ZEROS OF CHARACTERS ON PRIME ORDER ELEMENTS

by

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ABSTRACT. Suppose that G is a finite group, let χ be a faithful irreducible character of degree a power of p and let P be a Sylow p- subgroup of G. If $\chi(x) \neq 0$ for all elements of G of order p, then P is cyclic or generalized quaternion.

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Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$ be an irreducible complex character of G. If χ is non-linear, W. Burnside's theorem on zeros asserts that there exists $x \in G$ such that $\chi(x) = 0$. It has been recently proved (using the classification) that there always exists $x \in G$ of primer power order such that $\chi(x) = 0$ (**[2]**). Is it possible to find a zero of χ among the elements of G of prime order? Since the elements of prime order can generate a proper normal subgroup of the group G, it is natural to assume in our question that χ is faithful. Even in this case, the answer is no, as shown by any quaternion group. (It is worth mentioning that there is strong evidence that the answer is yes for simple groups, however.) On the other hand, it seems that groups G having a faithful character χ with no zeros on elements of prime order have a restricted Sylow subgroup structure (for those primes dividing $\chi(1)$).

THEOREM A. *Suppose that $\chi \in \mathrm{Irr}(G)$ is faithful of degree a power of p, and let $P \in \mathrm{Syl}_p(G)$. If $\chi(x) \neq 0$ for all $x \in G$ of order p, then P is cyclic or generalized quaternion.*

Proof. Let Z_p be the Sylow p-subgroup of $Z = \mathbf{Z}(G)$. Write $\chi_Z = \chi(1)\lambda$, where $\lambda \in \mathrm{Irr}(Z)$ is faithful. Notice that Z is cyclic. Also, $Z_p \subseteq \mathbf{Z}(P)$.

Let $u \in P$ of order p satisfying $[P, u] \subseteq Z_p$. We claim that $u \in Z_p$. Suppose that $\mathbf{C}_{G/Z_p}(uZ_p) > \mathbf{C}_G(u)/Z_p$. In this case, there exists $y \in G$ such that $1 \neq [u, y] = z \in Z_p$. Then

$$\chi(u) = \chi(y^{-1}uy) = \chi(uz) = \chi(u)\lambda(z)\,,$$

and we deduce that $\chi(u) = 0$, because $\lambda(z) \neq 1$. Hence, $\mathbf{C}_{G/Z_p}(uZ_p) = \mathbf{C}_G(u)/Z_p$. Since $P/Z_p \subseteq \mathbf{C}_{G/Z_p}(uZ_p)$, we deduce that $\mathbf{C}_G(u)$ contains P. In this case, $(\chi(1), |G : \mathbf{C}_G(u)|) = 1$, and by Burnside's Theorem (3.8) of **[1]**, we deduce that $\chi(u) = 0$ or $|\chi(u)| = \chi(1)$. Since $\chi(u) \neq 0$, we have that $|\chi(u)| = \chi(1)$. Then $u \in Z \cap P = Z_p$, as claimed.

In particular, $\Omega_1(\mathbf{Z}(P)) \subseteq Z_p$. Therefore, $\mathbf{Z}(P)$ is cyclic. Now, suppose that $v \in \mathbf{Z}_2(P)$ has order p, and let $x \in P$. Then

$$[x, v]^p = [x, v^p] = 1\,,$$

and we deduce that $[x, v]$ is an element of order p in $\mathbf{Z}(P)$. Hence $[x, v] \in Z_p$. By the claim, we have that $v \in Z_p$. We conclude that $\Omega_1(\mathbf{Z}_2(P)) \subseteq Z_p$, and therefore that $\Omega_1(\mathbf{Z}_2(P))$ is cyclic. By (4.5) of **[4]**, we have that every abelian normal subgroup of P is cyclic. Therefore, by (4.3) of **[4]**, we deduce that P is cyclic, generalized quaternion, or dihedral or semidihedral of order greater than or equal to 16.

Suppose now that P is dihedral or semidihedral of order greater than or equal to 16. In both cases, P has a cyclic subgroup C of index 2 and an involution h outside C. Also $\mathbf{Z}(P) \subseteq P' \subseteq C$ and $|\mathbf{Z}(P)| = 2$. In this case, we have that $\mathbf{Z}(P) = Z_p$. We can write $\chi_{\mathbf{Z}(P)} = \chi(1)\mu$, where $1 \neq \mu \in \mathrm{Irr}(\mathbf{Z}(P))$. Now, let $\theta \in \mathrm{Irr}(P)$ be an irreducible constituent of χ_P. Therefore θ lies over μ, and in particular, we deduce that $\theta(1) > 1$. Since $|P : C| = 2$ and C is abelian, we have that there exists $\tau \in \mathrm{Irr}(C)$ such that $\theta = \tau^C$. Then $\theta(h) = 0$ for all irreducible constituents θ of χ_P. Therefore $\chi(h) = 0$, and this proves the theorem. ∎

We remind the reader that the character χ in Theorem A above does not vanish on any element of G of prime order (for instance, by Theorem (2.1) of **[3]**).

The hypothesis in Theorem A of $\chi(1)$ being a power of p cannot be removed. For instance, $PSL(2,7)$, which has dihedral Sylow 2-subgroups, has a faithful irreducible character χ of degree 6 which does not vanish on any element of order 2. However, χ does vanish on the elements of order 3. Furthermore, we have the following.

EXAMPLE B. *There exists a group G of order $2^6 \cdot 3^4$ having a faithful character $\chi \in \mathrm{Irr}(G)$ of degree 6 such that $\chi(x) \neq 0$ for all elements of G of squarefree order, and such that no Sylow subgroup of G is cyclic or generalized quaternion.*

Our group G is constructed in the following way. We let C be a cyclic group of order 36. Let K be the subgroup of C of order 6. Now, let V the direct product of 4 copies of the cyclic group of order 2, and let H be cyclic of order 9. Then C acts on $H \times V$ in the following way. We will have that K is in the kernel of the action. Also, the cyclic group C/K of order 6 acts on H like its automorphism group, and acts like

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \in \mathrm{GL}(4,2)$$

on V. By elementary but somewhat tedious calculations, the reader can check that the semidirect product $G = (H \times V)C$ satisfies the conclusions of Example B.

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