EXTENDING PALFY'S THEOREM ON PRIME DIVISORS OF CHARACTER DEGREES

ALEXANDER MORETÓ AND PHAM HUU TIEP

1. Introduction

This paper concerns the arithmetical structure of the character degrees of a finite group. A useful tool in these problems is *the character degree graph*. Given a group G, the vertices in this graph are the prime divisors of the degrees of the complex irreducible characters of G, and two vertices p and q are joined by an edge if pq divides the degree of some irreducible character of G. This graph was first introduced in 1988 by O. Manz, R. Staszewski and W. Willems [**?**]. Since then many results have been proved. Usually, these theorems indicate that the character degrees of a finite group are closely related in certain sense. For a long time the main focus was on the case of solvable groups, but in very recent years these results are being extended to arbitrary finite groups. We refer the reader to [**?**] for a survey on character degree graphs.

One of the main results on the character degree graph is Palfy's theorem, which asserts that given 3 vertices of the character degree graph of a solvable group, at least two of them are joined or equivalently, without using the graph theoretic terminology, given a set π of 3 prime divisors of character degrees of a solvable group, we can choose two of them such that their product divides some character degree (see [**?**] or [**?**, Theorem 18.7]). Of course, the solvability hypothesis cannot be removed (consider for instance the alternating group A_5 and $\pi = \{2, 3, 5\}$). But we can prove the following result for arbitrary finite groups.

Main Theorem. *Let G be a finite group and π a set of four prime divisors of irreducible character degrees of G. Then there are primes p and q in π such that pq divides the degree of some irreducible character of G.*

Our proof of Main Theorem depends on the classification of finite simple groups and the Deligne-Lusztig theory of irreducible characters of finite groups of Lie type. Once a theorem for character degrees is proved, it is natural to ask if the same result for

Date: May 31, 2006.

The first author's research is partially supported by the Spanish Ministerio de Educación y Ciencia, the FEDER, the Generalitat Valenciana and the Programa Ramón y Cajal. The second author gratefully acknowledges the support of the NSA and the NSF.

conjugacy classes holds (and conversely). In this case, an analogue of Main Theorem for conjugacy class sizes has been recently proved by S. Dolfi [?]. His result, which also depends on the classification of finite simple groups, shows that if we consider conjugacy class sizes instead of character degrees, then we can take the set π of cardinality 3.

We will prove a few results related to finite simple groups in §2. With the help of these theorems, we complete the proof of Main Theorem in §3.

2. Simple groups

We begin by proving several lemmas. The first one is a well-known observation.

Lemma 2.1. *Let Φ and Ψ be two permutation representations of a finite cyclic group A with the same character. Then they have the same number of regular orbits.*

Proof. Let χ denote the character of Φ and let λ denote a faithful irreducible character of A. Consider any orbit $\mathcal{O}$ of A and let ρ denote the permutation character of A acting on this orbit. Observe that, if $|\mathcal{O}| = n/d < n := |A|$, then ρ is trivial at the (unique) subgroup of order d of A and so $(\rho, \lambda) = 0$. Therefore the number of regular orbits of Φ equals (χ, λ), whence the statement follows. □

The next lemma is convenient when we work with semisimple elements in exceptional finite groups of Lie type. If $\mathcal{G}$ is an algebraic group then $\mathcal{G}^\circ$ denotes the connected component of $\mathcal{G}$.

Lemma 2.2. *Let $\mathcal{G}$ be a simple algebraic group defined over a field of characteristic $\ell > 0$ and F a Frobenius map on $\mathcal{G}$ such that $\mathcal{G}^F$ is an exceptional finite group of Lie type. Let $p \neq \ell$ be a prime such that either $p \geq 11$ or $p > \mathrm{rank}(\mathcal{G}) + 1$. Let $s \in \mathcal{G}^F$ be any nontrivial p-element. Then $\mathcal{C} := C_{\mathcal{G}}(s)$ is connected and p divides $|(Z(\mathcal{C})^\circ)^F|$; in particular, $\mathcal{C}$ cannot be semisimple.*

Proof. By [?, Cor. E-II.4.6], the condition $p > 3$ implies that $\mathcal{C}$ is connected. Now we can write $\mathcal{C} = \mathcal{M}\mathcal{S}$, where $\mathcal{M} := [\mathcal{C}, \mathcal{C}]$ is semisimple and $\mathcal{S} := Z(\mathcal{C})^\circ$. It remains to prove that p divides $|\mathcal{S}^F|$. First we observe that $(p, |Z(\mathcal{M})|) = 1$. (Assume the contrary and write $\mathcal{M} = \mathcal{M}_1 * \ldots * \mathcal{M}_n$ as a central product of simple algebraic groups $\mathcal{M}_1, \ldots, \mathcal{M}_n$. Then $Z(\mathcal{M}) = Z(\mathcal{M}_1) * \ldots * Z(\mathcal{M}_n)$ and so we may assume that p divides $|Z(\mathcal{M}_1)|$. But this is a contradiction, since $\mathcal{M}_1$ is a simple algebraic group of rank at most $\mathrm{rank}(\mathcal{G}) \leq 8$ and so its centre has order at most $\mathrm{rank}(\mathcal{G}) + 1 \leq 9$.) In particular, $(p, |\mathcal{M}^F \cap \mathcal{S}^F|) = 1$, as $\mathcal{M}^F \cap \mathcal{S}^F \leq Z(\mathcal{M})$. Clearly, $s \in Z(\mathcal{C})^F$. Assume that $(p, |\mathcal{S}^F|) = 1$. It is well known (cf. [?] for instance) that $|\mathcal{C}^F| = |\mathcal{M}^F| \cdot |\mathcal{S}^F|$. Since p is coprime to $|\mathcal{M}^F \cap \mathcal{S}^F| = |\mathcal{C}^F/(\mathcal{M}^F * \mathcal{S}^F)|$, $s \in \mathcal{M}^F * \mathcal{S}^F$. But p is coprime to $|\mathcal{S}^F|$ by our assumption, hence $s \in \mathcal{M}^F$. Thus $s \in \mathcal{M} \cap Z(\mathcal{C}) = Z(\mathcal{M})$ and so p divides $|Z(\mathcal{M})|$, a contradiction. □

In what follows we use *primitive prime divisors* (p.p.d.) as defined in [?]. Also, $\Phi_n(x)$ denotes the n^{th} cyclotomic polynomial.

Lemma 2.3. *Let $\mathcal{G}$ be a simple algebraic group defined over a field of characteristic $\ell > 0$ and F a Frobenius map on $\mathcal{G}$ such that $G := \mathcal{G}^F$ is an exceptional finite group of Lie type. Then G contains semisimple elements s_1 and s_2 such that $p_1 := |s_1|$ and $p_2 := |s_2|$ are different primes, and one of the following statements holds.*

(i) *Assume $G = {}^2B_2(r)$ with $r = \ell^f \geq 8$. Then p_1 is a p.p.d. of $\ell^{4f} - 1$ and $|C_G(s_1)| = r \pm \sqrt{2r} + 1$. Furthermore, p_2 is a p.p.d. of $\ell^f - 1$ and $|C_G(s_2)| = r - 1$.*

(ii) *Assume $G = {}^2G_2(r)$ with $r = \ell^f \geq 27$. Then p_1 is a p.p.d. of $\ell^{6f} - 1$ and $|C_G(s_1)| = r \pm \sqrt{3r} + 1$. Furthermore, p_2 is a p.p.d. of $\ell^f - 1$ and $|C_G(s_2)| = r - 1$.*

(iii) *Assume $G = {}^2F_4(r)$ with $r = \ell^f > 2$. Then p_1 is a p.p.d. of $\ell^{12f} - 1$ and $|C_G(s_1)| = r^2 + r + 1 \pm (\sqrt{2r^3} + \sqrt{2r})$. Furthermore, p_2 is a p.p.d. of $\ell^{6f} - 1$ and $|C_G(s_2)| = r^2 - r + 1$.*

(iv) *Assume $G = {}^3D_4(r)$ with $r = \ell^f > 2$. Then p_1 is a p.p.d. of $\ell^{12f} - 1$ and $|C_G(s_1)| = r^4 - r^2 + 1$. Furthermore, p_2 is a p.p.d. of $\ell^{6f} - 1$ and $|C_G(s_2)|$ is coprime to $|C_G(s_1)|$.*

(v) *Assume $G = G_2(r)$, resp. $F_4(r)$, $E_8(r)$, with $r = \ell^f > 2$. Then p_i is a p.p.d. of $\ell^{m_i f} - 1$ and $|C_G(s_i)| = \Phi_{m_i}(r)$, with $i = 1, 2$. Furthermore, $(m_1, m_2) = (6, 3)$, $(12, 8)$, $(30, 24)$, resp.*

(vi) *Assume $G = E_6(r)$ with $r = \ell^f$. Then p_1 is a p.p.d. of $\ell^{9f} - 1$ and $|C_G(s_1)| = \Phi_9(r)$. Furthermore, p_2 is a p.p.d. of $\ell^{12f} - 1$ and $|C_G(s_2)| = \Phi_{12}(r)\Phi_3(r)$.*

(vii) *Assume $G = {}^2E_6(r)$ with $r = \ell^f$. Then p_1 is a p.p.d. of $\ell^{18f} - 1$ and $|C_G(s_1)| = \Phi_{18}(r)$. Furthermore, p_2 is a p.p.d. of $\ell^{12f} - 1$ and $|C_G(s_2)| = \Phi_{12}(r)\Phi_6(r)$.*

(viii) *Assume $G = E_7(r)$ with $r = \ell^f$. Then p_i is a p.p.d. of $\ell^{m_i f} - 1$ and $|C_G(s_1)| = \Phi_{m_i}(r)\Phi_2(r)$, with $i = 1, 2$ and $(m_1, m_2) = (18, 14)$.*

Proof. The conjugacy classes of ${}^2B_2(r)$ are completely known, see e.g. [?]. Using this information one readily gets (i). Consider the case of (iii). Using the information given in [?] about $|C_G(s)|$ and $|s|$, one arrives at (iii). The case of (ii) is handled similarly using [?].

In all the remaining cases, let $\mathcal{G}_{sc}$ denote the algebraic group of simply connected type corresponding to $\mathcal{G}$. Then F lifts to a Frobenius map on $\mathcal{G}_{sc}$ which we also denote by F. Let $\pi : \mathcal{G}_{sc} \to \mathcal{G}$ denote the natural projection. Since p_i is coprime to $|Z(\mathcal{G}_{sc})|$, $g \in \mathcal{G}$ is of order p_i if and only if $\pi^{-1}(g) \subset \mathcal{G}_{sc}$ contains a (unique) element g_* of order p_i; and moreover $g, g' \in \mathcal{G}$ of order p_i are conjugate in $\mathcal{G}$ if and only if the corresponding elements g_*, g'_* of order p_i in $\pi^{-1}(g)$ and $\pi^{-1}(g')$ are conjugate in $\mathcal{G}_{sc}$. As their centralizers are connected by Lemma **??**, the conjugacy of these elements in the finite group $\mathcal{G}^F$ (resp. $\mathcal{G}_{sc}^F$) is equivalent to their conjugacy in the algebraic group

$\mathcal{G}$ (resp. $\mathcal{G}_{sc}$). Thus $|g^{\mathcal{G}^F}| = |g_*^{\mathcal{G}_{sc}^F}|$. Finally, observe that $|\mathcal{G}^F| = |\mathcal{G}_{sc}^F|$. It follows that $|C_{\mathcal{G}^F}(g)| = |C_{\mathcal{G}_{sc}^F}(g_*)|$. We have shown that the isogeny class of $\mathcal{G}$ does not matter for the conclusions of Lemma **??**.

Now we consider the case of (iv). Clearly, p_i as chosen in (iv) is $\geq 7 > \mathrm{rank}(\mathcal{G})+1$ and so we can apply Lemma **??**. The order of $(Z(\mathcal{C})^\circ)^F$ is listed explicitly in [**?**] for all semisimple conjugacy classes. In particular, we see that the divisibility of $|(Z(\mathcal{C})^\circ)^F|$ by p_1 implies that $|C_G(s_1)| = \Phi_{12}(r)$. Furthermore, the divisibility of $|(Z(\mathcal{C})^\circ)^F|$ by p_2 implies that $|C_G(s_2)|$ is one of the following: $(r^3+1)(r \pm 1)$, $(r^2-r+1)^2$, $(r^3+1)r(r^2-1)$, $(r^3+1)r(r^2-1)(r^2-r+1)$, $r^3(r^6-1)(r \pm 1)$, $r^4(r^6-1)(r^2-1)$; and all these possible orders are coprime to $\Phi_{12}(r)$. Next we consider the case of (vi). Then $p_i \geq 11$. Using [**?**], we see that the divisibility of $|(Z(\mathcal{C})^\circ)^F|$ by p_i implies that $|C_G(s_1)| = \Phi_9(r)$ and $|C_G(s_2)| = \Phi_{12}(r)\Phi_3(r)$. The cases of ${}^2E_6(r)$, $F_4(r)$, and $G_2(r)$ are dealt with in an entirely similar manner. Now assume that we are in the case of (viii). Then $p_i \geq 19$. If the semisimple element s_i is not regular, then one can check that $|(Z(\mathcal{C})^\circ)^F|$ as listed in [**?**] is not divisible by p_i, a contradiction. Hence s_i is regular. In this case $C_G(s_i)$ is a maximal torus, and its order is given in [**?**]. In particular, s_1 corresponds to type E_7 and s_2 corresponds to type $E_7(a_1)$ in [**?**], and we arrive at (viii). Similarly, if $G = E_8(r)$, then s_1 corresponds to type E_8 and s_2 corresponds to type $E_8(a_1)$ in [**?**], and we arrive at (v). □

Next we prove an analogue of Lemma **??** for classical groups.

Lemma 2.4. *Let $\mathcal{G}$ be a simple algebraic group defined over a field of characteristic ℓ and F a Frobenius map on $\mathcal{G}$ such that $G := \mathcal{G}^F$ is a classical finite group of Lie type. Then G contains semisimple elements s_1 and s_2 such that $p_1 := |s_1|$ and $p_2 := |s_2|$ are different primes. Moreover, if S denotes the finite Lie-type group of simply connected type corresponding to G, then one of the following statements holds.*

(i) *Assume $S = \mathrm{SL}_n(r)$ with $n \geq 2$, $r = \ell^f \geq \ell^5$, $(n,r) \neq (2,2^6)$. Then p_1 is a p.p.d. of $\ell^{nf}-1$ and $|C_G(s_1)| = (r^n-1)/(r-1)$. Furthermore, p_2 is a p.p.d. of $\ell^{(n-1)f}-1$ and $|C_G(s_2)| = r^{n-1}-1$.*

(ii) *Assume $S = \mathrm{SU}_n(r)$ with $n \geq 3$, n odd, $r = \ell^f \geq \ell^5$. Then p_1 is a p.p.d. of $\ell^{2nf}-1$ and $|C_G(s_1)| = (r^n+1)/(r+1)$. Furthermore, p_2 is a p.p.d. of $\ell^{2(n-2)f}-1$, $|C_G(s_2)| = r(r^{n-2}+1)(r^2-1)$ if $n \geq 5$ and $|C_G(s_2)| = (r+1)^2$ if $n = 3$.*

(iii) *Assume $S = \mathrm{SU}_n(r)$ with $n \geq 4$, n even, $r = \ell^f \geq \ell^5$. Then p_1 is a p.p.d. of $\ell^{2(n-1)f}-1$ and $|C_G(s_1)| = r^{n-1}+1$. Furthermore, p_2 is a p.p.d. of $\ell^{2(n-3)f}-1$, $|C_G(s_2)| = r^3(r^{n-3}+1)(r^3+1)(r^2-1)$ if $n \geq 6$ and $|C_G(s_2)| = (r+1)^3$ if $n = 4$.*

(iv) *Assume $S = \mathrm{Sp}_{2n}(r)$ or $\mathrm{Spin}_{2n+1}(r)$ with $n \geq 2$, $r = \ell^f \geq \ell^5$. Then p_1 is a p.p.d. of $\ell^{2nf}-1$ and $|C_G(s_1)| = r^n+1$. Furthermore, p_2 is a p.p.d. of $\ell^{2(n-1)f}-1$ and $|C_G(s_2)| = r(r^{n-1}+1)(r^2-1)$.*

(v) *Assume* $S = \mathrm{Spin}_{2n}^{-}(r)$ *with* $n \geq 4$, $r = \ell^f \geq \ell^5$. *Then* p_1 *is a p.p.d. of* $\ell^{2nf} - 1$ *and* $|C_G(s_1)| = r^n + 1$. *Furthermore,* p_2 *is a p.p.d. of* $\ell^{2(n-1)f} - 1$ *and* $|C_G(s_2)| = (r^{n-1} + 1)(r - 1)$.

(vi) *Assume* $S = \mathrm{Spin}_{2n}^{+}(r)$ *with* $n \geq 4$, $r = \ell^f \geq \ell^5$. *Then* p_1 *is a p.p.d. of* $\ell^{2(n-1)f} - 1$ *and* $|C_G(s_1)| = (r^{n-1} + 1)(r + 1)$. *Furthermore,* p_2 *is a p.p.d. of* $\ell^{nf} - 1$ *and* $|C_G(s_2)| = r^n - 1$ *if* n *is odd, and* p_2 *is a p.p.d. of* $\ell^{(n-1)f} - 1$ *and* $|C_G(s_2)| = (r^{n-1} - 1)(r - 1)$ *if* n *is even.*

Proof. In all cases, p_i is chosen to be larger than $|Z(\mathcal{G})|$. So arguing as in the proof of Lemmas **??** and **??**, we see that the conclusions of Lemma **??** do not depend on the choice of the isogeny class of $\mathcal{G}$. Thus in the proof of the lemma we can choose the isogeny class that is most convenient for our arguments. We also denote by $\mathbb{F}$ an algebraically closed field of characteristic ℓ and choose $\alpha, \beta \in \mathbb{F}^\times$ such that $|\alpha| = p_1$, $|\beta| = p_2$. In all the following cases, it is straightforward to find the order of the centralizers in question.

Assume $G = \mathrm{SL}_n(r)$. Then we choose s_1 to be $\mathcal{G}$-conjugate to $\mathrm{diag}(\alpha, \alpha^r, \ldots, \alpha^{r^{n-1}})$, and s_2 to be $\mathcal{G}$-conjugate to $\mathrm{diag}(1, \beta, \beta^r, \ldots, \beta^{r^{n-2}})$.

Assume $G = \mathrm{SU}_n(r)$ with n odd. Then we choose s_1 to be $\mathcal{G}$-conjugate to

$$\mathrm{diag}(\alpha, \alpha^{-r}, \ldots, \alpha^{(-r)^{n-1}}).$$

Furthermore, we choose s_2 to be $\mathcal{G}$-conjugate to

$$\mathrm{diag}(1, 1, \beta, \beta^{-r}, \ldots, \beta^{(-r)^{n-3}})$$

if $n \geq 5$, and to $\mathrm{diag}(1, \beta, \beta^{-1})$ if $n = 3$.

Assume $G = \mathrm{SU}_n(r)$ with n even. Then we choose s_1 to be $\mathcal{G}$-conjugate to

$$\mathrm{diag}(1, \alpha, \alpha^{-r}, \ldots, \alpha^{(-r)^{n-2}}).$$

Furthermore, we choose s_2 to be $\mathcal{G}$-conjugate to

$$\mathrm{diag}(1, 1, 1, \beta, \beta^{-r}, \ldots, \beta^{(-r)^{n-4}})$$

if $n \geq 6$, and to $\mathrm{diag}(1, \beta, \beta^2, \beta^{-3})$ if $n = 4$.

Assume $G = \mathrm{Sp}_{2n}(r)$ or $\mathrm{SO}_{2n}^{-}(r)$. Then we choose s_1 to be $\mathcal{G}$-conjugate to

$$\mathrm{diag}(\alpha, \alpha^r, \ldots, \alpha^{r^{2n-1}}),$$

and s_2 to be $\mathcal{G}$-conjugate to $\mathrm{diag}(1, 1, \beta, \beta^r, \ldots, \beta^{r^{2n-3}})$ (in particular, the fixed point subspace of s_2 is a 2-subspace of type $+$ in the case of $\mathrm{SO}_{2n}^{-}(r)$). Observe that $\mathrm{SO}_{2n}^{\pm}(r) = \mathrm{Spin}_{2n}^{\pm}(r)$ when r is even.

Assume $G = \mathrm{SO}_{2n+1}(r)$. Then we choose s_1 to be $\mathcal{G}$-conjugate to

$$\mathrm{diag}(1, \alpha, \alpha^r, \ldots, \alpha^{r^{2n-1}}),$$

and s_2 to be $\mathcal{G}$-conjugate to

$$\mathrm{diag}(1,1,1,\beta,\beta^r,\ldots,\beta^{r^{2n-3}}).$$

Assume $G = \mathrm{SO}^+_{2n}(r)$. Then we choose s_1 to be $\mathcal{G}$-conjugate to

$$\mathrm{diag}(1,1,\alpha,\alpha^r,\ldots,\alpha^{r^{2n-3}})$$

(in particular, the fixed point subspace of s_1 is a 2-subspace of type $-$). Furthermore, if n is odd we choose s_2 to be $\mathcal{G}$-conjugate to

$$\mathrm{diag}(\beta,\beta^r,\ldots,\beta^{r^{n-1}},\beta^{-1},\beta^{-r},\ldots,\beta^{-r^{n-1}}).$$

If n is even we choose s_2 to be $\mathcal{G}$-conjugate to

$$\mathrm{diag}(1,1,\beta,\beta^r,\ldots,\beta^{r^{n-2}},\beta^{-1},\beta^{-r},\ldots,\beta^{-r^{n-2}}),$$

in particular, the fixed point subspace of s_1 is a 2-subspace of type $+$. $\square$

The first statement in the following proposition is known, see e.g. Lemma 6(a) of [**?**].

Proposition 2.5. *Assume that a finite group A acts coprimely and faithfully on a finite simple group S. Then A has a regular orbit on the conjugacy classes of S. In fact, if S is nonabelian in addition, then A has at least two regular orbits on the conjugacy classes of S or, equivalently, on the irreducible characters of S.*

Proof. The case of abelian S is obvious, so we will assume that S is nonabelian and moreover A is nontrivial. Using the description of $\mathrm{Out}(S)$ (cf. [**?**]), we see that S is a finite group of Lie type defined over a field $\mathbb{F}_r$ with $r = \ell^f$ for some prime ℓ. Since $\mathrm{Out}(S)$ is solvable, all Hall subgroups of $\mathrm{Out}(S)$ (of given order) are conjugate, and so we may assume that A is a subgroup of field automorphisms of S. By the Schur-Zassenhaus theorem applied to the semidirect product $S : A$, we may assume that A is generated by a fixed field automorphism σ of order d (induced by the map that raises any element of $\overline{\mathbb{F}}_r$ to its $\ell^{f/d}$-power). It is well known (cf. for instance [**?**]) that the permutation actions of A on the set $\mathrm{cl}(S)$ of conjugacy classes of S and on $\mathrm{Irr}(S)$ have the same character. Hence by Lemma **??** it suffices to show that A has two regular orbits on $\mathrm{cl}(S)$. Thus we aim to show that there are at least two classes s^S each of which is not fixed by any element τ of prime order say p of A. Observe that, since $(p,|S|) = 1$, p is coprime to $|g^S|$ for any $g \in S$. Assuming g^S is τ-invariant, we then see that τ fixes some representative $h \in g^S$ and so $h \in C_S(\tau)$; in particular, $|g| = |h|$ divides $|C_S(\tau)|$. Thus it suffices to find two non-conjugate elements $s \in S$ such that $|s|$ is coprime to $|C_S(\tau)|$ for any such a τ.

We can find a simple simply connected algebraic group $\mathcal{G}$ defined over a field of characteristic ℓ and a Frobenius map F on $\mathcal{G}$ such that $S = \mathcal{G}^F/Z$ for $Z := Z(\mathcal{G}^F)$. One can extend τ to an automorphism of $\mathcal{G}^F$ (that clearly stabilizes Z) which we also denote by τ. Let D denote the complete inverse image of $C_S(\tau)$ in $\mathcal{G}^F$. Then the map $d \mapsto d^{-1}\tau(d)$ yields a homomorphism $D \to Z$ with kernel equal to $C_{\mathcal{G}^F}(\tau)$. It follows that $|C_S(\tau)|$ divides $|C_{\mathcal{G}^F}(\tau)|$. Now we apply Lemmas **??** and **??** to $\mathcal{G}^F$, and consider the elements s_1, s_2 constructed in these two lemmas. Notice that $C_{\mathcal{G}^F}(\tau)$ is a finite Lie-type group of the same type as of $\mathcal{G}^F$, but defined over $\mathbb{F}_{r^{1/p}}$. Moreover, either $(\mathcal{G}^F, p) = ({}^2B_2(r), 3)$ and $f \geq 3$, or $p \geq 5$ and $f \geq 5$. Hence it is straightforward to check that both p_1 and p_2 (as defined in Lemmas **??** and **??**) are coprime to $|C_{\mathcal{G}^F}(\tau)| \cdot |Z|$, and so they are coprime to $|C_S(\tau)|$. We will need this conclusion in the proof of Theorem **??**. Now we can choose s to be the image in S of the element s_i, $i = 1, 2$. □

In order to prove Main Theorem we also need the following two results.

Theorem 2.6. *Let p and q be two different primes and put $\pi = \{p, q\}$. Let S be a finite simple nonabelian group and assume that $S \lhd G \leq \mathrm{Aut}(S)$, where $|G/S| = p$, p does not divide $|S|$ and q divides $|S|$. Assume pq does not divide $\chi(1)$ for every $\chi \in \mathrm{Irr}(G)$. Then S is a finite simple group of Lie type in characteristic q, and G does not have any abelian subgroup H with $|H|_\pi = |G|_\pi$.*

Proof. 1) The assumption on p again implies that S is a finite simple group of Lie type defined over a field $\mathbb{F}_r$ for some $r = \ell^f$ with $f \geq 5$, or $f = 3$ and $S = {}^2B_2(r)$. So we can find a simple simply connected algebraic group $\mathcal{G}$ defined over a field of characteristic ℓ and a Frobenius map F on $\mathcal{G}$ such that $S = \mathcal{G}^F/Z$ for $Z := Z(\mathcal{G}^F)$. Let the pair $(\mathcal{G}^*, F^*)$ be dual to $(\mathcal{G}, F)$, and set $H := \mathcal{G}^{*F^*}$. According to the Deligne-Lusztig theory (cf. [**?**], [**?**]), the irreducible characters of $\mathcal{G}^F$ are partitioned into rational series $\mathcal{E}(\mathcal{G}^F, (s))$ which are indexed by $\mathcal{G}^{*F^*}$-conjugacy classes (s) of semisimple elements $s \in \mathcal{G}^{*F^*}$. As in the proof of Theorem **??**, we may assume that $G = \langle S, \sigma\rangle$, where σ is a field automorphism of order p of S. One can then extend σ to automorphisms of $\mathcal{G}^F$ and $\mathcal{G}^{*F^*}$ which we also denote by σ.

2) Let $s \in \mathcal{G}^{*F^*}$ be a semisimple element such that $|s|$ is coprime to $|C_{\mathcal{G}^{*F^*}}(\sigma)|$. Then arguing as in the proof of Proposition **??**, we see that the $\mathcal{G}^{*F^*}$-conjugacy class (s) of s in $\mathcal{G}^{*F^*}$ is not σ-invariant, whence $\sigma(s)$ and s are not $\mathcal{G}^{*F^*}$-conjugate. By [**?**], if $\psi \in \mathcal{E}(\mathcal{G}^F, (s))$, then $\psi^\sigma \in \mathcal{E}(\mathcal{G}^F, (\sigma(s)))$. It follows that every $\psi \in \mathcal{E}(\mathcal{G}^F, (s))$ is not σ-invariant.

Assume furthermore that $(|s|, |Z|) = 1$. Claim that every $\psi \in \mathcal{E}(\mathcal{G}^F, (s))$ is trivial at Z. Indeed, consider any generalized Deligne-Lusztig character $R^{\mathcal{G}}_{\mathcal{T},\theta}$ belonging to $\mathcal{E}(\mathcal{G}^F, (s))$. Here $\mathcal{T}$ is an F-stable maximal torus of $\mathcal{G}$, and θ is an irreducible character of $\mathcal{T}^F$. As shown in [**?**], $|\theta| = |s|$ and so θ is coprime to $|Z|$. In particular, θ is trivial at $\mathcal{T}^F \cap Z$. According to [**?**, Prop. 3.6.8], $Z = Z(\mathcal{G})^F$ (and so $Z \leq \mathcal{T}^F$ as $Z \leq C_{\mathcal{G}}(\mathcal{T}) = \mathcal{T}$). Now using the formula for the values of $R^{\mathcal{G}}_{\mathcal{T},\theta}$ at any semisimple element [**?**, Prop. 7.5.3], we see that $R^{\mathcal{G}}_{\mathcal{T},\theta}(z) = R^{\mathcal{G}}_{\mathcal{T},\theta}(1)$ for all $z \in Z$. As recorded in

[?], any $\psi \in \mathcal{E}(\mathcal{G}^F, (s))$ is a linear combination (with rational coefficients) of the $R^{\mathcal{G}}_{\mathcal{T},\theta}$ belonging to $\mathcal{E}(\mathcal{G}^F(s))$ and a class function that vanishes at semisimple elements of $\mathcal{G}^F$. It follows that $\psi(z) = \psi(1)$ for all $z \in Z$, as stated.

3) Now we can apply Lemmas **??** and **??** to $\mathcal{G}^{*F^*}$ and find semisimple elements s_i of order p_i, $i = 1, 2$. As shown in the proof of Theorem **??**, p_i is coprime to $|C_S(\sigma)| \cdot |Z|$, and so it is coprime to $|C_{\mathcal{G}^{*F^*}}(\sigma)| \cdot |Z|$ (as S is a normal subgroup of index $|Z|$ in $\mathcal{G}^{*F^*}$). Consider any $\psi_i \in \mathcal{E}(\mathcal{G}^F, (s_i))$. By 2), ψ_i is in fact an irreducible character of S which is not σ-invariant. Since $G/S = \langle\sigma\rangle$ is cyclic of order p, $\chi_i := \mathrm{Ind}_S^G(\psi_i)$ is an irreducible character of G of degree divisible by p. By the assumptions, q does not divide χ_i for $i = 1, 2$. Since $\psi_i(1)$ is divisible by $|\mathcal{G}^{*F^*} : C_{\mathcal{G}^{*F^*}}(s_i)|_{\ell'}$, it follows that q is coprime to $|\mathcal{G}^{*F^*} : C_{\mathcal{G}^{*F^*}}(s_i)|_{\ell'}$ for $i = 1, 2$. (We use the notation $N_{\ell'}$ to denote the ℓ'-part of an integer N.)

4) Here we consider the case $q \neq \ell$. Since q divides $|\mathcal{G}^{*F^*}|_{\ell'}$, the conclusion of 3) implies that q divides $|C_{\mathcal{G}^{*F^*}}(s_i)|_{\ell'}$ but not $|\mathcal{G}^{*F^*} : C_{\mathcal{G}^{*F^*}}(s_i)|_{\ell'}$, for $i = 1, 2$. In the cases (i) – (v) of Lemma **??**, $|C_{\mathcal{G}^{*F^*}}(s_1)|$ and $|C_{\mathcal{G}^{*F^*}}(s_2)|$ are coprime, a contradiction. If $S = \mathrm{PSL}_2(r)$, then q divides $(2, r-1)$ and so q divides $|\mathcal{G}^{*F^*} : C_{\mathcal{G}^{*F^*}}(s_1)|_{\ell'}$, again a contradiction. If $S = \mathrm{PSU}_n(r)$ with $n \geq 6$ even, then q divides $(r^3+1)(r^2-1)$ and so q divides $|\mathcal{G}^{*F^*} : C_{\mathcal{G}^{*F^*}}(s_1)|_{\ell'}$, a contradiction. In all other cases, q divides $r^2 - 1$ and so q divides $|\mathcal{G}^{*F^*} : C_{\mathcal{G}^{*F^*}}(s_1)|_{\ell'}$, again a contradiction.

5) We have shown that $q = \ell$. Notice that the Sylow ℓ-subgroups of S can be abelian only when $S = \mathrm{PSL}_2(r)$. Assume $S = \mathrm{PSL}_2(r)$. In this case, it is easy to check that no Sylow ℓ-subgroup of S can be centralized by σ. Now assume that H is any subgroup of G with $|H|_\pi = |G|_\pi$. Then we may assume that $\sigma \in H$, whence H is not abelian. □

Lemma 2.7. *Let S be a finite simple nonabelian group. Let p and q be different prime divisors of $|S|$ and $\pi := \{p, q\}$. If pq does not divide $\chi(1)$ for every $\chi \in \mathrm{Irr}(S)$, then for any $H \leq S$ such that $|H|_\pi = |S|_\pi$, H is not abelian.*

Proof. By Corollary 1.2 of [?], we know that the hypothesis holds for some pair of primes p and q only if S is M_{11}, M_{23}, J_1, $\mathrm{PSL}_2(r)$ where $r \geq 4$, $\mathrm{PSL}_3(r)$ where $r = 4$ or some prime bigger than 3 divides $r - 1$, $\mathrm{PSU}_3(r)$ where some prime bigger than 3 divides $r + 1$, A_8, or a Suzuki simple group.

Assume S is one of the aforementioned simple groups that satisfies the assumptions, but not the conclusion of the lemma. Then the abelian subgroup H contains a Sylow p-subgroup P of S. Clearly,

(i) P is abelian,

(ii) $C_S(x)$ cannot be a p-subgroup; and moreover,

(iii) pq divides $|C_S(x)|_\pi$

for any nontrivial $x \in P$.

First consider the case $S = \mathrm{PSL}_2(r)$. We may assume that $r \geq 4$ and $r \neq 5$. The condition (ii) implies that $(pq, r) = 1$. Interchanging p and q if necessary, we may assume that $p \neq 2$ and $p|(r - \epsilon)$ for some $\epsilon = \pm 1$. By (iii), pq divides $(r - \epsilon)/2$. But this yields a contradiction, since S has an irreducible character of degree $r - \epsilon$ (as $r \neq 5$).

Next assume that $S = \mathrm{PSU}_3(r)$. The condition (i) implies that $(pq, r) = 1$. Interchanging p and q if necessary, we may assume that $p \neq d := (3, r + 1)$. By (iii), pq divides $(r^2 - 1)(r + 1)/d$, $(r + 1)^2/d$, $(r^2 - 1)/d$, or $(r^2 - r + 1)/d$. But this yields a contradiction, since S has irreducible characters of degrees $(r^2 - 1)(r + 1)$ and $r^2 - r + 1$ (cf. [?]) for instance). The cases $S = \mathrm{PSL}_3(r)$ or $S = {}^2B_2(r)$ can be treated similarly.

Assume $S = M_{23}$. The condition (i) implies that $p, q > 2$, whereas the condition (ii) implies that $p, q \neq 11, 23$. Thus $\pi \subset \{3, 5, 7\}$. So we arrive at a contradiction, as S has irreducible characters of degrees 45, 231, and 770, cf. [?]. The cases $S \in \{A_8, M_{11}, J_1\}$ can be dealt with similarly. □

Next, we prove Main Theorem for simple groups and we determine the simple counterexamples to the analog of Palfy's theorem.

Lemma 2.8. *Let S be a finite simple group and π a set of four prime divisors of irreducible character degrees of S. Then there are primes p and q in π such that pq divides the degree of some irreducible character of S. The same result holds when $|\pi| = 3$ except if $S = \mathrm{PSL}_2(l)$ for some prime power l.*

Proof. Again using Corollary 1.2 of [?], we see that it suffices to check the result for a short list of simple groups. There are explicit formulas for the character degrees of these groups, so it is a routine to complete the proof of the lemma. □

In the proof of Main Theorem, we will also use the following known result.

Lemma 2.9. *The set of coprime automorphisms of a finite simple group S is a cyclic central subgroup of* $\mathrm{Out}(S)$.

Proof. This follows from, for instance, Lemma 1.3 of [?]. □

3. Proof of Main Theorem

We will use the following result on permutation groups.

Lemma 3.1. *Let G be a permutation group on a finite set Ω. Let p and q be distinct primes. Then there exist $\Gamma_1, \Gamma_2 \subseteq \Omega$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and*

$$\{p, q\} \cap \pi(|G|) \subseteq \pi(|G : G_{\Gamma_1} \cap G_{\Gamma_2}|),$$

where G_{Γ_i} is the stabilizer of Γ_i and given an integer n, $\pi(n)$ is the set of prime divisors of n.

Proof. This is Lemma 8 of [?]. $\square$

As an immediate consequence, we have the following.

Corollary 3.2. *Assume that a group G acts on a finite set Ω. Let p and q be distinct primes. Then there exist $\Gamma_1, \Gamma_2 \subseteq \Omega$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and*

$$\{p, q\} \cap \pi(|G/\operatorname{Ker}_G(\Omega)|) \subseteq \pi(|G : G_{\Gamma_1} \cap G_{\Gamma_2}|).$$

Finally, we need the following elementary result.

Lemma 3.3. *Let G be a nonabelian solvable group with $G = O^{r'}(G)$, where r is a prime. Then there exists $\chi \in \operatorname{Irr}(G)$ nonlinear of r-power degree and r-power order.*

Proof. Since $|G/G'|$ is a power of r, all irreducible characters of G have r-power order. It suffices to show that G has a nonlinear irreducible character of r-power degree. Write $X = O^r(G)$. We may assume that G/X is abelian. Let X/Y be a chief factor of G. Observe that X/Y is an elementary abelian r'-group. Since $G = O^{r'}(G)$, we deduce that $C_{G/Y}(X/Y) < G/Y$. It follows that there exists $\lambda \in \operatorname{Irr}(X/Y)$ that is not G-invariant. Hence the degree of any irreducible character of G lying over λ is a power of r, as desired. $\square$

Now we are ready to complete the proof of Main Theorem, which we restate. We will use the Ito-Michler theorem, which asserts that all the character degrees of a finite group G are coprime to p if and only if G has a normal abelian Sylow p-subgroup (see Theorem 12.33 of [?] and [?]).

Theorem 3.4. *Let G be a finite group and π a set of four prime divisors of irreducible character degrees of G. Then there are primes p and q in π such that pq divides the degree of some irreducible character of G.*

Proof. Write $\pi = \{p, q, r, s\}$. Let A be the product of the normal abelian Sylow subgroups of G and R the largest normal solvable subgroup of G. Of course $A \leq R$ and, by the Ito-Michler theorem, all the primes in π divide $|G/A|$. Also, by Palfy's theorem, we may assume that $|R/A|$ is a multiple of at most 2 of the primes in π. In particular, we may assume that

(1) at least two of the four primes in π divide $|G/R|$.

Now, we define a few other normal subgroups of G. First, we define S to be the subgroup of G such that S/R is the socle of G/R. Of course, S/R is a direct product of nonabelian simple groups. It is also clear that $C_{G/R}(S/R) = 1$, so that G/R is isomorphic to a subgroup of $\operatorname{Aut}(S/R)$ and G/S is isomorphic to a subgroup of

$\mathrm{Out}(S/R)$. If $S/R = S_1^{a_1} \times \cdots \times S_t^{a_t}$ for some nonabelian simple groups S_i such that S_j is not isomorphic to S_k if $j \neq k$, then it is well-known that $\mathrm{Out}(S/R)$ is isomorphic to

$$\Gamma = \mathrm{Out}(S_1) \wr \Sigma_{a_1} \times \cdots \times \mathrm{Out}(S_t) \wr \Sigma_{a_t}.$$

In particular, G/S is isomorphic to a subgroup of Γ. We have that $\Delta = \mathrm{Out}(S_1)^{a_1} \times \cdots \times \mathrm{Out}(S_t)^{a_t}$ is a normal subgroup of Γ, so $T/S = \Delta \cap G/S$ is a normal subgroup of G/S. It is well-known that the outer automorphisms of a nonabelian simple group S of order coprime to $|S|$ form a cyclic central subgroup of $\mathrm{Out}(S)$ (see Lemma **??**). Write $\mathrm{COut}(S)$ to denote this subgroup and $\Lambda = \mathrm{COut}(S_1)^{a_1} \times \cdots \times \mathrm{COut}(S_t)^{a_t}$. Then $U/S = \Lambda \cap G/S$ is a normal subgroup of G/S and $U \leq T$. Observe that all the prime divisors of $|T/U|$ divide $|S/R|$.

Also, G/T is a permutation group on the simple groups that appear in the decomposition $S/R = S_1^{a_1} \times \cdots \times S_t^{a_t}$. We will write this group as $S/R = D_1 \times D_2 \times \cdots \times D_{a_1+\cdots+a_t}$, where $D_1, \ldots, D_{a_1}$ are isomorphic to S_1, $D_{a_1+1}, \ldots, D_{a_1+a_2}$ are isomorphic to S_2 and so on. In order to simplify the notation, we will look at the group G/T as a permutation group on the set $\Omega = \{1, 2, \ldots, a_1 + \cdots + a_t\}$. In fact, we will look at G as a group that is acting on Ω, where $\mathrm{Ker}_G(\Omega) = T$.

We may also assume that

$$\text{if } p, q \text{ divide } |S/R| \text{ then } pq \text{ divides some } |D_i|. \tag{2}$$

(Indeed, assume $p||D_i|$ and $q||D_j|$ for some $i \neq j$. Then we can find $\chi_i \in \mathrm{Irr}(D_i)$ and $\chi_j \in \mathrm{Irr}(D_j)$ such that $p|\chi_i(1)$ and $q|\chi_j(1)$. In this case, an irreducible character of G/R lying above $\chi_i \otimes \chi_j \in \mathrm{Irr}(S/R)$ has degree divisible by pq.)

Now, we split the proof of the theorem into several steps.

Step 1. We may assume that at most one of the primes in π divides $|G/T|$.

Assume that, for instance, pq divides $|G/T|$. By Corollary **??**, we know that there exist disjoint subsets Γ_1 and Γ_2 of Ω such that pq divides $|G : G_{\Gamma_1} \cap G_{\Gamma_2}|$. Of course, Γ_i corresponds to the normal subgroup $N_i = \times_{j \in \Gamma_i} D_j$ of S/R and $N_1 \cap N_2 = R/R$. For each of the simple groups S_i that are direct factors of S/R choose nonprincipal characters $\varphi_{i1}, \varphi_{i2} \in \mathrm{Irr}(S_i)$ such that $\varphi_{i1} \neq \varphi_{i2}$. Let $\varphi_1 \in \mathrm{Irr}(N_1)$ be the product of $|\Gamma_1|$ characters of the form φ_{i1}, one for each of the direct factors D_j that appear in N_1. Similarly, let $\varphi_2 \in \mathrm{Irr}(N_2)$ be the product of $|\Gamma_2|$ characters of the form φ_{i2}, one for each of the direct factors D_j that appear in N_2. Then we may view $\varphi = \varphi_1 \otimes \varphi_2$ as an irreducible character of S/R (similar idenfications will be made later in this proof). It is easy to see that the size of the G-orbit of this character is a multiple of pq. It follows from Clifford's theorem that pq divides the degree of some irreducible character of G. Hence, we may assume that at most one of the primes in π divides $|G/T|$, as desired.

Step 2. We may assume that $G = T$.

Assume now that, for instance, p divides $|G/T|$. As before, we consider disjoint subsets Γ_1 and Γ_2 of Ω such that p divides $|G : G_{\Gamma_1} \cap G_{\Gamma_2}|$. We also define the normal subgroups N_1 and N_2 of S/R as before. By (**??**) and Step 1, there is some prime not equal to p in π, say q, such that q divides $|T/R|$.

Assume first that q divides $|S/R|$. If q divides $|S_j|$ for some S_j in N_k (for $k = 1$ or 2), then it suffices to argue as before choosing φ_{jk} of degree a multiple of q. If q does not divide $|N_1N_2|$, then we argue as before but now we take $\varphi = \varphi_1 \otimes \varphi_2 \otimes \mu \in \mathrm{Irr}(S/R)$, where $\mu \in \mathrm{Irr}(S_l)$ for some l is a character whose degree is a multiple of q. As before, the degree of the characters of G lying over φ is a multiple of pq.

Hence, we may assume that q divides $|U/S|$ but q does not divide $|S/R|$. Then there exists $uS = u_1 \dots u_{a_1+\dots+a_t}S \in U/S$ of order q, with $u_i \in \mathrm{COut}(D_i)$. Assume that, for instance, the order of u_1S is q, i.e., u_1S is an automorphism of D_1 of order q. If $1 \in \Gamma_1$, then it suffices to consider $\varphi_{i1}, \varphi_{i2}$ as in Step 1 in such a way that the characters φ_{i1} lie in a regular orbit under the action of $\mathrm{COut}(S_i)$ (this is possible by Proposition **??**). The degree of the irreducible characters of G lying over the character φ defined in Step 1 is a multiple of pq. Similarly, if $1 \in \Gamma_2$, then it suffices to consider $\varphi_{i1}, \varphi_{i2}$ as in Step 1 in such a way that the characters φ_{i2} lie in a regular orbit under the action of $\mathrm{COut}(S_i)$. Finally, if $1 \notin \Gamma_1 \cup \Gamma_2$, then we define $\varphi_{i3} \in \mathrm{Irr}(S_i)$ lying in a regular orbit under the action of $\mathrm{COut}(S_i)$ and such that $\varphi_{i3} \notin \{\varphi_{i1}, \varphi_{i2}\}$ (this is possible as $|\mathrm{Irr}(S_i)| \geq 4$ and we can take the characters φ_{i1}, for instance, lying in nonregular orbits under the action of $\mathrm{COut}(S_i)$, if necessary). We now define the characters

$$\varphi_3 = \bigotimes_{i \in \Omega \setminus (\Gamma_1 \cup \Gamma_2)} \varphi_{i3} \in \mathrm{Irr}(S/R)$$

and

$$\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \in \mathrm{Irr}(S/R).$$

One can easily see that both p and q divide the degree of any irreducible character of G lying over φ. Step 2 follows.

Step 3. We may assume that $G = U$.

It suffices to note that all the prime divisors of $|T/U|$ divide $|S/R|$.

Step 4. There is at most one prime in π that divides $|U/S|$.

Assume that at least two of the primes in π, say p and q, divide $|U/S|$. We claim that pq divides the degree of some irreducible character of U. The group U/S is abelian, so we may assume that it is a cyclic group of order pq. Let $x = x_1 \dots x_{a_1+\dots+a_t}$ be a generator of U/S, where each x_i is an outer automorphism of D_i of order some divisor of pq. If some of the x_i's has order pq, then the result follows from Proposition **??**. Otherwise, we can find some i such that $|x_i| = p$ and some j such that $|x_j| = q$. If $\tau_1 \in \mathrm{Irr}(D_i)$ is not x_i-invariant and $\tau_2 \in \mathrm{Irr}(D_j)$ is not x_j-invariant (and they exist by Proposition **??**), then $\tau_1 \otimes \tau_2 \in \mathrm{Irr}(S/R)$ induces irreducibly to G and its degree is a multiple of pq.

Step 5. We may assume that $G = S$.

Arguing as before, we may assume that $U/S = \langle x \rangle$ is cyclic of order p. Replacing G by S if necessary, we may assume that p does not divide $|S/A|$. We know by (**??**) that some of the primes in π, say q, divides $|S/R|$. If q divides $|D_i|$ then, as one can easily see, we may assume that x does not act trivially on D_i. (Otherwise x acts nontrivially on some D_j with $j \neq i$. Take $\chi_j \in \mathrm{Irr}(D_j)$ which is not x-invariant, $\chi_i \in \mathrm{Irr}(D_i)$ of degree divisible by q. Then some irreducible character of G/R lying above $\chi_i \otimes \chi_j \in \mathrm{Irr}(S/R)$ has degree divisible by pq.) But we may assume that x fixes all the irreducible characters of D_i whose degree is a multiple of q. By Theorem **??**,

$$U/R \text{ does not have any abelian subgroup } Y \text{ with } |Y|_\sigma = |U/R|_\sigma \tag{3}$$

for $\sigma = \{p, q\}$.

Assume first that all 4 primes in π divide $|U/R|$. Then by (**??**) we may assume that qrs divides $|D_i|$ for some i. By our assumption, p does not divide $|D_i|$. By Lemma **??**, D_i is isomorphic to $\mathrm{PSL}_2(l)$ for some prime power l and we may assume that q divides $l+1$, r divides l and s divides $l-1$. We also have that x does not act trivially on D_i. It is easy to see (say by using Theorem **??**) that pw, for some $w \in \{q, r, s\}$, divides the degree of some irreducible character of U/R.

Hence, we may assume that some of the primes in π does not divide $|U/R|$. Thus r, for instance, divides $|R/A|$ and does not divide $|U/R|$. Put $X = O^{r'}(R)$. By Lemma **??**, X has some nonlinear irreducible character γ of r-power degree and r-power order. Since $|U/X|$ is coprime to r, γ extends to its inertia subgroup I in U (by Corollary 8.16 of [**?**]). Let $\hat{\gamma}$ be such an extension. If p or q divides $|U : I|$, then by Clifford's theorem $(\hat{\gamma})^U \in \mathrm{Irr}(U)$ has degree divisible by pr or qr. Hence we may assume that $|I|_\sigma = |U|_\sigma$. Now (**??**) implies that I/X cannot have an abelian Hall $\{p, q\}$-subgroup. By the Ito-Michler theorem it follows that either p or q divides the degree of some irreducible character β of I/X. Now, by Gallagher's theorem (Corollary 6.17 of [**?**]) and Clifford's theorem $(\hat{\gamma}\beta)^U \in \mathrm{Irr}(U)$ is a character whose degree is a multiple of pr or qr.

Consequently, we may assume that none of the primes in π divides $|U/S|$ and so we may replace G by S, whence Step 5 follows.

Step 6. Completion of the proof.

Recall that now we have $G = S$. Again by (**??**) and (**??**) we may assume that pq divides $|D_i|$ for some i. Now if r and s both divide $|S/R|$, then we may assume $pqrs$ divides $|D_i|$ by (**??**), and so we are done by Lemma **??**. Thus we may assume that r divides $|R/A|$ but does not divide $|S/R|$. Set $X = O^{r'}(R)$ and consider $\gamma \in \mathrm{Irr}(X)$ of

r-power degree and r-power order as in Step 5. Arguing as in Step 5, we see that γ extends to (an irreducible character $\hat{\gamma}$ of) its inertia group I in G, and that $|I|_\sigma = |G|_\sigma$ for $\sigma = \{p, q\}$. Applying Lemma **??** to D_i, we may assume that D_i has no abelian subgroup Y with $|Y|_\sigma = |D_i|_\sigma$. It follows that none of G/R, I/R, and I/X can have an abelian Hall $\{p, q\}$-subgroup. By the Ito-Michler theorem, either p or q divides the degree of some irreducible character β of I/X. Consequently, $(\hat{\gamma}\beta)^G \in \mathrm{Irr}(G)$ has degree divisible by pr or qr. $\square$

References

[Bo] A. Borel, R. Carter, C. W. Curtis, N. Iwahori, T. A. Springer, and R. Steinberg, '*Seminar on Algebraic Groups and Related Finite Groups*', Lect. Notes in Math., vol. **131**, Springer-Verlag, Berlin, 1970.

[Bu] R. Burkhardt, Über die Zerlegungszahlen der Suzukigruppen $Sz(q)$, *J. Algebra* **59** (1979), 421 – 433.

[C1] R. Carter, Conjugacy classes in the Weyl group, *Compositio Math.* **25** (1972), 1 – 59.

[C2] R. Carter, '*Finite Groups of Lie type: Conjugacy Classes and Complex Characters*', Wiley, Chichester, 1985.

[CD] C. Casolo and S. Dolfi, Products of primes in conjugacy class sizes and irreducible character degrees, submitted.

[Atlas] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, '*An ATLAS of Finite Groups*', Clarendon Press, Oxford, 1985.

[De1] D. I. Deriziotis, Centralizers of semisimple elements of the Chevalley groups E_7 and E_8, *Tokyo J. Math.* **6** (1983), 191 – 216.

[De2] D. I. Deriziotis, '*Conjugacy Classes and Centralizers of Semisimple Elements in Finite Groups of Lie Type*', Vorlesungen aus dem Fachbereich Mathematik der Universität Essen, Heft 11, 1984.

[Do] S. Dolfi, On independent sets in the class graph of a finite group, *J. Algebra* (to appear).

[Ge] M. Geck, Irreducible Brauer characters of the 3-dimensional special unitary groups in non-describing characteristic, *Comm. Algebra* **18** (1990), 563 – 584.

[GW] D. Gluck and T. R. Wolf, Brauer's height conjecture for p-solvable groups, *Trans. Amer. Math. Soc.* **282** (1984), 137–152.

[GLS] D. Gorenstein, R. Lyons and R. Solomon, '*The Classification of the Finite Simple Groups*', Number 3, Mathematical Surveys and Monographs, Amer. Math. Soc., Providence, 1994.

[H] G. Hiss, Regular and semisimple blocks of finite reductive groups, *J. London Math. Soc.* **41** (1990), 63 – 68.

[Is] I. M. Isaacs, '*Character Theory of Finite Groups*', Academic Press, New York, 1976.

[Le] M. L. Lewis, An overview of graphs associated with character degrees and conjugacy class sizes in finite groups, *Rocky Mountain J. Math.* (to appear).

[L1] G. Lusztig, '*Characters of Reductive Groups over a Finite Field*', Annals of Math. Studies **107**, Princeton Univ. Press, Princeton, 1984 .

[L2] G. Lusztig, On the representations of reductive groups with disconnected centre, in: '*Orbites Unipotentes et Représentations, I.*', *Astérisque* **168** (1988), pp. 157 – 166.

[MSW] O. Manz, R. Stazewski, and W. Willems, On the number of components of a graph related to character degrees, *Proc. Amer. Math. Soc.* **103** (1988), 31 – 37.

[MW] O. Manz and T. R. Wolf, '*Representations of Solvable Groups*', Cambridge University Press, Cambridge, 1993.

[Mi] G. O. Michler, A finite simple group of Lie type has p-blocks with different defects, $p \neq 2$, *J. Algebra* **104** (1986), 220 − 230.

[NT] G. Navarro and Pham Huu Tiep, Rational irreducible characters and rational conjugacy classes in finite groups, submitted.

[Pa] P. P. Pálfy, On the character degree graph of solvable groups I: three primes, *Period. Math. Hungar.* **36** (1998), 61 − 65.

[Sh] K. Shinoda, The conjugacy classes of the finite Ree groups of type (F_4), *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **22** (1975), 1 − 15.

[TZ] Pham Huu Tiep and A. E. Zalesskiĭ, Minimal polynomials of semisimple elements in cross characteristic representations of finite groups of Lie type, (in preparation).

[Wa] H. N. Ward, On Ree's series of simple groups, *Trans. Amer. Math. Soc.* **121** (1966), 62−89.

[Wh] D. L. White, Degree graphs of simple orthogonal and symplectic groups, *J. Algebra* (to appear).

[Zs] K. Zsigmondy, Zur Theorie der Potenzreste, *Monath. Math. Phys.* **3** (1892), 265 − 284.

Departament d'Àlgebra, Universitat de València, 46100 Burjassot, València, Spain

E-mail address: `Alexander.Moreto@uv.es`

Department of Mathematics, University of Florida, Gainesville, FL 32611, U.S.A.

E-mail address: `tiep@math.ufl.edu`