Proposition 3.1 for alternating groups. We may hence assume that S is of Lie type. Assume that G is almost simple with socle S and satisfies P_k. Then S satisfies $P_{f_3(k)}$ by Lemma 3.4, so $f_3(k) \ge n/2 - 5$ by Proposition 3.2, where n denotes the rank of S. Thus $n \le 2(f_3(k)+5)$. By Lemma 2.4 we also have that

$$|\operatorname{cd}(G)| \le k(|\operatorname{cd}(S)| - 1) + |\operatorname{cd}(G/S)|.$$

Furthermore, $|\operatorname{cd}(S)| \le f_2(n)$, where n denotes the Lie-rank of S, again by Proposition 3.2. Since G/S is solvable and satisfies property P_k, [1, Th. A] implies that the quantity $|\operatorname{cd}(G/S)|$ is bounded by a function $g(k)$. (Alternatively, this follows directly from the explicit knowledge of the structure of $\mathrm{Out}(S)$.) Hence

$$\begin{aligned} |\operatorname{cd}(G)| \le& k(|\operatorname{cd}(S)| - 1) + |\operatorname{cd}(G/S)| \\ \le& k\, f_2(n) + g(k) \le k\, f_2(2(f_3(k)+5)) + g(k) \end{aligned}$$

is k-bounded. □

Lemma 3.6. *There exists an integer valued function f_5 such that if S is a nonabelian simple group that satisfies property P_k, then there exists a prime divisor p of $|S|$ such that the number of indices of the maximal subgroups of S of p'-index is at most $f_5(k)$.*

Proof. First assume that S is of Lie type, with Weyl group W of rank n. If S satisfies property P_k then $k \ge f_1(n)$ by Proposition 3.2. If M is a maximal subgroup of S, then either $|S : M|$ is divisible by the defining characteristic p of S, or M is a maximal parabolic subgroup (see for example [3, Thm. 3.1.3]). Conjugacy classes of parabolic subgroups correspond to subsets of the set of nodes of the Dynkin diagram for W, hence the number of classes of maximal parabolic subgroups of S is at most equal to n if W has rank n. Since $f_1(n)$ tends to infinity, this proves the claim for groups of Lie type.

It remains to consider alternating groups. Here, by [6, Th.] maximal subgroups of odd index of $\mathfrak{A}_n$, $n \ge 9$, are either intransitive $(\mathfrak{S}_k \times \mathfrak{S}_{n-k}) \cap \mathfrak{A}_n$ or imprimitive $\mathfrak{S}_k \wr \mathfrak{S}_{n/k} \cap \mathfrak{A}_n$. Clearly, the number of such subgroups is bounded in n, and n is bounded in k by Proposition 3.1. □

Proposition 3.7. *Let S be a finite nonabelian simple group, $S \not\cong \mathrm{L}_2(3^f)$. Then there exist nonlinear characters $\varphi, \psi \in \mathrm{Irr}(S)$ of different degrees that extend to $\mathrm{Aut}(S)$.*

Proof. For sporadic groups and the Tits group, this can be checked from the known character tables. For the alternating groups $\mathfrak{A}_n$, $n \geq 7$, take the characters parameterized by the partitions $(n-1,1)$ and $(n-2,2)$. (Note that $\mathfrak{A}_6 = \mathrm{L}_2(9)$, and $\mathfrak{A}_5$ can be checked easily.) For S of Lie type it is shown in [9, Thm. 2.4] that all unipotent characters extend to their inertia groups in $\mathrm{Aut}(S)$, and it is known that there are at least two nonlinear unipotent characters of different degree which are invariant in $\mathrm{Aut}(S)$ if S is not of type A_1.

So finally assume that $S = \mathrm{L}_2(q)$ for $q \geq 7$, $q \not\equiv 0 \pmod 3$. Let χ_1 denote the Steinberg character of S, of degree q. This extends to $\mathrm{Aut}(S)$ by [2]. Let χ_2 be a character corresponding to a semisimple element of order 3 in the dual group, of degree $q+\epsilon$ if $q \equiv \epsilon \pmod 3$. It is shown in [5, Lemma 15.2] that χ_2 extends to $\mathrm{Aut}(S)$. □

It can be checked easily that the groups $\mathrm{L}_2(3^f)$ constitute counterexamples to the conclusion of the proposition. Still, we have the following:

Corollary 3.8. *Let S be a nonabelian finite simple group. Then there exist non-linear $\varphi, \psi \in \mathrm{Irr}(S)$ of different degree such that φ extends to $\mathrm{Aut}(S)$ and ψ extends to its inertia subgroup in $\mathrm{Aut}(S)$.*

Finally, we will need the following well-known fact.

Lemma 3.9. *Let S be a nonabelian simple group. Write n to denote the rank of S if S is of Lie type and put $n = 3$ otherwise. Let π be the set of primes smaller than $\max\{4, n+2\}$. Then $\mathrm{Out}(S)$ has a cyclic and normal Hall π'-subgroup.*

Proof. If S is not of Lie type, then $\mathrm{Out}(S)$ is cyclic, so we may assume that S is of Lie type.

The group of diagonal automorphisms is either the Klein four group or it is cyclic of order at most $n+1$. Thus its automorphism group is a π-group. The group of graph automorphisms has order divisible by 6, so again it is a π-group. Since the group of outer automorphisms is a semidirect product of the group of diagonal automorphisms with the field and graph automorphisms, a Hall π'-subgroup of $\mathrm{Out}(S)$ consists of some field automorphisms that form a normal subgroup in $\mathrm{Out}(S)$. □

4 Proof of Theorem A

Now, we are ready to prove Theorem A. We split the proof in two parts. The argument of the first part is essentially contained in [1]

Theorem 4.1. *There exists an integer-valued function f_6 such that for any finite group G satisfying property P_k and which has some nonabelian solvable quotient, we have $|\operatorname{cd}(G)| \leq f_6(k)$.*

Proof. Let L be a maximal normal subgroup of G such that G/L is a nonabelian solvable group. By Lemma 12.3 of [4], G/L is a p-group for some prime p, or a Frobenius group with elementary abelian kernel and cyclic complement. (Note that an abelian Frobenius complement is cyclic.)

Assume first that G/L is a p-group. We know that G has at most k character degrees that are divisible by p, so it suffices to bound the character degrees of G that are p'-numbers. If $\chi \in \operatorname{Irr}(G)$ has p'-degree, then χ_L is also irreducible (by Corollary 11.29 of [4], for instance). Hence, by Gallagher's Theorem (Corollary 6.17 of [4]), $\chi\psi \in \operatorname{Irr}(G)$ for any $\psi \in \operatorname{Irr}(G/L)$. If we take a nonlinear ψ, it follows that for any $d \in \operatorname{cd}(G)$ that is a p'-number, $d\psi(1)$ also belongs to $\operatorname{cd}(G)$. This implies that the number of character degrees of G that are p'-numbers bigger than 1 is at most k. Therefore, we have proved that $|\operatorname{cd}(G)| \leq 2k+1$.

Now, we assume that G/L is a Frobenius group with elementary abelian kernel K/L of order q^a and cyclic complement of order f. Then $\operatorname{cd}(G/L) = \{1, f\}$. We know that G has at most k character degrees that are divisible by q, so it suffices to bound the number of irreducible characters of G of q'-degree. Assume that $\varphi \in \operatorname{Irr}(L)$ is nonlinear of q'-degree and extends to K (otherwise, the irreducible characters of G lying over φ would have degrees divisible by q). Since the Frobenius complement is cyclic, by Corollaries 11.22 and 11.31 of [4], we deduce that φ extends to its inertia subgroup in G. Now, by Clifford's Correspondence and Gallagher's Theorem, we obtain that $f\varphi(1) \in \operatorname{cd}(G)$. It follows that the number of degrees of the nonlinear irreducible characters of L of q'-degree that extend to K is at most k. By Lemma 2.4, we obtain that the number of character degrees of G lying over these irreducible characters is at most k^2. It remains to consider the number of character degrees of G lying over linear irreducible characters of L. In other words, it remains to bound $|\operatorname{cd}(G/L')|$. But G/L' is solvable and satisfies property P_k, so the result follows from Theorem A of [1]. $\square$

Given a normal subgroup N of a finite group G, we write $\operatorname{Irr}(G|N)$ to denote the set of irreducible characters of G which do not contain N in their kernel. The notation $\operatorname{cd}(G|N)$ stands for the degrees of the characters in $\operatorname{Irr}(G|N)$.

Theorem 4.2. *There exists an integer-valued function f_7 such that for any finite group G satisfying property P_k and which has no nonabelian solvable quotients, we have $|\operatorname{cd}(G)| \leq f_7(k)$.*

Proof. Let L be a maximal normal subgroup of G such that G/L is not abelian. Write $K = G'L$. Observe that K/L is a nonabelian chief factor of G and $C/L = C_{G/L}(K/L) = L/L$. We have that K/L is a direct product of, say l, copies of a nonabelian simple group S. Since $C = L$, we have that G/L is isomorphic to a subgroup of $\Gamma = \mathrm{Aut}(S) \wr \mathfrak{S}_l$. Put $U = \mathrm{Aut}(S)^l \cap G \trianglelefteq G$, so that G/U is isomorphic to a subgroup of $\mathfrak{S}_l$.

Now, we split the proof in several steps.

Step 1. We claim that l is k-bounded.

Since G/K is abelian, we have that G/U is an abelian permutation group on the set $\Omega = \{S_1, \ldots, S_l\}$ consisting by l copies of S. By Lemma 2.7, G/U has a regular orbit on $\mathcal{P}(\Omega)$, say the orbit of $\Delta \subseteq \Omega$. Since $\Omega - \Delta$ also lies in a regular orbit we may assume, without loss of generality, that $|\Delta| \geq |\Omega|/2$. Thus, it suffices to show that $|\Delta|$ is k-bounded. We may assume, without loss of generality, that $\Delta = \{S_1, \ldots, S_u\}$ for some $u \geq l/2$.

By Corollary 3.8, S_i has two nonlinear irreducible characters φ_i and ψ_i of different degrees such that φ_i extends to $\mathrm{Aut}(S_i)$ and ψ_i extends to its inertia subgroup in $\mathrm{Aut}(S_i)$. (We identify the characters φ_i and φ_j for every i, j and the same with the characters ψ_i and ψ_j.) For each $1 \leq k \leq u$, we consider

$$\mu_k = \varphi_1 \times \cdots \times \varphi_k \times 1_{S_{k+1}} \times \cdots \times 1_{S_u} \times \psi_{u+1} \times \cdots \times \psi_l \in \mathrm{Irr}(K/L).$$

The number of different degrees among these characters is u. By the way we have chosen them, their inertia subgroup is contained in U. In fact, the inertia subgroup $I := I_G(\mu_k)$ of μ_k is

$$I = \mathrm{Aut}(S_1) \times \cdots \times \mathrm{Aut}(S_u) \times I_{\mathrm{Aut}(S_{u+1})}(\psi_{u+1}) \times \cdots \times I_{\mathrm{Aut}(S_l)}(\psi_l) \cap U,$$

which does not depend on k.

Since φ_i, ψ_i and 1_{S_i} extend to their inertia subgroup in $\mathrm{Aut}(S_i)$, we conclude that μ_k extends to its inertia subgroup in G for every k. Write $\hat{\mu}_k \in \mathrm{Irr}(I)$ to denote the extension of μ_k. By Clifford's correspondence, $\hat{\mu}_k^G \in \mathrm{Irr}(G)$. These u induced irreducible characters provide u different character degrees of G all of them being multiples of $\varphi(1)$. It follows that $|\Delta| = u \leq k$, as desired.

Step 2. We may assume that G has a nonabelian almost simple quotient with socle S.

Since G/U is a subgroup of $\mathfrak{S}_l$ and l is k-bounded, the number of divisors of the order of this group is k-bounded, and by Lemmas 2.5 and 2.3 we may assume that $G = U$. This implies that G has a normal subgroup M such that G/M is almost simple with socle S.

Write $R/M = \mathrm{Soc}(G/M)$. We know from Lemma 3.5 that $|\operatorname{cd}(G/M)| \leq f_4(k)$. It remains to prove that $|\operatorname{cd}(G|M)|$ is bounded by some integer valued function of k.

We split the set $\operatorname{cd}(G|M)$ in three subsets. Write $\operatorname{cd}_1(G|M)$ to denote the set of degrees of the irreducible characters of G lying over irreducible characters of M that are not R-invariant, $\operatorname{cd}_2(G|M)$ to denote the set of degrees of the irreducible characters of G lying over irreducible characters of M that are R-invariant but do not extend to R and $\operatorname{cd}_3(G|M)$ to denote the set of degrees of the irreducible characters of G lying over nonprincipal irreducible characters of M that extend to R. Then

$$\operatorname{cd}(G|M) = \operatorname{cd}_1(G|M) \cup \operatorname{cd}_2(G|M) \cup \operatorname{cd}_3(G|M),$$

and it suffices to bound $|\operatorname{cd}_i(G|M)|$ for $i = 1, 2, 3$. This is what we will do in the next three steps.

Step 3. $|\operatorname{cd}_1(G|M)|$ is k-bounded.

Since G/M satisfies property P_k, Lemma 3.4 implies that R/M satisfies property $P_{f_3(k)}$. Since the characters of M we are looking at are not R-invariant, their inertia subgroup in R is contained in some maximal subgroup of R that contains M. In particular, by Clifford's Correspondence, the degrees of the irreducible characters of R lying over the characters of M we are considering are multiples of the index of some maximal subgroup of R/M. Since R is normal in G, the same happens with the degrees of the irreducible characters of G lying over these characters of M.

Now, Lemma 2.4 implies that for any maximal subgroup U/M of R/M, the number of character degrees of G that are multiples of $|R/U|$ is at most k. By Lemma 3.6, there is a prime p such that the number of indices of maximal subgroups of S of p'-index is at most $f_5(f_3(k))$. Hence, the number of character degrees of G lying over these characters of M is at most $f_5(f_3(k))k + k$ (the latter k corresponds to the at most k character degrees that are multiples of p).

Step 4. $|\operatorname{cd}_2(G|M)|$ is k-bounded.

Now, we consider the characters of M that are R-invariant but do not extend to R. Recall that R satisfies property $P_{f_3(k)}$. Using a character triple isomorphism (Theorem 11.28 of [4]), Lemma 3.3 and Proposition 3.1, it follows that there is a set of primes of k-bounded size such that any character degree of R arising from any of these characters of M is a multiple of some of them. The same happens with the degrees of the irreducible characters of G lying over these characters of M. As before, we obtain that the number of character degrees of G that is obtained this way is k-bounded.

Step 5. $|\operatorname{cd}_3(G|M)|$ is k-bounded.

Finally, we consider the irreducible characters of M that extend irreducibly to R. Let ψ be such a character. If ψ is linear, then the character degrees of G lying over ψ belong to $\mathrm{cd}(G/G'')$. By Benjamin's result, the cardinality of this set is k-bounded. Hence, we may assume that ψ is not linear. Let K/R be a Hall π'-subgroup of G/R, where π is the same as in Lemma 3.9. By Proposition 3.2, $|\pi|$ is bounded in terms of k and by Lemma 2.6, it suffices to bound the number of character degrees of G that are π'-numbers. Hence, we may assume that $G = K$.

Since R/M is perfect, it follows from Gallagher's Theorem that ψ has a unique extension $\hat{\psi} \in \mathrm{Irr}(R)$. Since G/R is cyclic, $\hat{\psi}$ extends to its inertia subgroup in G. Using Clifford's Theorem, we have shown that for any $\psi \in \mathrm{Irr}(M)$ that extends to R, $\psi(1)|G : I_G(\hat{\psi})| \in \mathrm{cd}(G)$, where $\hat{\psi}$ is the unique extension of ψ to R. But by Gallagher's theorem, $\psi(1)|G : I_G(\hat{\psi})|d \in \mathrm{cd}(G)$, where $1 < d$ is the degree of some irreducible character of R/M that extends to $\mathrm{Aut}(R/M)$. Therefore, the number of different possibilities for $\psi(1)|G : I_G(\hat{\psi})|$ as ψ runs over the nonlinear irreducible characters of M that extend to R is at most k.

Write $d_1, \ldots, d_w$ for the $w \leq k$ different possibilities for $\psi(1)|G : I_G(\hat{\psi})|$ as ψ runs over the nonlinear irreducible characters of M that extend to R. We deduce from Clifford's Correspondence and Gallagher's Theorem that the character degrees of G lying over irreducible characters of M that extend to R have the form $d_i t$, where t is a character degree of some almost simple group with socle S. It follows from our hypothesis that for any i, the number of degrees of the form $d_i t$ is at most k. We conclude that the number of character degrees of G lying over nonlinear irreducible characters of M that extend to R is at most k^2. This completes the proof. □

Proof of Theorem A. It suffices to take $f = \max\{f_6, f_7\}$. □

References

[1] D. Benjamin, Coprimeness among irreducible character degrees of finite solvable groups. *Proc. Amer. Math. Soc.* **125** (1997) 2831–2837.

[2] W. Feit, Extending Steinberg characters. Pp. 1–9 in: *Linear algebraic groups and their representations*, Contemp. Math. 153, Amer. Math. Soc., Providence, 1993.

[3] D. Gorenstein, R. Lyons and R. Solomon, *The classification of the finite simple groups, Number 3.* Mathematical Surveys and Monographs, Amer. Math. Soc., Providence, 1994.

[4] I. M. Isaacs, *Character Theory of Finite Groups*. Dover, New York, 1994.

[5] I. M. Isaacs, G. Malle, G. Navarro, A reduction theorem for the McKay conjecture. Submitted, 2005.

[6] M. W. Liebeck, J. Saxl, The primitive permutation groups of odd degree. *J. London Math. Soc.* **31** (1985), 250–264.

[7] M. W. Liebeck, A. Shalev , Character degrees and random walks in finite groups of Lie type. *Proc. London Math. Soc.* **90** (2005), 61–86.

[8] G. Lusztig, *Characters of reductive groups over a finite field*. Annals of Math. Studies, 107. Princeton Univ. Press, Princeton, NJ, 1984.

[9] G. Malle, Extensions of unipotent characters and the inductive McKay condition. In preparation, 2007.

[10] G. Malle, A. E. Zalesskiĭ, Prime power degree representations of quasi-simple groups. *Archiv Math.* **77** (2001), 461–468.

[11] O. Manz, T. R. Wolf, *Representations of Solvable Groups*. Cambridge University Press, Cambridge, 1993.

[12] J. K. McVey, *Bounding the number of character degrees using generalized relative primeness conditions*. Ph.D. Thesis, Kent State University, 2001.

[13] J. K. McVey, Prime divisibility among character degrees. *Comm. Algebra* **32** (2004), 3391–3402.

[14] A. Moretó, A proof of Huppert's ρ-σ conjectures for nonsolvable groups. *Int. Math. Research Notices* **54** (2005), 3375–3383.