A finiteness condition on normal subgroups of nilpotent groups

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1 Introduction

It is an elementary fact that for every group G any subgroup contained in the center or containing G' is normal in G. What can be said about a group if these are its only normal subgroups? Of course, all abelian and all simple groups enjoy this property, so we have groups of this kind at both ends of the spectrum. We have studied in [1] the finite nilpotent groups satisfying this condition, that we call the strong condition on normal subgroups. It is easy to reduce this study to the case of finite p-groups, and then we proved that the nilpotency class is at most 3 and, furthermore, that when the class of G is 3 then $|G : Z(G)| \leq p^3$ and $|G| \leq p^5$. This has been established independently by G. Silberberg in [6]. On the other hand, there are no bounds even for the index of the center when the class is 2, as extraspecial p-groups show. In [1] we also dealt with finite p-groups satisfying a slightly weaker hypothesis, namely that for any normal subgroup N of G, either $G' \leq N$ or $|NZ(G) : Z(G)| \leq p$. In this case the nilpotency class is at most 4, but the order of G cannot be bounded if the class is greater than 2. However, it is still possible to bound the index of the center, more precisely we get that $|G : Z(G)| \leq p^6$. I.M. Isaacs [3] has generalized our results in the following way. Suppose G is a finite p-group and that $G' \leq N$ or $|NZ(G) : Z(G)| < p^a$ holds for any normal subgroup N of G, where a is a fixed positive integer. Then the class of G is at most $a + 2$, and Isaacs has proved that if the class exceeds 2 then $|G : Z(G)| \leq p^{3a}$, and that this bound is sharp for odd p.

In this paper we extend this kind of results to the realm of infinite groups. Let n be a positive integer or ∞. We say that a group G satisfies condition $\mathfrak{C}_n$, or that G is a $\mathfrak{C}_n$-group, if either $G' \leq N$ or $|NZ(G) : Z(G)| < n$ for any normal subgroup N of G. Our purpose is to obtain information about the index $|G : Z(G)|$ from this finiteness condition on normal subgroups. Again we focus our attention on nilpotent groups.

We first consider the case when n is a positive integer. In this setting, the natural generalization of Isaacs' result seems to be to prove that the index of the center is n-bounded for the nilpotent $\mathfrak{C}_n$-groups of class greater than 2. (We will say that a certain invariant associated to a family of groups is n-bounded if it can be uniformly bounded by a function of n for all groups in the family. We may speak similarly of boundedness in terms of more than one parameter.) However, this fails to hold as stated. For instance, any p-group of maximal class of order p^4 satisfies the strong condition, so it is a $\mathfrak{C}_2$-group. Nevertheless, there is no absolute bound for $|G : Z(G)| = p^3$ as p ranges over all primes. This example suggests that we first prove that $|G : Z(G)|$ is finite and then somehow introduce in our bound for $|G : Z(G)|$ one or more "relevant" primes, perhaps the primes dividing that index. In fact, we have obtained the following result.

Theorem A. *Let n be a positive integer and G a nilpotent $\mathfrak{C}_n$-group of class $c > 2$. Then G is central-by-finite. Furthermore, if p is the smallest prime dividing $|G : Z_{c-1}(G)|$ then $|G : Z(G)|$ is (p, n)-bounded.*

It is possible to give actual bounds in Theorem A, but we have not tried to obtain accurate bounds as Isaacs does in the case of finite p-groups.

Next we consider condition $\mathfrak{C}_\infty$. It is clear that central-by-finite groups are $\mathfrak{C}_\infty$-groups and, in view of all the previous results, it seems reasonable to ask whether any nilpotent $\mathfrak{C}_\infty$-group of class greater than 2 is conversely central-by-finite. This question turns out to be well targeted, but the answer brings a small surprise with it: this is true, but we need that the class is greater than 3. Thus we get the following theorem.

Theorem B. *Let G be a nilpotent group of class greater than 3. Then G is a $\mathfrak{C}_\infty$-group if and only if it is central-by-finite.*

Observe that infinite extraspecial p-groups have class 2 and satisfy the strong condition, so *a fortiori* they are $\mathfrak{C}_\infty$-groups. However, they are not central-by-finite. On the other hand, we can construct a nilpotent $\mathfrak{C}_\infty$-group of class 3 with infinite central index as follows. Let $C = \langle x \rangle$ be an infinite cyclic group and let D be the direct product of three copies of Prüfer's C_{p^∞} group. Then C acts on D by means of $(a, b, c)^x = (a, ab, bc)$ and the corresponding semidirect product satisfies all the desired conditions.

Our next two theorems show that, even if nilpotent $\mathfrak{C}_\infty$-groups of class 2 or 3 need not be central-by-finite, it is possible to characterize in several different ways which of these groups are central-by-finite. We call a group G **Prüfer-free** if there are no normal subgroups $N \leq K$ of G such that $K/N \cong C_{p^\infty}$.

Theorem C. *Let G be a nilpotent $\mathfrak{C}_\infty$-group of class 2. Then the following conditions are equivalent:*

(i) *G is central-by-finite.*

(ii) *$G/Z(G)$ is finitely generated.*

Theorem D. *Let G be a nilpotent $\mathfrak{C}_\infty$-group of class 3. Then the following conditions are equivalent:*

(i) *G is central-by-finite.*

(ii) *G' is finite.*

(iii) $G/Z_2(G)$ *is finitely generated.*

(iv) $G/Z_2(G)$ *is a torsion group.*

(v) $G/Z_2(G)$ *is a Prüfer-free group.*

For a $\mathfrak{C}_\infty$-group G of class 2, however, we cannot conclude that G is central-by-finite if G' is finite or if $G/Z(G)$ is a torsion group or a Prüfer-free group. This is shown again by infinite extraspecial p-groups.

The proof of all the theorems above relies on the following result about capable groups which, we think, has some interest by itself. Recall that a group P is **capable** if there exists a group G such that $P \cong G/Z(G)$.

Theorem E. *Let P be a nilpotent capable group such that $|P'| = n < \infty$. Then $|P : Z(P)|$ is n-bounded.*

Isaacs [2] has proved this same result for arbitrary finite capable groups. By a famous theorem of Schur, if G is central-by-finite then G' is finite. It is also well-known that the converse is not true, even for nilpotent groups, consider for example an infinite extraspecial p-group. Thus Theorem E proves that we can obtain a partial converse to Schur's Theorem with the additional hypothesis of G being nilpotent and capable. It would be nice to remove the nilpotency hypothesis, as Isaacs does in the finite case. We have not attempted to do this, however.

We close this introduction by mentioning another related problem we have considered, which is the study of $\mathfrak{C}_\infty$-groups in the category of profinite groups. In this case, condition $\mathfrak{C}_\infty$ is only required for closed normal subgroups and then it is very easy to prove the following.

Theorem F. *Let G be a profinite group. Then G is a $\mathfrak{C}_\infty$-group if and only if it is central-by-finite.*

2 Nilpotent capable groups

In this section we prove Theorem E, which is a cornerstone for our results about $\mathfrak{C}_n$-groups. Before proceeding, let us mention some facts that we will use freely throughout the paper. If G is a group and $H \leq Z_2(G)$ then $[H^n, G] = [H, G]^n = [H, G^n]$ for all $n \in \mathbb{N}$ and, as a consequence, $\exp[H, G] = \exp HZ(G)/Z(G)$. In particular, the order modulo $Z(G)$ of any element $x \in Z_2(G)$ coincides with the exponent of the commutator subgroup $[x, G]$.

The key to the proof of Theorem E is the following preparatory lemma.

Lemma 2.1. *Let P be a capable group such that $|P'| = n < \infty$. Then there exists another capable group, $Q = H/Z(H)$, satisfying the following conditions:*

(i) $|Q'| = |P'|$ *and* $|Q : Z(Q)| = |P : Z(P)|$.

(ii) *For any $h \in H$ such that its image in Q has prime order and lies in $Q' \cap Z(Q)$, the size of the conjugacy class of h in H is n-bounded.*

Furthermore, if P is nilpotent then Q can be taken to be nilpotent.

Proof. Since P is capable, we may assume that there exists a group G such that $P = G/Z(G)$. Write $Z = Z(G)$.

Let p be any prime dividing $|P' \cap Z(P)|$. Let T/Z be the subgroup generated by the elements of order p in $P' \cap Z(P)$, and write $T_1/Z, \dots, T_k/Z$ to denote the different subgroups of order p of T/Z. It is clear that $k < n$.

Let $U = [T, G]$ and $U_i = [T_i, G]$ for $1 \le i \le k$. Since $T \le Z_2(G)$, all these subgroups are central in G and

$$\exp U = \exp[T, G] = \exp T/Z = p.$$

Hence we can view U as a vector space over the field with p elements. Note that all the U_i are non-trivial, so we can find for every i a maximal subspace M_i of U such that $U_i \not\le M_i$. In particular $|U : M_i| = p$. Put $M = \cap_{i=1}^{k} M_i$. Then $|U : M| \le p^k < n^n$ and $U_i \not\le M$ for every i. In the rest of the proof we write $T_{(p)}$, $U_{(p)}$ and $M_{(p)}$ instead of T, U and M to remark their dependence on the prime p.

Now, let N denote the product of all the $M_{(p)}$ as p runs over the prime divisors of $|P' \cap Z(P)|$. Put $H = G/N$ and $Q = H/Z(H)$. Our aim is to prove that Q and H satisfy the conditions in the statement of the lemma. Observe that we can identify Q with a quotient of P. More precisely, if $Z(H) = X/N$ and we define $R = X/Z$ then $Q \cong P/R$. Since $N \le Z$, it follows that $X \le Z_2(G)$ and consequently $R \le Z(P)$.

Let us see that $P' \cap R = 1$. Otherwise, there is an element $a \in P' \cap R$ of prime order p. In particular, p is a divisor of $|P' \cap Z(P)|$. Since $a \in R = X/Z$, we can write $a = xZ$ with $x \in X$ and then $[x, G] \le N$. But $[x, G]$ has exponent p, so necessarily $[x, G] \le M_{(p)}$. This is a contradiction with the choice of $M_{(p)}$, since $a \in P' \cap Z(P)$. Now the condition $P' \cap R = 1$ has several consequences. In the first place, we deduce that the canonical epimorphism from P onto P/R induces an isomorphism between P' and $(P/R)'$. Thus P' and Q' are isomorphic and in particular $|Q'| = |P'|$. Secondly, two elements of P commute modulo R if and only if they commute in P. Hence

$Z(P/R) = Z(P)/R$ and we derive that $|Q : Z(Q)| = |P : Z(P)|$. This proves (i).

Now, let $h \in H$ be such that its image in Q has prime order p and lies in $Q' \cap Z(Q)$. If $h = gN$ with $g \in G$, it follows from the previous remarks that its image in P, gZ, lies in $P' \cap Z(P)$ and has order p. Consequently $g \in T_{(p)}$ and $[g, G] \le U_{(p)}$. Then

$$[h, H] \le U_{(p)}N/N \cong U_{(p)}/U_{(p)} \cap N = U_{(p)}/M_{(p)},$$

which has order at most n^n. Therefore the number of conjugates of h is n-bounded. $\square$

After this lemma, we can prove Theorem E following some of the ideas of Isaacs in [2].

Theorem 2.2. *Let P be a nilpotent capable group such that $|P'| = n < \infty$. Then $|P : Z(P)|$ is n-bounded.*

Proof. We argue by induction on n. The case $n = 1$ is clear, so suppose $n > 1$. Let us write $P = G/Z(G)$ and $Z = Z(G)$. According to the previous lemma, we may assume that the size of the conjugacy class of $g \in G$ is n-bounded whenever its image in P has prime order and lies in $P' \cap Z(P)$. Since P is nilpotent and non-abelian, we have that $P' \cap Z(P) \neq 1$ and consequently there is at least one element g as above. Let $C = C_G(g)$, which is a subgroup of G of n-bounded index, $X = Z(C)$ and $Q = C/X$. Note that Q is nilpotent and capable. Since $g \in G'Z \setminus Z$ belongs to X, we have that $|Q'| < |P'|$ and by the induction hypothesis $|Q : Z(Q)|$ is n-bounded.

Let $Z(Q) = V/X$, so that V has n-bounded index in G. We can find elements $g_1, \ldots, g_r \in G$ such that $G = \langle g_1, \ldots, g_r, C\rangle$, where r is n-bounded. Put $S/X = V/X \cap (\cap_{i=1}^{r} C_Q(g_iX))$. Since $|Q : C_Q(g_iX)| \le |Q'| < n$ for all i, it follows that $|G : S|$ is n-bounded. On the other hand, it is clear that S/X is central in G/X. Hence

$$[G, S, S] \le [X, S] \le [X, C] = 1$$

and, by Hall's three subgroup lemma, $[S', G] = 1$. Thus $S' \le Z$ and $A = S/Z$ is an abelian subgroup of P of n-bounded index. If $P = \langle b_1, \ldots, b_s, A\rangle$ then $A \cap (\cap_{i=1}^{s} C_P(b_i))$ is a central subgroup of P whose index is n-bounded. This proves the theorem. $\square$

The following lemma is well-known, but we include its short proof here for the convenience of the reader.

Lemma 2.3. *Any capable divisible abelian group is torsion-free. As a consequence, if a quotient of a group G by a central subgroup is a torsion divisible abelian group then G is abelian.*

Proof. Let $D = G/Z(G)$ be a divisible abelian group and suppose D has an element $xZ(G)$ of finite order $n > 1$. Obviously, G is nilpotent of class 2 and hence the exponent of the commutator subgroup $[x, G]$ is also n. Since $[x, G]$ is abelian and its elements are all commutators of the form $[x, y]$, there exists $y \in G$ such that $[x, y]$ has order n. Let p be any prime divisor of n. Since D is divisible, we can find an element $g \in G$ such that $(gZ(G))^p = yZ(G)$. Then $[x, g]^p = [x, g^p] = [x, y]$ and $[x, g]$ has order pn, contrary to the fact that $\exp[x, G] = n$. $\square$

3 Nilpotent $\mathfrak{C}_n$-groups

In this section we prove the rest of the theorems stated in the introduction. For this purpose, we need a series of lemmas. The first one shows that the classes of groups that we are handling are closed for quotients.

Lemma 3.1. *Let G be a $\mathfrak{C}_n$-group and N a normal subgroup of G. Then G/N is also a $\mathfrak{C}_n$-group.*

Proof. Let K/N be a normal subgroup of G/N and assume that $(G/N)' \not\leq K/N$. Then $G' \not\leq K$, so $|KZ(G) : Z(G)| < n$. Write $Z(G/N) = X/N$. Since $Z(G) \leq X$ we obtain that

$$|K/N \cdot Z(G/N) : Z(G/N)| = |KX : X| \leq |KZ(G) : Z(G)| < n$$

and G/N is a $\mathfrak{C}_n$-group. $\square$

However, the class of $\mathfrak{C}_n$-groups is not closed for direct products. It suffices to observe that an infinite extraspecial p-group E satisfies the strong condition but $E \times E$ does not even satisfy condition $\mathfrak{C}_\infty$.

Lemma 3.2. *Let G be an abelian p-group and suppose that G has finitely many elements of order p. Then G is isomorphic to a direct product of finitely many Prüfer groups and a finite abelian p-group.*

Proof. For every $i \geq 1$, let G_i denote the subgroup formed by the elements of G of order at most p^i. Suppose some subgroup G_i is infinite. By Prüfer's First Theorem (see Theorem 10.1.5 in [4]), G_i is a direct product of infinitely many cyclic groups of p-power order. But then we can find infinitely many elements of order p in G, contrary to our assumption. It follows that G has finitely many elements of each possible order and, by Exercise 4.3.5 in [5], G satisfies the minimal condition. Now Kuroš' characterization of the abelian groups with the minimal condition (Theorem 4.2.11 in [5]) shows that G has the specified structure. $\square$

The following two lemmas will play a fundamental role in obtaining Theorems A, B and D, as will become evident in their proof.

Lemma 3.3. *Let G be a nilpotent $\mathfrak{C}_\infty$-group of class 2. If $G' \cong C_{p^\infty}$ then $G/Z(G)$ is not finitely generated, not a torsion group and not a Prüfer-free group.*

Proof. Let us write $Z = Z(G)$. First of all, if $G/Z = \langle x_1 Z, \ldots, x_r Z\rangle$ is finitely generated then $G' = \langle [x_i, x_j] \mid i, j = 1, \ldots, r\rangle$ is also finitely generated and cannot be isomorphic to C_{p^∞}.

Let us assume now, by way of contradiction, that G/Z is a torsion group. Since G has class 2, the exponent of any commutator subgroup $[x, G]$ coincides with the order of x modulo Z, so it is finite. In fact, since $[x, G]$ is contained in $G' \cong C_{p^\infty}$, this subgroup is cyclic of p-power order. In particular, any element of G/Z has p-power order and G/Z is a p-group.

Suppose that G/Z has finitely many elements of order p. By Lemma 3.2, we can decompose G/Z as the direct product of a divisible group K/Z and a finite group L/Z. Then $G = KL$ and $G' = K'[L, G]$. Since G' is a Prüfer group and $\exp[L, G] = \exp L/Z < \infty$, we necessarily have that $G' = K'$. But, on the other hand, if we apply Lemma 2.3 to the torsion divisible abelian group K/Z it follows that K is abelian. Therefore $G' = K' = 1$, which is a contradiction, since G has class 2.

Hence G/Z has infinitely many elements of order p. Let T/Z be the subgroup of G/Z generated by the elements of order p. For any $x \in T$, the subgroup $[x, G]$ has exponent p and, since it is contained in a Prüfer group, it has order p. It follows that the conjugacy class of x in G has p elements, i.e., that $|G : C_G(x)| = p$. With this property in mind, we are going to construct an abelian subgroup A of T such that A/Z is infinite. Let $A_1 = \langle x_1, Z\rangle$, where x_1 is any element of $T \setminus Z$. In general, suppose that we have already built $A_i = \langle x_1, \ldots, x_i, Z\rangle$. Now

$$C_G(A_i) = \bigcap_{j=1}^{i} C_G(x_j)$$

is a subgroup of finite index of G, so $T \cap C_G(A_i)$ has also finite index in T. Since T/Z is infinite and A_i/Z is finite, we can choose an element $x_{i+1} \in (T \cap C_G(A_i)) \setminus A_i$. Define then $A_{i+1} = \langle x_{i+1}, A_i\rangle$. Observe that A_{i+1} is abelian and $|A_i/Z| < |A_{i+1}/Z|$. Then $A = \cup_{i\geq 1} A_i$ is an abelian group contained in T and A/Z is infinite.

Since A is abelian and $G' \cong C_{p^\infty}$ is a subgroup of A, we deduce from Baer's Theorem (4.1.3 of [5]) that $A = G' \times B$ for some subgroup B of A. Let P be the unique subgroup of order p of G' and $N = BP$. Since $[B, G] \leq [T, G]$ has exponent p, it follows that $[B, G] \leq P$ and N is normal in G. It is clear that $G' \not\leq N$ and, on the other hand, $|NZ : Z| = |A : Z| = \infty$. This contradicts the assumption that G is a $\mathfrak{C}_\infty$-group.

Thus we have proved that G/Z is not a torsion group. Let $x \in G$ be an element of infinite order modulo Z. Then $[x, G]$ has not finite exponent and consequently $[x, G] = G'$. Now the map $g \mapsto [x, g]$ is a homomorphism from G onto G'. It follows that $G/C_G(x) \cong G' \cong C_{p^\infty}$ and G/Z is not Prüfer-free. This concludes the proof of the lemma. □

Lemma 3.4. *Let G be a nilpotent $\mathfrak{C}_\infty$-group of class 3 and $P = G/Z(G)$. If P' is finite then P is finite.*

Proof. According to Theorem E, the finiteness of P' implies that $P/Z(P)$ is also finite, so we only need to show that $Z(P)$ is finite. Let us first see that $Z(P)$ is a torsion group. Otherwise, let $a \in Z(P)$ be an element of infinite order. If $a = xZ(G)$ then $N = \langle x, Z(G) \rangle$ is normal in G and $|N : Z(G)| = \infty$. Hence $G' \leq N$, that is, $P' \leq \langle a \rangle$. This is a contradiction, since P' is a non-trivial finite group.

Let π be the (finite) set of primes dividing the order of P'. Since $Z(P)$ is a torsion abelian group, we can write $Z(P) = A \times B$, where A is a π-group and B is a π'-group. Then $B \cap P' = 1$ and, using that G is a $\mathfrak{C}_\infty$-group, we derive that B is finite. Now fix a prime $p \in \pi$ and let A_p be a Sylow p-subgroup of A. Suppose A_p has infinitely many elements of order p. Since P' is finite, we can use some of the elements of order p in A_p to build an infinite subgroup C such that $C \cap P' = 1$. This contradicts that G is a $\mathfrak{C}_\infty$-group. Hence A_p has only finitely many elements of order p and, by Lemma 3.2, A_p is a direct product of finitely many copies of C_{p^∞} and a finite abelian p-group. Since this can be done for every prime $p \in \pi$, we deduce that $Z(P) = D \times E$, where D is a torsion divisible abelian group and E is a finite abelian group. Write $D = N/Z(G)$. Then Lemma 2.3 assures that N is abelian. On the other hand, $|G : N| = |P : D| = |P : Z(P)||E| < \infty$ and there exists $k \in \mathbb{N}$ such that $G^k \leq N$. Since $N \leq Z_2(G)$, we have that

$$[N^k, G] = [N, G^k] \leq [N, N] = 1$$

and $N^k \leq Z(G)$. Then $D^k = 1$ and D is divisible, so $D = 1$ and $Z(P) = E$ is finite, as desired. □

For ease of reference, we state apart the following result, which is Exercise 4.1.2 in [5].

Lemma 3.5. *Let G be an abelian group all of whose proper subgroups are finite. Then G is either finite or a Prüfer group.*

After these preliminary lemmas, we proceed now to prove the rest of our main theorems. We begin by proving Theorems B, C and D together. In these three theorems we have to show that some conditions on a group G are equivalent to G being central-by-finite. If G is central-by-finite then it is clear that the rest of the conditions hold, so we only need to prove the reverse implications.

Theorem 3.6. *Let G be a nilpotent $\mathfrak{C}_\infty$-group and assume that one of the following conditions holds:*

(i) *G has class 2 and $G/Z(G)$ is finitely generated.*

(ii) *G has class 3 and G' is finite.*

(iii) *G has class 3 and $G/Z_2(G)$ is either finitely generated, a torsion group or a Prüfer-free group.*

(iv) *G has class greater than 3.*

Then G is central-by-finite.

Proof. First of all, let us see that if (i) holds then G is central-by-finite. Since $G/Z(G)$ is finitely generated, it suffices to see, as is well-known, that G' is finite. Now G' is a finitely generated abelian group in this case, so it is a direct product of cyclic groups. Suppose one of these cyclic factors, say $\langle z\rangle$, is infinite. Then there exist $x, y \in G$ such that $z = [x, y]$, and both x and y have infinite order modulo $Z(G)$. Now consider $N = \langle x^2\rangle[x^2, G]$. It is clear that N is normal in G and that $|NZ(G) : Z(G)| = \infty$. Since G is a $\mathfrak{C}_\infty$-group, it follows that $G' \leq N$ and, in particular, $z \in N$. Then $z \in N \cap G' = (\langle x^2\rangle \cap G')[x^2, G]$, by Dedekind's Law. But x^2 has infinite order modulo G', so $\langle x^2\rangle \cap G'$ has infinite index in $\langle x^2\rangle$. It follows that $\langle x^2\rangle \cap G' = 1$ and $z \in [x^2, G]$. Hence $z = [x^2, g] = [x, g]^2$ for some $g \in G$ and z is a square in G'. This is a contradiction. We conclude that G' has no infinite cyclic factors and consequently G' is finite, as desired.

Next we see that (ii) and (iii) imply central-by-finiteness. Let us suppose that G has class 3, and write $P = G/Z(G)$. We have to see that P is finite. Since $G' \leq Z_2(G)$, any subgroup N such that $Z(G) \leq N < G'Z(G)$ is normal in G. It follows from condition $\mathfrak{C}_\infty$ that $|N : Z(G)| < \infty$, and therefore any proper subgroup of the abelian group P' is finite. By Lemma

3.5, we deduce that P' is either finite or a Prüfer group. In the second case G' cannot be finite and, by applying Lemma 3.3 to P, we also get that $G/Z_2(G)$ is not finitely generated, a Prüfer-free group or a torsion group. So under any of the hypotheses in (ii) and (iii) we necessarily have that P' is finite. Now, Lemma 3.4 yields that P is finite.

Finally, assume that G has class $c > 3$. Since $G' \not\leq Z_{c-2}(G)$, we have that $|Z_{c-2}(G) : Z(G)| < \infty$. Thus if we want to prove that G is central-by-finite, it suffices to show that $|G : Z_{c-2}(G)|$ is finite or, equivalently, that the quotient $Q = G/Z_{c-3}(G)$ is central-by-finite. By Lemma 3.1, Q is a $\mathfrak{C}_\infty$-group of class 3, and we have to prove that $P = Q/Z(Q)$ is finite. Now we can argue as above to deduce that P' is finite or a Prüfer group. If P' is a Prüfer group then

$$\exp \gamma_3(Q) = \exp[Q', Q] = \exp Q'Z(Q)/Z(Q) = \exp P' = \infty$$

and, in particular, $\gamma_3(Q)$ is infinite. But $\gamma_3(Q) = \gamma_3(G)Z(G)/Z(G)$ and $G' \not\leq \gamma_3(G)$, so we reach a contradiction with G being a $\mathfrak{C}_\infty$-group. Hence P' is finite and, by Lemma 3.4, P is also finite. This concludes the proof of the theorem. $\square$

If G is a $\mathfrak{C}_\infty$-group of class greater than 3, by combining the previous theorem with Schur's Theorem we obtain that the derived subgroup of G is finite. This is not true for groups of class 3, since the example in the introduction has a Prüfer derived subgroup. However, we can prove the following result.

Theorem 3.7. *Let G be a $\mathfrak{C}_\infty$-group of class 3. Then G' is a torsion group.*

Proof. Let $P = G/Z(G)$. As in the proof of the previous theorem, P' is either finite or a Prüfer group. If P' is finite then, by Lemma 3.4, P is also finite, and we deduce from Schur's Theorem that G' is finite.

Thus we may assume that P' is isomorphic to C_{p^∞}. Then $G'/G' \cap Z(G) \cong C_{p^\infty}$ is a torsion group and consequently $\gamma_3(G) = [G', G]$ is also a torsion group. So we only need to prove that $A = G'/\gamma_3(G)$ is a torsion group. Put $B = (G' \cap Z(G))/\gamma_3(G)$. In the rest of the proof, we will use several times the following fact: if $X \subseteq A$ generates A modulo B then X is actually a generating set of A. To see this, we lift the set X to a subset S of G. Then the subgroup $N = \langle S \rangle \gamma_3(G)$ is normal in G and $|NZ(G) : Z(G)| = |G'Z(G) : Z(G)| = \infty$. Thus $G' = N$ and $A = \langle X \rangle$.

Now, for all $i \geq 1$, choose elements $x_i \in A$ of order p^i modulo B. Since $A/B \cong C_{p^\infty}$, these elements generate A modulo B. It follows from the previous remark that $A = \langle x_i \mid i \geq 1 \rangle$. For any $n \in \mathbb{N}$, the elements x_i^n also generate A modulo B, so $A = \langle x_i^n \mid i \geq 1 \rangle = A^n$. Thus A is a divisible abelian group. Let $y_1 \in A$ be an element of order p modulo B. We can find recursively elements y_i in A such that $y_{i+1}^p = y_i$ for all $i \geq 1$. As before, $A = \langle y_i \mid i \geq 1 \rangle$. If y_1 has finite order then A is a torsion group, as desired. Suppose otherwise that the order of y_1 is infinite. Take n coprime to p and let $y \in A$ be an n-th root of y_1. Since $A/\langle y_1^p \rangle \cong C_{p^\infty}$, it follows that $y \in \langle y_1 \rangle$, which is impossible. $\square$

Consider the semidirect product $(\mathbb{Q} \times \mathbb{Q}) \rtimes \mathbb{Q}$, where the action is given by $(a, b)^x = (a, b + ax)$. This is a torsion-free group that satisfies the strong condition on normal subgroups. Thus the derived subgroup of a $\mathfrak{C}_\infty$-group of class 2 need not even be a torsion group.

On the other hand, observe that there is no analog to Theorem 3.7 with $G/Z(G)$ in place of G'. Just take into account the example above for class 2 and the example in the introduction for class 3.

We provide now the proof of Theorem A.

Theorem 3.8. *Let n be a positive integer and G a nilpotent $\mathfrak{C}_n$-group of class $c > 2$. Then G is central-by-finite. Furthermore, if p is the smallest prime dividing $|G : Z_{c-1}(G)|$ then $|G : Z(G)|$ is (p, n)-bounded.*

Proof. We can argue as in the last paragraph of the proof of Theorem 3.6 to reduce ourselves to the case that G has class 3. If $P = G/Z(G)$ then any proper subgroup of P' has order less than n. So, by Lemma 3.5, P' is necessarily finite. Hence we may apply Lemma 3.4 to derive that P is finite, that is, that G is central-by-finite.

Now, let p be the smallest prime dividing $|P : Z(P)|$ and let us see that the order of P is (p, n)-bounded. If $a \in P$ has order p modulo $Z(P)$ then $\exp[a, P] = p$ and we derive that p also divides $|P'|$. Since P' is a finite abelian group, we can find a subgroup Q of P' such that $|P' : Q| = p$. Then $|P'| = |P' : Q||Q| < pn$ is (p, n)-bounded. It follows from Theorem E that $k = |P : Z(P)|$ is also (p, n)-bounded, so it suffices to prove that $|Z(P)|$ is (p, n)-bounded.

Let us take $R \leq Z(P)$ maximal with respect to $P' \cap R = Q$. Since G satisfies condition $\mathfrak{C}_n$, we get that $|R| < n$. By the choice of R, the quotient $Z(P)/R$ cannot have any other subgroup of prime order than $P'R/R$. Hence $Z(P)/R$ is cyclic. In particular, $Z(P)$ can be generated by fewer than n elements. Put $R = N/Z(G)$. Then we have that

$$\begin{aligned}[Z_2(G), G]^{kn} &= [Z_2(G), G^k]^n \leq [Z_2(G), Z_2(G)]^n = [Z_2(G), N]^n \\ &= [Z_2(G), N^n] \leq [Z_2(G), Z(G)] = 1,\end{aligned}$$

where in the second equality we have used that $Z_2(G)/N$ is cyclic. It follows that $\exp Z(P) = \exp Z_2(G)/Z(G) \leq kn$. Thus $Z(P)$ is an abelian group whose number of generators and exponent are (p, n)-bounded, so its order is also (p, n)-bounded. This proves the theorem. $\square$

Let us conclude with the easy proof of Theorem F.

Theorem 3.9. *Let G be a profinite group. Then G is a $\mathfrak{C}_\infty$-group if and only if it is central-by-finite.*

Proof. Suppose G is a profinite $\mathfrak{C}_\infty$-group. If $G' \not\leq N$ for some normal open subgroup N of G then $|NZ(G) : Z(G)| < \infty$. Since $|G : N| < \infty$, it follows that $|G : Z(G)| < \infty$, as we wanted to prove. Otherwise $G' \leq N$ and G/N is abelian for all normal open subgroups N. Since G is the inverse limit of these abelian quotients, we get that G is even abelian in this case. $\square$

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