On the Origin of This Work

Alexander Moretó

While working on a paper with Attila Maróti and Juan Martínez Madrid, I became convinced that picky elements should play an important role in the representation theory of finite groups. This led me to formulate the picky conjecture in June 2023 and, subsequently, the subnormalizer conjecture in the winter of 2024. I found these conjectures exciting, but I believed that something was missing: the right conjecture should imply the McKay conjecture. I was unable to find such a conjecture myself.

In June 2026, I decided to test ChatGPT's mathematical abilities by asking it to prove the strong subnormalizer conjecture for groups of odd order. After extensive work, it produced an elaborate proposed proof. I was not particularly concerned with whether the argument was completely correct. What impressed me was the mathematical creativity involved: ChatGPT produced numerous ideas that I would never have found myself, and that I believe most representation theorists would not have found either. At a time when AI's capacity for original research in representation theory was still widely doubted, this experience convinced me that a fundamental change was already taking place.

On September 8, 2026, OpenAI announced a proposed solution to the Navier–Stokes Millennium Prize Problem, accompanied by a formalization in Lean. Nevertheless, I continued to hear that AI was merely reproducing or recombining existing mathematics. In particular, even if it could solve problems formulated by humans, it supposedly could not formulate genuinely new and important mathematical conjectures.

On September 12, I decided to test precisely this claim. I asked ChatGPT to find what I had been unable to find myself: a generalization of the subnormalizer conjecture that would imply McKay's conjecture.

ChatGPT proposed several candidates, found counterexamples to them, and, in the course of this investigation, discovered counterexamples to the original picky and subnormalizer conjectures. Eventually, it formulated Conjecture B, which implies McKay's conjecture and has survived extensive searches for counterexamples.

All the mathematical work in the manuscript that follows was generated by ChatGPT. I claim no credit for these results or for Conjecture B. If this conjecture, or some slight modification of it, is eventually proved, the only credit I believe I deserve is for having believed that something along these lines should exist.

I have not checked the mathematical results in the manuscript, and I do not intend to do so.

CHARACTER VALUES AND SUBNORMALIZERS

CHATGPT

Abstract. The picky and subnormalizer conjectures predict correspondences between nonzero character values. We give counterexamples for every prime and propose an integral replacement. It combines a height-zero comparison, a Clifford-theoretic refinement, and, at odd primes, weighted value lattices involving every ordinary height. We prove the complete conjecture for solvable groups and for groups with a normal p-complement, when p is odd. We prove its weighted part for $\mathrm{PSU}_3(q)$, $\mathrm{PSp}_4(q)$, $\mathrm{PSp}_6(q)$ and $\mathrm{PGSp}_6(q)$ in odd defining characteristic, and give exact signed correspondences for an infinite family of proper-defect symmetric-group blocks. These families include nonzero values at unbounded positive heights. A solvable counterexample shows that the weighted assertion cannot be extended to two. At two we prove the height-zero and Clifford-level assertions for every subnormalizer and visible block pair in $(5^{1+2} \rtimes Q_8)^n \rtimes T$, where $T \leq \mathfrak{S}_n$ is a transitive 2-group with its permutation action and the quaternion action is symplectic. The conjecture implies McKay and maximal-defect Alperin–McKay, and recovers the strong subnormalizer conjecture for p-solvable groups at odd primes and groups with abelian Sylow p-subgroups.

1. Introduction

Alperin's Main Problem asks for rules that determine character values from local subgroup information. The picky and subnormalizer conjectures in [24] propose such rules for values at p-elements. For a p-element x of a finite group K, write

$$\mathrm{Irr}^x(K) = \{\chi \in \mathrm{Irr}(K) : \chi(x) \neq 0\}.$$

Following Casolo [3], for a subgroup $U \leq G$ put

$$S_G(U) = \{g \in G : U \text{ is subnormal in } \langle U, g\rangle\}, \qquad \mathrm{Sub}_G(U) = \langle S_G(U)\rangle.$$

The set $S_G(U)$ need not be a subgroup. For a p-subgroup U, Lemma 5.1 gives the useful equivalent description

$$\mathrm{Sub}_G(U) = \langle N_G(T) : T \in \mathrm{Syl}_p(G),\ U \leq T\rangle.$$

For cyclic U, this is Malle's Proposition 2.6 [20]. We abbreviate $\mathrm{Sub}_G(\langle x\rangle)$ to $\mathrm{Sub}_G(x)$. An element is *picky* if it belongs to a unique Sylow p-subgroup P; then $\mathrm{Sub}_G(x) = N_G(P)$.

The weak subnormalizer conjecture asks for a bijection from $\mathrm{Irr}^x(G)$ to $\mathrm{Irr}^x(\mathrm{Sub}_G(x))$, preserving the p-parts of the degrees and the fields generated by the values at x. The weak picky conjecture is its assertion for picky elements. The strong form asks that corresponding values agree up to sign. For symmetric groups, Martínez Madrid [22, Theorems A and B] proved the strong picky conjecture at every prime and the strong subnormalizer conjecture at two. Our first result disproves even the required equality of cardinalities.

Theorem A. (a) *For every prime p, there are infinitely many finite groups G with a picky p-element x, whose unique Sylow subgroup P satisfies $x \notin P'$, such that*

$$|\,\mathrm{Irr}^x(G)| < |\,\mathrm{Irr}^x(N_G(P))|.$$

(b) *There is a solvable group G with a picky 2-element x such that*

$$|\,\mathrm{Irr}^x(G)| \neq |\,\mathrm{Irr}^x(\mathrm{Sub}_G(x))|.$$

2020 *Mathematics Subject Classification.* 20C15, 20C20, 20D10.

Key words and phrases. Character values, subnormalizers, Clifford theory, blocks, magic representations, solvable groups.

Theorem 2.3 proves (a), and Proposition 2.5 proves (b). Cancellation in cyclic wreath products is responsible for both failures: values at a base element are sums of products, and different sign patterns can change their zero sets.

We replace the ungraded comparison by integral conditions that retain height-zero irreducibles and constrain values at every ordinary height. There are three comparisons. The first matches height-zero characters after allowing integer combinations of positive-height values. The second uses restriction to a normal subgroup to define a finer grading of the characters. The third controls how values at different ordinary heights may be combined. We define these terms before stating the conjecture.

Fix

$$1 \neq Q \leq P \in \mathrm{Syl}_p(G), \qquad H = \mathrm{Sub}_G(Q).$$

A pair of p-blocks (b, c) is *visible for* Q if b has a defect group D with $Q \leq D \leq P$, and c is its Brauer correspondent in H, with defect group D. The required normalizer containment follows from Lemma 5.2. A p-element $u \in H$ is *eligible* if $\mathrm{Sub}_G(u) \leq H$. Put

$$X_H = \{us : u \in H \text{ is eligible}, \ s \in C_G(u)_{p'}\}, \qquad \mathrm{ev}_Y \chi = (\chi(y))_{y \in Y}.$$

Here $C_G(u)_{p'}$ is the set of p'-elements of the centralizer. Lemma 5.2 gives $C_G(u) \leq H$, so $X_H \subseteq H$. We allow $u = 1$; it is eligible exactly when $H = G$. Write

$$E_H = \{u \in H : u \text{ is an eligible } p\text{-element}\}.$$

Recall that the *p-section* of u in a group K consists of the elements whose p-part is K-conjugate to u. Its classes are represented by us, with $s \in C_K(u)_{p'}$. Thus X_H is a union of entire p-sections of H, and eligibility gives $C_H(u) = C_G(u)$ on each one. The comparison on p-sections, and on reduced p-sections, was proposed in [24, Section 4]. Here part (a) retains the entire sections but permits integral corrections from positive-height characters. Parts (b) and (c) concern only their p-elements.

For a block d of $K = G, H$, let $\mathrm{Irr}_0(K, d)$ be its height-zero characters and put

$$\Lambda_{K,d,+} = \mathbb{Z}\{\mathrm{ev}_{X_H} \chi : \chi \in \mathrm{Irr}(K, d), \ h_d(\chi) > 0\}.$$

Here a profile is simply the list of values on the indicated set. The integer span of profiles consists of their finite sums with integer coefficients; we call it a value lattice. Thus the correction allowed above is an integer combination of positive-height profiles, with the same coefficients at every argument. No rational coefficients are allowed.

Recall that the ordinary height of a character in a block with defect group D is specified by

$$\chi(1)_p = |K : D|_p p^{h_d(\chi)}.$$

Height zero therefore means the smallest degree p-part permitted by the defect group. In a full-defect block it means p'-degree.

Choose one normal subgroup from the global group:

$$R = R_p(G) = \begin{cases} \mathrm{Rad}_p(G), & p \text{ odd}, \\ 1, & p = 2, \end{cases} \qquad N_K = K \cap R \quad (K = G, H).$$

Here $\mathrm{Rad}_p(G)$ is the largest normal p-solvable subgroup. A group is p-solvable if it has a normal series whose factors are p-groups or groups of order prime to p. This is one choice made in G; in the local group we use its intersection with H, not a new radical chosen in H. We group characters in a block together when their restrictions to N_K have the same irreducible constituents. Clifford's theorem says that these constituents form one K-orbit. Let $\mathcal{O}_K(\chi)$ be the K-orbit of the constituents of χ_{N_K}, and set

$$\mathcal{C}_{K,d}(\chi) = \{\psi \in \mathrm{Irr}(K, d) : \mathcal{O}_K(\psi) = \mathcal{O}_K(\chi)\}.$$

The *Clifford level* is

$$\delta^R_{K,d}(\chi) = v_p(\chi(1)) - \min_{\psi \in \mathcal{C}_{K,d}(\chi)} v_p(\psi(1)) = h_d(\chi) - \min_{\psi \in \mathcal{C}_{K,d}(\chi)} h_d(\psi). \tag{1.1}$$

In words, the level is the height above the least height in this restriction family. Level zero means attaining that least height; it need not mean ordinary height zero. If G is p-solvable and p is odd,

then $N_K = K$, every family has one member, and every Clifford level is zero. At two, $N_K = 1$, so a family is a whole block and Clifford level is ordinary height. For example, if the heights in a restriction family are $2, 2, 5$, the corresponding levels are $0, 0, 3$.

This choice of radical interpolates between two comparisons. On a p-solvable group at an odd prime, level zero retains all characters, so part (b) asks for an exact signed comparison at every height. When the radical is trivial, it reduces to height zero modulo positive-height values. In between, the level measures the variation of height among characters lying over the same orbit of lower constituents. At two, taking $R = G$ for solvable groups would again require an exact comparison of all nonzero pure profiles, contrary to Theorem A(b). We therefore use $R = 1$ at that prime. For a finite family Y of eligible p-elements, define

$$F_1^R(K, d; Y) = \mathbb{Z}\{\mathrm{ev}_Y \chi : \chi \in \mathrm{Irr}(K, d),\ \delta_{K,d}^R(\chi) > 0\}, \tag{1.2}$$

$$A_0^R(K, d; Y) = \{\chi \in \mathrm{Irr}(K, d) : \delta_{K,d}^R(\chi) = 0,\ \mathrm{ev}_Y \chi \notin F_1^R(K, d; Y)\}. \tag{1.3}$$

The set A_0^R retains exactly the level-zero characters whose profiles do not become zero after allowing positive-level corrections. We keep their irreducible labels: two distinct characters with identical values are still two members of this set. For an eligible u, define the ordinary-height weighted lattice

$$\mathcal{M}_r(K, d; u) = \sum_{\chi \in \mathrm{Irr}(K,d)} p^{\max\{0, r - h_d(\chi)\}} \mathbb{Z}\chi(u), \qquad r \geq 0. \tag{1.4}$$

At level r, a value of height $h < r$ is multiplied by p^{r-h}; a value of height at least r is left unchanged. Equality of these spans therefore compares all heights, not only the height-zero characters.

Conjecture B (Weighted subnormalizer conjecture)**.** *For every finite group G, prime p, $1 \neq Q \leq P \in \mathrm{Syl}_p(G)$, and visible pair (b, c) for $H = \mathrm{Sub}_G(Q)$, the following hold.*

(a) Ordinary height zero. $\Lambda_{G,b,+} = \Lambda_{H,c,+}$. *There are a bijection*

$$f_b : \mathrm{Irr}_0(G, b) \longrightarrow \mathrm{Irr}_0(H, c)$$

and signs $\varepsilon_\chi \in \{1, -1\}$ such that

$$\mathrm{ev}_{X_H} \chi - \varepsilon_\chi \mathrm{ev}_{X_H} f_b(\chi) \in \Lambda_{H,c,+}. \tag{1.5}$$

(b) Clifford level zero. *On the set E_H of all eligible p-elements,*

$$F_1^R(G, b; E_H) = F_1^R(H, c; E_H). \tag{1.6}$$

There are a bijection

$$f_0 : A_0^R(G, b; E_H) \longrightarrow A_0^R(H, c; E_H)$$

and signs $\varepsilon_\chi \in \{1, -1\}$ such that

$$\chi(1)_p = f_0(\chi)(1)_p, \qquad \mathrm{ev}_{E_H} \chi - \varepsilon_\chi \mathrm{ev}_{E_H} f_0(\chi) \in F_1^R(H, c; E_H). \tag{1.7}$$

(c) All ordinary heights, at odd primes. *If p is odd, then for every eligible u and every integer $r \geq 2$,*

$$\mathcal{M}_r(G, b; u) = \mathcal{M}_r(H, c; u). \tag{1.8}$$

In (a) and (b), one sign and one integral correction work on the whole specified set of arguments. The lattices are used without saturation. The matching in (b) restricts to every subset of E_H, with the same signs and correction coefficients; see Proposition 5.8. No compatibility between the matchings in (a) and (b) is required. Part (a) retains all height-zero characters, including those with zero mixed profile. Part (c) is an elementwise assertion; the coefficients used to compare its lattices may depend on u.

Proposition 6.4 proves the equalities in (c) for $r = 0, 1$ from the general value comparison and (a). Thus the extra requirement begins at $r = 2$. The weights in (1.4) use ordinary height. Proposition 6.5 shows that (c), together with the first two levels, is equivalent to integral reconstruction of each value in both directions, with coefficients satisfying $\chi(1)_p \mid a_{\psi\chi}\psi(1)_p$. This imposes a condition at every height occurring in a visible block.

Theorem C. *Conjecture B implies the following assertions.*

(a) *The Alperin–McKay conjecture holds for blocks of maximal defect, and the McKay conjecture holds for every finite group.*

(b) *The strong subnormalizer conjecture holds for p-solvable groups when p is odd.*

(c) *The strong subnormalizer conjecture holds for groups with abelian Sylow p-subgroups, at every prime.*

Proposition 6.1 proves (a), and Corollary 6.3 proves (b) and (c). Taking $Q = P$ gives $H = N_G(P)$, and part (a) of the conjecture supplies the height-zero bijections even when no nonidentity element is eligible. For exact values, the Clifford levels are all zero when G is p-solvable and p is odd. The same holds in blocks of abelian defect by the height-zero theorem. The correction in part (b) then vanishes. For smaller nonabelian defect, passing from a subnormalizer of the defect group to its normalizer requires an additional argument; the counting conclusion in Theorem C(a) concerns maximal defect.

We distinguish the complete conjecture from its weighted part in the positive results below. Throughout part (b), the prime is the defining prime, not an arbitrary divisor of the group order.

Theorem D. *Let p be an odd prime.*

(a) *Conjecture B holds if G is solvable or has a normal p-complement.*

(b) *Part* (c) *of Conjecture B holds if*

$$G = \mathrm{PSU}_3(q), \quad \mathrm{PSp}_4(q), \quad \mathrm{PSp}_6(q), \quad \textit{or} \quad \mathrm{PGSp}_6(q), \qquad q = p^f.$$

In fact, the weighted equalities hold for every $r \geq 0$, every nontrivial Q, and every visible pair.

The normal p-complement in (a) need not be solvable. Part (b) asserts the weighted comparison, not the complete conjecture. Its proofs include nonzero values of height f, with f arbitrary. At the eligible elements, the surviving values have heights $0, f$. These results test the divisibility of positive-height values required by the weighted comparison; they do not compare several distinct positive heights at one element.

For solvable groups at odd primes we construct an exact signed correspondence on all eligible mixed sections, preserving degree p-parts. Its relative form allows an invariant irreducible character of a normal subgroup. Clifford induction handles proper inertia groups. An invariant constituent either has the extension used by restriction, or is fully ramified. In the latter case an ordinary source and a smaller relative correspondence absorb a linear twist. The proof occupies Sections 7–11. Groups with a normal p-complement are treated separately in Section 12. The family proofs appear in Sections 13–19.

There is also a proper-defect result for symmetric groups. We use $\mathfrak{S}_n$ for the symmetric group on n letters. Blocks are labelled by their p-cores, and characters by partitions.

Theorem E. *Let p be odd, $a \geq 2$, $m = p^a$, and $0 < t < m$. Let τ be a p-core of size t. In $G = \mathfrak{S}_{m+t}$, let $Q \cong C_p^a$ act regularly by translations on m letters and fix the other t. Then*

$$H = \mathrm{Sub}_G(Q) = \mathfrak{S}_m \times \mathfrak{S}_t.$$

The block b labelled by τ and the block $c = B_0(\mathfrak{S}_m) \otimes b_\tau(\mathfrak{S}_t)$ form a visible pair. Matching partitions with the same ordered p-quotient gives a height-preserving bijection $f : \mathrm{Irr}(G, b) \to \mathrm{Irr}(H, c)$ and signs ε_χ such that

$$\chi(u) = \varepsilon_\chi f(\chi)(u) \qquad \textit{for every eligible } p\textit{-element } u.$$

Consequently B(c) *holds for this pair at every $r \geq 0$.*

This theorem concerns the specified block pairs, not all blocks of all symmetric groups. For each fixed odd prime, it gives proper nonabelian-defect blocks with nonzero values at unbounded positive heights, even at eligible elements of fixed order p^2; see Corollary 20.4.

The restriction to odd primes in B(c) is necessary. In Section 3 we construct a solvable group with a normal odd complement and a visible pair for which, at one eligible element v of order sixteen,

$$\mathcal{M}_{80}(G, b; v) = 2^{15}\mathbb{Z}, \qquad \mathcal{M}_{80}(H, c; v) = 2^{16}\mathbb{Z}.$$

Parts (a) and (b) hold for this same pair. They therefore do not imply the weighted assertion at two.

There is nevertheless a uniform positive result at two for the family containing our normal-complement counterexample.

Theorem F. *Let $E = 5^{1+2}$ have exponent five, let Q_8 act on E by the symplectic action in* (2.7)*, and let $T \leq \mathfrak{S}_n$ be a transitive 2-group. Then B*(a),(b) *hold at two for $G = (E \rtimes Q_8)^n \rtimes T$, for every nontrivial Q and every visible pair.*

Theorem 4.1 proves this statement by classifying all subnormalizers and using integral quaternion substitutions. In particular, $S \wr C_4$ fails weak subnormalizer but satisfies the replacement for all its subnormalizers, including those smaller than the one used in the counterexample. This result does not settle B(a),(b) for all solvable groups at two.

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2. Counterexamples to the ungraded conjectures

The unitary construction gives an infinite family for every prime. At two, a normal-complement example and a solvable picky example show that the ungraded assertions also fail in solvable groups. The common calculation is the following cyclic wreath-product formula.

2.1. **Cyclic words and a unitary family.** Write $\mathrm{Irr}^x(K) = \{\chi \in \mathrm{Irr}(K) : \chi(x) \neq 0\}$. A p-element is picky if it belongs to exactly one Sylow p-subgroup. For such an element, with unique Sylow P, $\mathrm{Sub}_K(x) = N_K(P)$. Indeed the Sylow-normalizer description of the subnormalizer then has just one generating normalizer. We compare the numbers of characters with nonzero value, as required by the ungraded picky and subnormalizer conjectures.

Lemma 2.1. *Let $W = L^n \rtimes C_n$, with cyclic permutation of the factors. A rotation orbit of a word $\boldsymbol{\theta} \in \mathrm{Irr}(L)^n$, of length d, gives exactly n/d irreducible characters. Their degree and their common value at $x = (x_1, \ldots, x_n)$ are*

$$d \prod_i \theta_i(1), \qquad \sum_{\boldsymbol{\rho} \in \mathrm{Orb}(\boldsymbol{\theta})} \prod_i \rho_i(x_i). \tag{2.1}$$

Proof. Use the same representation for equal letters. Permutation of its tensor factors extends the tensor character to its inertia group $L^n \rtimes C_{n/d}$. The n/d linear twists give all irreducibles above the orbit after Clifford induction. Their restrictions to the base are the same sum of its distinct conjugate tensor characters, proving (2.1). □

Letters remain distinct when their profiles agree. Summing all n rotations repeats every distinct term n/d times and has the same zero set. For $x = (z, \dots, z, y)$, if every $\theta_i(z) \neq 0$, put $r_i = \theta_i(y)/\theta_i(z)$. Formula (2.1) becomes

$$\chi(x) = \frac{d}{n} \prod_i \theta_i(z) \sum_i r_i. \tag{2.2}$$

Thus a small ratio sum can count zeros in an enormous group.

The proofs give actual unequal supports, rather than infer them from one vanishing character. An unequal support rules out the original pointwise bijection, and hence any subgroup formulation requiring that bijection at a generator of a cyclic subgroup.

Let $q = p^f \geq 3$, $3 \nmid q+1$, and set

$$S = \mathrm{PSU}_3(q), \quad P \in \mathrm{Syl}_p(S), \quad B = N_S(P) = P \rtimes C_{q^2-1}. \tag{2.3}$$

Here P is TI, $|P| = q^3$, $P' = Z(P)$ has order q, and $|P/P'| = q^2$. Fix $1 \neq z \in Z(P)$ and $y \in P \setminus P'$. They represent the two nonidentity unipotent classes.

Lemma 2.2. *There are decompositions* $\mathrm{Irr}(S) = \mathcal{A} \sqcup \mathcal{C} \sqcup \{\mathrm{St}\}$ *and* $\mathrm{Irr}(B) = \mathcal{A}_0 \sqcup \mathcal{C}_0$ *with:*

- $|\mathcal{A}| = |\mathcal{A}_0| = q^2$*, all of* p'*-degree and nonzero at* z*. Some* $\alpha \in \mathcal{A}$ *has degree* $q^2 - q + 1$ *and profile* $(-(q-1), 1)$ *at* (z, y)*.*
- $\mathcal{A}_0$ *has* $q^2 - 1$ *linear characters with profile* $(1, 1)$*, and one character* η *of degree* $q^2 - 1$*, with profile* $(q^2 - 1, -1)$*.*
- $|\mathcal{C}| = |\mathcal{C}_0| = q + 1$*; their degree* p*-parts are* q*. Every member vanishes at* y *and is nonzero at* z*. A character* $\gamma \in \mathcal{C}$ *has degree* $q(q-1)$ *and profile* $(-q, 0)$*. Every member of* $\mathcal{C}_0$ *has that degree and profile.*
- $\mathrm{St}(1) = q^3$*, and its profile is* $(0, 0)$*.*

Proof. The global data are the unipotent columns of Simpson–Frame [29, Table 2], with $\delta = -1, d = 1, r = q+1, s = q-1, t = q^2 - q + 1$. Their notation is $\mathrm{PSU}(3, q^2)$. The Steinberg entry on a nonidentity unipotent class is zero; the printed nonzero entry in that position is not used.

Locally, P has q^2 linear characters and $q-1$ nonlinear characters of degree q, indexed by the nontrivial central characters. The cyclic complement is transitive on each of the two nonprincipal sets. The linear orbit induces η, with the stated values. A nonlinear character has inertia quotient of order $q+1$, so its extensions give $q+1$ irreducibles of degree $q(q-1)$. At z their value is $q \sum_{\lambda \neq 1} \lambda(z) = -q$, and they vanish outside the centre. □

Theorem 2.3. *For the data in* (2.3)*, put* $n = q + 1$*,* $W = S \wr C_n$*,* $\Sigma = P^n$*, and*

$$H = B \wr C_n, \qquad x = (\underbrace{z, \dots, z}_{q}, y). \tag{2.4}$$

Then x *is picky,* $x \notin \Sigma'$*,* $\mathrm{Sub}_W(x) = H$*, and* $|\mathrm{Irr}^x(W)| < |\mathrm{Irr}^x(H)|$*. For every prime there are infinitely many such examples.*

Proof. The cyclic top has order prime to p. Every Sylow subgroup lies in the base and is a product of Sylow subgroups of S. All coordinates of x are nonidentity, so the TI property makes Σ its unique Sylow subgroup. The last coordinate lies outside P', proving $x \notin \Sigma'$.

Discard global words containing Steinberg; they give zero at every rotation. Words entirely in $\mathcal{C}$, and locally in $\mathcal{C}_0$, also give zero. Every remaining local word is nonzero. In fact, with k letters from $\mathcal{C}_0$, and $r = n - k \geq 1$ other letters, of which a are linear, its ratio sum is

$$a - \frac{r-a}{q^2-1} = \frac{q^2 a - r}{q^2 - 1} \neq 0, \qquad 1 \leq r \leq q+1 < q^2. \tag{2.5}$$

The product of the z-values is nonzero.

Arbitrary bijections $\mathcal{A} \to \mathcal{A}_0$, $\mathcal{C} \to \mathcal{C}_0$ preserve rotation orbits, stabilizers and the discarded all-$\mathcal{C}$ words. They match the entire supply of potentially nonzero characters. It suffices to find one additional global zero. The word $\gamma \otimes \alpha^{\otimes(q-1)} \otimes 1$ has trivial stabilizer, since its unique $\mathcal{C}$ letter cannot move. Its induced irreducible has degree $(q+1)q(q-1)(q^2-q+1)^{q-1}$, of p-part q, and value

$$\gamma(z)\alpha(z)^{q-1}\left(1+(q-1)\frac{\alpha(y)}{\alpha(z)}\right)=0. \tag{2.6}$$

Thus the support inequality is strict. Taking $q=3^f$ for $p=3$, or $q=p^{2f}$ for $p\neq 3$, gives the last assertion. □

The mismatch already occurs among degree-part q characters. They have one $\mathcal{C}$ letter and q letters from $\mathcal{A}$, giving $(q+1)(q^2)^q$ characters on each side before zeros are removed. All local ones are nonzero; (2.6) is a global zero.

2.2. **Solvable examples at two.** At two the same cancellation can occur in a solvable base group. We give both a normal-complement example and a picky example. The normal-complement example also shows why oddness is needed for the exact correspondence in Section 12.

Let $L=5^{1+2}$ have exponent five, with coordinates $(a,z)\in\mathbb{F}_5^2\times\mathbb{F}_5$ and product $(a,z)(b,t)=(a+b,z+t+\frac{1}{2}\det(a,b))$. The matrices

$$u=\begin{pmatrix}2&0\\0&3\end{pmatrix},\qquad v=\begin{pmatrix}0&1\\-1&0\end{pmatrix} \tag{2.7}$$

generate $P_0\cong Q_8$ and act by fixing the second coordinate. Set $S=L\rtimes P_0$, $C=Z(L)$, $t=u^2$, $w=uv$. Every nonidentity element of P_0 acts fixed point freely on L/C, so $C_L(a)=C$ for $1\neq a\in P_0$. Consequently P_0 is TI and $B=N_S(P_0)=C\times P_0$.

The four degree-five characters of L are invariant. For each, its canonical extension has source on P_0

$$\vartheta=\rho_{P_0}-1_{P_0}-\beta,\qquad \vartheta(t,u,v,w)=(1,-1,-1,-1), \tag{2.8}$$

where β has degree two. To check this directly, the cyclic formula gives $|\vartheta(a)|=1$ for $a\neq 1$. If β has multiplicity m, then $\vartheta(t)=5-4m$, forcing $m=1$. The norm $(25+7)/8=4$ then forces three distinct linear constituents. Their missing linear character is 1, since determinant one on P_0 makes it trivial. This proves (2.8). Also $\vartheta\beta=\rho_{P_0}+\beta$.

The twenty odd-degree characters of S therefore have profiles $(1,a,b,c)$ on (t,u,v,w), with $a,b,c\in\{1,-1\}$: each even-parity profile $abc=1$ occurs once and each odd-parity profile occurs four times. Locally the four even-parity profiles each occur five times. On each side there are five degree-part-two characters with profile $(-2,0,0,0)$. The other three characters of S have degree eight and vanish at all four elements; they come from the three free P_0-orbits on the nontrivial linear characters of L.

Theorem 2.4. *With the data above, let*

$$W=S\wr C_4,\quad \Sigma=P_0^4\rtimes C_4,\quad x=(t,u,v,w),\quad H=B\wr C_4. \tag{2.9}$$

Then W is solvable, has normal odd complement L^4, and has order $4\cdot 1000^4$. Moreover

$$\mathrm{Sub}_W(x)=H,\quad |\operatorname{Irr}^x(W)|=63320>50360=|\operatorname{Irr}^x(H)|. \tag{2.10}$$

The element x belongs to 125 Sylow subgroups, so this is a subnormalizer counterexample, not a picky one.

Proof. The group formula (12.2), whose proof is independent of the parity of p, gives $\mathrm{Sub}_W(x)=\langle\Sigma, C_{L^4}(x)\rangle$. Here $C_{L^4}(x)=C^4$ is Σ-invariant, so this is $C^4\Sigma=H$. The centralizer of Σ in L^4 is diagonal C. Sylow complements containing x are parametrized by $C_{L^4}(x)/C_{L^4}(\Sigma)$, giving $5^3=125$.

For words with four odd letters, the four rotated products are signs $q_0,\dots,q_3$. Their product is the product of the letter parities. For any fixed letter parities they are uniform among the eight sign vectors of that total parity. Indeed write each letter as $(1,(-1)^a,(-1)^b,(-1)^{a+b+\delta})$. The map from the eight free bits to the four product exponents has image the even-parity subspace: it contains the independent vectors $(0,1,0,1),(0,0,1,1),(1,0,0,1)$.

Locally all total parities are even, so precisely one quarter of the 20^4 odd words have nonzero rotation sum. Globally the even- and odd-parity word counts are

$$E_0 = \frac{20^4 + 12^4}{2} = 90368, \qquad O_0 = \frac{20^4 - 12^4}{2} = 69632. \tag{2.11}$$

Every odd-parity vector has nonzero sum; one quarter of the even-parity vectors do. The nonzero word counts are therefore 92224 globally and 40000 locally. The next argument converts words to irreducibles, retaining their different stabilizers.

There are twenty constant odd words, all nonzero, each giving four irreducibles. This contributes 80 on each side. A period-two word $(\alpha, \gamma, \alpha, \gamma)$ has nonzero value exactly when the parities of its two letters agree. There are $4^2 + 16^2 = 272$ such words globally and $20^2 = 400$ locally. Removing constants leaves 252 and 380. A period-two orbit has two words and gives two characters, so these are already the character counts of degree part two.

The period-four odd words contribute $(92224 - 272)/4 = 22988$ and $(40000 - 400)/4 = 9900$. A word with one even letter has one nonzero rotated term: that letter must occur at t. All such words have period four and give $5 \cdot 20^3 = 40000$ characters of degree part eight. Two even letters force zero, since only one coordinate is t. A letter of zero profile also forces zero. Table 1 collects the counts.

Table 1. Nonzero values in the first solvable example at two

Degree 2-part	W	H
1	80	80
2	252	380
4	22988	9900
8	40000	40000
Total	63320	50360

This completes the count in Theorem 2.4. □

The group has precisely the normal-complement structure of Section 12, at $p = 2$. The failure here concerns even the numbers of nonzero rows, so no selection of signs can repair the old weak subnormalizer assertion. The local cyclic trace formula still holds, but reduction modulo two cannot identify a common sign. Formula (2.8) displays the incompatible signs before induction. This support calculation concerns the ungraded correspondence.

To obtain a picky element in a solvable group, use an odd cyclic top. Retain S, P_0, t, u, v, w, and put

$$W = S \wr C_{15}, \quad \Sigma = P_0^{15}, \quad M = B \wr C_{15}, \quad x = (t, t, t, u, w, u, w, t, u, v, w, u, v, v, w). \tag{2.12}$$

Every coordinate is picky in S, and the top has odd order. Hence x is picky, its unique Sylow is Σ, and $\mathrm{Sub}_W(x) = M$.

Proposition 2.5. *The nonzero counts in the degree-part-two slice are*

$$5068402196480000000 \quad \textit{and} \quad 2048000000000000000 \tag{2.13}$$

globally and locally. The total nonzero counts are also unequal, as computed below.

Proof. Such a word has one even letter and fourteen odd letters; the unique even position makes its stabilizer trivial. Fix that position at zero. There are $T_0 = 5 \cdot 20^{14}$ characters. Only the shifts in $T = \{0, 1, 2, 7\}$ survive, giving value $-2\sum_{s \in T} q_s$ with signs q_s. An odd global letter is a linear character α of Q_8, multiplied on u, v, w by $(-1)^e$, where $e = 0, 1$ has multiplicities $1, 4$. Locally $e = 0$, with multiplicity five.

The linear choices make (q_s) uniform within a parity coset of $\mathbb{F}_2^4$. Their image contains the independent vectors $(0, 0, 1, 1), (0, 1, 1, 0), (1, 1, 0, 0)$. The flags change parity exactly in positions $\{2, 5, 6, 7, 8, 9, 10, 13\}$ of the fourteen odd letters. Their parity bias is therefore $c = (3/5)^8$ globally, and $c = 1$ locally. Each sign vector has probability $(1 + c\prod q_s)/16$. The sum is zero for exactly six

vectors, all of even parity, so the zero proportions are $3(1+(3/5)^8)/8$ and $3/4$. Subtracting from T_0 proves (2.13).

Words with a zero-profile letter vanish. Let k count the remaining even letters. For $k \geq 5$ no rotation survives, since only four coordinates of x are central. For $k = 2$, two shifts survive only for an adjacent pair of even positions; the two odd-letter products agree with probability one half on both sides. The four other translation types of pairs in T have one surviving term. The common nonzero count is $(9/2)5^2 20^{13}$. For the adjacent pair $\{0, 1\}$, the shifts are zero and one; the factor in coordinate two compares t with u. Averaging its linear character makes the correlation zero, proving the probability one half just used.

For $k = 3, 4$ at most one rotation survives: the four triples in T are pairwise nonconjugate under cyclic translation. Nonzero words have trivial stabilizer; a period-five word with three even positions would require a translate of $\{0, 5, 10\}$, which T does not contain. Their counts are $4 \cdot 5^3 20^{12}$ and $5^4 20^{11}$, respectively.

All odd-degree characters are nonzero at a two-element. Their common number, by periods $1, 3, 5, 15$, is

$$\kappa = 15 \cdot 20 + \frac{5}{3}(20^3 - 20) + \frac{3}{5}(20^5 - 20) + \frac{20^{15} - 20^5 - 20^3 + 20}{15}. \tag{2.14}$$

The complete nonzero counts are given in Table 2.

Table 2. Nonzero values in the solvable picky example at two

Degree 2-part	W	M
1	κ	κ
2	5068402196480000000	2048000000000000000
4	9216000000000000000	9216000000000000000
8	2048000000000000000	2048000000000000000
16	1280000000000000000	1280000000000000000
Total	18644935529815053056	15624533333335053056

The difference 3020402196480000000 is already the difference in (2.13). This is a failure of weak picky in a solvable group, and hence also a counterexample to the ungraded subnormalizer correspondence. Together with Theorem 2.3, this proves Theorem A. □

The base group is Evseev's example [6, Section 4.2]; the wreath counts follow from the calculation above.

3. The weighted comparison fails at two

The obstruction in this section concerns ordinary heights, not the choice of a Clifford family. It occurs in a group with a normal odd complement and at a single eligible element.

Theorem 3.1. *There are a solvable group G, a nontrivial 2-subgroup Q, $H = \mathrm{Sub}_G(Q)$, and a visible pair (b, c) with common defect $D \geq Q$, such that an eligible element v of order sixteen satisfies*

$$\mathcal{M}_{80}(G, b; v) = 2^{15}\mathbb{Z}, \qquad \mathcal{M}_{80}(H, c; v) = 2^{16}\mathbb{Z}.$$

Here

$$|G| = 5^{144}2^{235}, \quad |H| = 5^{16}2^{235}, \quad |Q| = 2^{224}, \quad |D| = 2^{227}.$$

The same discrepancy occurs at level 79.

3.1. **The source and its four values.** Write $Q_8 = \{1, t, \pm i, \pm j, \pm k\}$, where $t = -1$, and let β be its degree-two character. For every linear character λ,

$$\lambda(t) = 1, \quad \lambda(i)\lambda(j)\lambda(k) = 1, \qquad \beta(t) = -2, \quad \beta(i) = \beta(j) = \beta(k) = 0.$$

Let T be extraspecial of order 5^3 and exponent five. The symplectic quaternion action on $T/Z(T) \cong \mathbb{F}_5^2$ fixes the centre and has no nonzero fixed vector for a nonidentity quaternion element. A degree-five Heisenberg character extends to $T \rtimes Q_8$, with source ω satisfying

$$\omega(t) = 1, \qquad \omega(i) = \omega(j) = \omega(k) = -1.$$

These are traces of a genuine representation. For example, put $\zeta = e^{2\pi i/5}$, $g = \sum_{x\in\mathbb{F}_5} \zeta^{x^2}$ and $F = (\zeta^{xy}/g)_{x,y\in\mathbb{F}_5}$. On the Schrödinger space take generators $-P_2, -F$, where P_2 permutes the basis by multiplication by two. They normalize the Heisenberg shift and diagonal matrices, satisfy the quaternion relations, and have the stated traces by $g^2 = 5$ and $\sum_x \zeta^{2x^2} = -g$.

Let E be the central product of four copies of T, with matching central characters. It is extraspecial of order 5^9. Set $A = Q_8 \wr C_4$, of order 2^{14}. The tensor Schrödinger representation, with ordinary permutation of tensor factors, extends to $E \rtimes A$. Its source Ψ has degree 5^4. In the base of A put

$$a_0 = (t,t,t,t), \quad a_1 = (t,t,t,i), \quad a_2 = (t,t,t,j), \quad a_3 = (t,t,t,k).$$

Each acts fixed-point-freely on $E/Z(E)$, and their source values are $(1,-1,-1,-1)$.

The characters of A are parametrized by rotation orbits of words in $\mathrm{Irr}(Q_8)^4$, with a linear label on the word stabilizer. A word containing k copies of β, of orbit length $o \in \{1,2,4\}$, gives degree $o2^k$ and height $h = k + \log_2 o$. Thus, for every $\alpha \in \mathrm{Irr}(A)$,

$$v_2(\alpha(a_0)) = h, \qquad v_2(\alpha(a_s)) \geq h-2 \quad (1 \leq s \leq 3), \tag{3.1}$$

where $v_2(0) = \infty$. Indeed, a summand at a_s vanishes if its noncentral coordinate receives β; every other summand is divisible by 2^k. Every height is at most five. Height five occurs exactly for three β-letters and one linear letter λ, giving four characters α_λ with profiles

$$(\alpha_\lambda(a_0), \ldots, \alpha_\lambda(a_3)) = -8(4, \lambda(i), \lambda(j), \lambda(k)). \tag{3.2}$$

3.2. **A proper-defect pair and two wreath lifts.** Put

$$G_0 = E^4 \rtimes P_0, \quad P_0 = A \wr C_4, \quad Q_0 = D_0 = A^4, \quad H_0 = Z(E)^4 \rtimes P_0.$$

Choose a nontrivial central character κ on the first E-factor and trivial characters on the others. The corresponding irreducible θ of E^4 has degree 5^4, inertia group $E^4 \rtimes D_0$, and an extension supplied by Ψ. In characteristic two its matrix-algebra component is a matrix algebra over the group algebra of D_0, and hence has one block. Inducing gives a block b_0 of G_0 parametrized by $\mathrm{Irr}(D_0)$. The central colour gives a block c_0 of H_0 with the same parameters.

Here are the subgroup and block checks. We have $C_{E^4}(Q_0) = Z(E)^4$, $Q_0 \lhd P_0$, and $N_{G_0}(D_0) = H_0$. The Sylow subgroups containing Q_0 are exactly those in H_0; their normalizers lie there and generate it. Thus $H_0 = \mathrm{Sub}_{G_0}(Q_0)$. Brauer projection at D_0 retains the central colour idempotents. The global and local coefficients agree modulo two, since their ratio is a power of five. The image is nonzero. A two-subgroup outside D_0 moves the four colour positions and fixes no colour word, so its Brauer image vanishes. Consequently D_0 is the common defect group and b_0, c_0 are actual Brauer correspondents.

The element $u = (a_0, a_1, a_2, a_3)$ has order four and is eligible: its action on $G_0/H_0 \cong (E/Z(E))^4$ has just one fixed point. For a parameter $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_3)$, put $h = \sum_j h(\alpha_j)$ and $T_s = \prod_{j=0}^3 \alpha_j(a_{j+s})$, with suffixes modulo four. The corresponding local and global values are

$$L_{\boldsymbol{\alpha}} = T_0 + T_1 + T_2 + T_3, \qquad G_{\boldsymbol{\alpha}} = T_0 - T_1 - T_2 - T_3. \tag{3.3}$$

The degrees are respectively $4\prod_j \alpha_j(1)$ and $4 \cdot 5^4 \prod_j \alpha_j(1)$. Since the defect index has two-part four, both ordinary heights are h. Equation (3.1) gives

$$v_2(L_{\boldsymbol{\alpha}}), v_2(G_{\boldsymbol{\alpha}}) \geq h - 6. \tag{3.4}$$

At the maximum height twenty, (3.2) makes each T_s equal to $\pm 2^{14}$, and the product of their four signs is one. Hence

$$L_{\boldsymbol{\alpha}} \in \{0, \pm 2^{16}\}, \qquad G_{\boldsymbol{\alpha}} \in \{\pm 2^{15}\}.$$

Taking all four linear letters trivial attains 2^{16} locally and -2^{15} globally.

Now define

$$G_1 = G_0 \wr C_2, \quad H_1 = H_0 \wr C_2, \quad Q_1 = Q_0^2, \quad D_1 = D_0 \wr C_2,$$
$$G = G_1 \wr C_2, \quad H = H_1 \wr C_2, \quad Q = Q_1^2, \quad D = D_1 \wr C_2.$$

If τ_i are the outer swaps, put $v_1 = (u, 1)\tau_1$ and $v = (v_1, 1)\tau_2$. Write $G = E^{16} \rtimes P$. Then $Q = A^{16} \lhd P$, $C_{E^{16}}(Q) = Z(E)^{16}$, and the same Sylow argument gives $H = \mathrm{Sub}_G(Q)$. The colour has one κ in each original quartet; its stabilizer is exactly D. The extended matrix-algebra argument again gives single colour blocks b, c. Their Brauer coefficients agree: the central coefficient ratio is 5^{-96}, equal to one modulo two. At D, fixed colour words restrict distinctly to the fixed central subgroup, because each permutation orbit has length a power of two, invertible modulo five. Their common Brauer image is nonzero. A two-subgroup with nonzero image fixes a colour word and is therefore contained in a conjugate of D. This proves the common defect assertion. Also $N_G(D) \le H$, since its odd part lies in $C_{E^{16}}(D) \le C_{E^{16}}(Q)$. Thus the blocks correspond in H. The successive swap fixed-point equations reduce to those for u, so v is eligible and has order sixteen.

At one binary lift, distinct tensor factors induce characters vanishing outside the base. Equal factors $\chi \otimes \chi$ have two extensions, of value $\pm\chi(u)$ at $(u, 1)\tau$. Their ordinary height is $2h(\chi)$, since the defect index valuation doubles. After two lifts every nonzero value at v is an original value up to sign, of height $4h$; all other rows vanish. At weighted level eighty, an original row with $h \le 19$ has value divisible by

$$2^{80-4h+h-6} = 2^{74-3h}, \qquad 74 - 3h \ge 17.$$

At height twenty the weights are one and the exact images are $2^{16}\mathbb{Z}$ locally and $2^{15}\mathbb{Z}$ globally. The displayed top parameters attain both generators. At level seventy-nine the lower bound is $73 - 3h \ge 16$, with the same conclusion. This proves Theorem 3.1.

Corollary 3.2. *For every $n \ge 1$, direct products of these data give an eligible element v_n of order sixteen and*

$$\mathcal{M}_{80n}(G^n, b^{\otimes n}; v_n) = 2^{15n}\mathbb{Z}, \quad \mathcal{M}_{80n}(H^n, c^{\otimes n}; v_n) = 2^{16n}\mathbb{Z}.$$

Proof. Subnormalizers, eligibility and the block data pass to direct products. Every nonzero factor value has height at most eighty, so the weight of a tensor row at level $80n$ is the product of its factor weights at level eighty. Apply the theorem. □

Proposition 3.3. *Parts* (a) *and* (b) *of Conjecture B hold for the pairs in Theorem 3.1 and Corollary 3.2.*

Proof. On the quaternion character lattice define

$$T(\lambda) = -\lambda - \beta \quad (\lambda \text{ linear}), \qquad T(\beta) = \beta.$$

At every nonidentity quaternion element x, $T(\alpha)(x) = \omega(x)\alpha(x)$. Realize negative terms as odd graded modules and permute tensor factors with the Koszul sign. The supertrace on a cycle is the virtual character at its cycle product. This gives an integral map on the complete Clifford parameters of A, whose value at every eligible a is $\Psi(a)\alpha(a)$: all quaternion cycle products must be nonidentity for the action on $E/Z(E)$ to be fixed-point-free.

This map preserves positive height. A word with a β-letter retains one in every term. For a word of linear letters the unique term without a new β retains the word and its stabilizer, with a possible linear parity twist of the top label. All other terms have positive height. Thus the leading terms give a signed bijection on all height-zero rows. Applying the evaluated identity again gives the reverse reconstruction, since $\Psi(a)^2 = 1$. Odd central factors multiply both sides by the same scalar, so these are identities on the entire mixed domain.

Apply the map on the first inertia factor for b_0, c_0 and induce. Positive-height tensors retain such a factor, and leading height-zero terms give a bijection. Outside the base of the outer C_4 both colour blocks vanish. Inside the base, eligibility gives exactly the cycle-product conditions just used. This proves (a) for the first pair.

The identities pass through each binary lift by graded tensor squares. Distinct factors have height $h(\chi) + h(\psi) + 1$; equal-factor extensions have height $2h(\chi)$. A positive factor remains positive. Distinct height-zero factors have distinct leading mates, so their induced leading term is still positive. Equal height-zero factors have their two extensions matched bijectively, possibly

interchanged by parity. All other terms are positive, and the reverse map supplies the reverse lattice inclusion. Eligibility in the product coset action ensures that every cycle product is in the original mixed domain. Hence (a) holds after both lifts.

At two the chosen normal subgroup is one, so Clifford level is ordinary height. Restrict the mixed identities to pure elements and discard exactly the zero quotient classes. The signed leading map then restricts to the essential rows. Height-zero degree two-parts agree because the blocks have a common defect and the ambient two-parts agree. This proves (b). Tensoring the same two-way identities proves both assertions for direct products. □

4. The height-zero comparison for quaternion wreath products

The solvable counterexample in Theorem 2.4 does satisfy the replacement conjecture at two, for every nontrivial subgroup of its Sylow subgroup. The argument works with any transitive 2-group as permutation group. We retain the group S and the action in (2.7), writing

$$E = 5^{1+2}, \qquad Z_0 = Z(E), \qquad A = Q_8, \qquad S = E \rtimes A.$$

Here E has exponent five and A fixes Z_0.

Theorem 4.1. *Let $T \leq \mathfrak{S}_n$ be a transitive 2-group and let $G = S^n \rtimes T$, with T permuting the factors. Parts* (a) *and* (b) *of Conjecture B hold at $p = 2$ for every $1 \neq Q \leq P \in \mathrm{Syl}_2(G)$ and every visible block pair. In particular, this holds for $S \wr C_4$, including all its subnormalizers.*

We prove the theorem without enumerating the subgroups of P. Put

$$N = E^n, \quad Z = Z_0^n, \quad V = N/Z, \quad P = A^n \rtimes T, \quad K = Z \rtimes P.$$

The comparison with K uses the quaternion identity (2.8). A separate inflation argument passes from K to the actual subnormalizer, which may be smaller than K.

4.1. Subnormalizers and visible blocks.

Lemma 4.2. *For $Q \leq P$, if $C_V(Q) \neq 0$, then $\mathrm{Sub}_G(Q) = G$. If $C_V(Q) = 0$, then*

$$\mathrm{Sub}_G(Q) = M_Q \rtimes P, \qquad M_Q = \langle C_Z(Q)^P \rangle_{\mathbb{F}_5} \leq Z. \tag{4.1}$$

Proof. The normal-complement formula (12.2), valid also at two, gives $\mathrm{Sub}_G(Q) = \langle P, C_N(Q) \rangle$. Indeed, the Sylow subgroups containing Q are the P^c with $c \in C_N(Q)$; their normalizers lie in this generated group, and the reverse inclusion uses $C_G(Q) \leq \mathrm{Sub}_G(Q)$.

The $\mathbb{F}_5 P$-module V is irreducible. The central involution of each coordinate copy of A projects a submodule onto that coordinate plane, on which A is irreducible. Transitivity of T then gives every coordinate plane. In the coordinates used in (2.7), the underlying set of N is $V \oplus Z$, and P acts separately and linearly on these two spaces. Thus $C_N(Q)$ maps onto $C_V(Q)$. If the latter is nonzero, $N \cap \mathrm{Sub}_G(Q)$ maps onto V. Commutators of lifts of the coordinate planes generate Z, so this intersection is N. If $C_V(Q) = 0$, then $C_N(Q) = C_Z(Q)$, giving (4.1). □

We will use the following elementary inflation observation. It also identifies the Brauer correspondent, rather than only its degrees.

Lemma 4.3. *Let a 2-group P act on an elementary abelian group Z of odd order. Suppose that $M \leq Z$ is P-invariant and $C_Z(Q) \leq M$, where $Q \leq P$. Set $K = Z \rtimes P$ and $H = M \rtimes P$. Every block of K with a defect group $D \geq Q$ is inflated from its Brauer correspondent in H. This gives a bijection of its irreducible characters with those of the correspondent, preserving degrees, heights and values on H.*

Proof. Choose a P-invariant complement R to M in Z. Then $C_R(Q) = 1$. The averaging operators on R and its dual have the same rank, so the only Q-invariant linear character of R is trivial. Blocks of K are labelled by P-orbits of $\lambda \in \mathrm{Irr}(Z)$, with defect group P_λ: λ extends trivially over P_λ, and the inertia component is the group algebra of this 2-group, up to a matrix algebra. For a visible block choose λ with $Q \leq D = P_\lambda$. It kills R, as do all its conjugates. Restriction to M preserves its stabilizer, and identifies the Clifford parameters with those in $K/R \cong H$. Hence the entire block is inflated and degrees and heights are unchanged.

Moreover, $\mathrm{Sub}_K(Q) \le H$, so $N_K(D) \le H$. We have $C_R(D) = 1$. In characteristic two the trivial-character idempotent of R has Brauer image 1 at D. Multiplying the block idempotent in H by this idempotent gives the inflated block in K, with the same Brauer image in the common centralizer. This proves the correspondence assertion. □

Assume henceforth that $H = \mathrm{Sub}_G(Q) < G$. Thus $C_V(Q) = 0$. For $c \in \mathrm{Irr}(Z_0)$ define $\theta_c \in \mathrm{Irr}(E)$ by $\theta_1 = 1_E$, and, for $c \neq 1$, let θ_c be the unique degree-five character over c. Call these five labels colours. Every Q-invariant irreducible character of N is a tensor product of colours. To see this, the coordinates occupied by linear factors form a Q-invariant set. Their labels give a fixed vector in the dual of the sum of those coordinate planes. The invariant symplectic form identifies this dual with the original module, whose fixed space is zero. Every linear factor is therefore trivial.

For a colour word $e = (c_1, \ldots, c_n)$, put

$$\theta_e = \bigotimes_i \theta_{c_i}, \qquad \lambda_e = \bigotimes_i c_i, \qquad P_e = A^n \rtimes T_e,$$

where T_e is the stabilizer of e. Canonical extensions in the base, followed by tensor permutations, extend θ_e to $N \rtimes P_e$. Consequently the orbital component is a single block b_e of G. Its counterpart d_e in K is labelled by the orbit of λ_e.

Here are the defect and correspondence checks. In characteristic two, the terms of a nontrivial orbit of a 2-subgroup on the primitive idempotents of the odd normal subgroup cancel under Brauer projection. A nonzero image therefore requires a fixed label. In particular every visible block has a colour label, and any defect of b_e is contained in a conjugate of P_e. At P_e, its image is nonzero: $C_N(P_e) = Z^{T_e}$, and distinct invariant central labels restrict distinctly to this fixed subgroup, by coprime averaging. Thus b_e and d_e both have defect P_e.

More explicitly, at $z \in Z$ the coefficients of the primitive idempotents for θ_e and λ_e are

$$\frac{\theta_e(1)^2}{|N|}\lambda_e(z^{-1}), \qquad \frac{1}{|Z|}\lambda_e(z^{-1}).$$

Their ratio is a power of five, hence reduces to one modulo two. For a common defect $D \ge Q$, we have $C_N(D) \le Z$; the orbital block idempotents therefore have the same Brauer image. Also $N_G(D) \le H \le K$. It follows that d_e is the Brauer correspondent in K; Lemma 4.3 gives its correspondent c_e in H. By the same Brauer images this is the correspondent of b_e in H.

4.2. **Integral reconstruction and positive heights.** Let β denote the degree-two character of A, and use the canonical extensions $\widehat{\theta}_c$ just chosen. In the base blocks, define the following integral substitutions:

$$\widehat{\theta}_c\alpha \longmapsto -c\alpha - c\beta \quad (\alpha \in \mathrm{Lin}(A),\ c \neq 1), \qquad \widehat{\theta}_c\beta \longmapsto c\beta \quad (c \neq 1). \tag{4.2}$$

For the trivial colour use the identity on $\mathrm{Irr}(A)$. Equation (2.8) and $\vartheta\beta = \rho_A + \beta$ show that these are equalities of values at every za, with $z \in Z_0$ and $1 \neq a \in A$. The coefficient matrix is involutory. Hence the reverse substitutions give equalities on the same set, preserve positive height, and have a signed bijection as their height-zero part.

Write X_K for the mixed domain defined by $\mathrm{Sub}_G(u) \le K$. After conjugation in K, its 2-part lies in P, and eligibility is equivalent to $C_V(u) = 0$. On each permutation cycle this says that the quaternion cycle product is nonidentity. The odd part lies in $C_N(u) \le Z$. Thus all cycle products of a mixed element in this domain are of the form za to which (4.2) applies.

The irreducibles of a fixed colour block are parametrized by a T_e-orbit of words (χ_i) of base irreducibles of those colours and an irreducible character η of the word stabilizer $J \le T_e$. The tensor-permutation extension is used before induction. On either side their ordinary height is

$$h = \sum_i h(\chi_i) + v_2([T_e : J]) + v_2(\eta(1)). \tag{4.3}$$

Indeed the exterior index $[T : T_e]$ is already the defect index and cancels in the height formula.

To apply the substitutions to these parameters, realize a virtual character as a graded module, putting each negative summand in odd degree. Use graded permutations on tensor products, tensor with the top label η, and induce. The difference of the even and odd characters is an integral virtual

character. The supertrace on each cycle is the virtual character evaluated at its cycle product. Consequently this construction gives the required equality of mixed values on all of X_K, with one set of integer coefficients. Colour is preserved, so every constituent belongs to the indicated target block.

We check positive height at the level of irreducible constituents. A word containing a β-letter retains such a letter in every term. Every constituent lying over such a word has positive height by (4.3), irrespective of its stabilizer. For a word of linear letters, there is exactly one tensor summand containing no β. It retains the word and its stabilizer J. Its top label is η multiplied by the linear sign character arising from permutations of the odd factors, and its overall coefficient is a sign. Thus this leading term has the same height as the original character. All other summands contain a β, even if several words coalesce or their stabilizers increase. This proves preservation of positive height without any assumption that the new stabilizers equal the old ones.

For height-zero parameters the leading rule is bijective: it preserves the linear base labels and twists the top label by a fixed linear character. All remaining terms have positive height. The reverse substitutions have exactly the same properties and value identities. We obtain equality of the integral positive-height lattices and a signed bijection on all height-zero labels modulo this common lattice. No saturation is used, and labels with zero profile are retained.

Finally $X_H \subseteq X_K$. Lemma 4.3, with $M = M_Q$, identifies $\mathrm{Irr}(K, d_e)$ and $\mathrm{Irr}(H, c_e)$ with equal degrees and values on H. Restricting the preceding two-way identities to X_H therefore proves B(a) for the actual visible pair (b_e, c_e).

At two the prescribed normal subgroup is 1, so Clifford level is ordinary height. Projecting the mixed identities onto any finite eligible family Y gives equality of the positive-level lattices. A height-zero label has zero class modulo that lattice exactly when its mate does; the signed bijection hence restricts to the essential labels. The common defect and $|G|_2 = |H|_2 = |P|$ give equal degree 2-parts for these labels. This proves B(b). If $H = G$, both assertions are identities for every visible block. Theorem 4.1 is proved.

For $T = C_4$, let $\pi : P \to C_4$ be the projection and put $D = \pi(Q)$. In the proper case, (4.1) becomes

$$H = Z^D \rtimes P,$$

since C_4 is abelian. The three possibilities $|D| = 1, 2, 4$ give odd parts Z_0^4, $\{(a, b, a, b) : a, b \in Z_0\}$, and the diagonal Z_0, respectively. Thus the theorem includes the two smaller subnormalizers as well as the one in Theorem 2.4. The only other prime dividing $|G|$ is five; there N is a normal Sylow subgroup and every subnormalizer is G. Hence these groups satisfy Conjecture B at every prime. The theorem makes no assertion of weighted equality at two.

5. Eligibility, integral values and blocks

We collect the general facts used in both the formulation and the proof. Eligibility first becomes a unique-fixed-coset condition. This condition identifies the kernel of mixed evaluation and then makes block projection invisible on the chosen sections. None of the results in this section assumes solvability or restricts the prime.

Throughout, $\mathcal{C}(K) = \mathbb{Z}\,\mathrm{Irr}(K)$ is the group of ordinary virtual characters, $\chi(1)_p$ is the p-part of a degree, and v_p is its additive valuation. For a block d, h_d denotes ordinary height. We write $U^g = g^{-1}Ug$. Character values are complex.

5.1. **Subnormalizers of p-subgroups.** We justify the description used in the introduction for arbitrary p-subgroups. The proof also explains why it agrees with Casolo's definition, rather than extending it to a new group-theoretic object.

Lemma 5.1. *For every p-subgroup $Q \leq G$,*

$$\langle S_G(Q) \rangle = \langle N_G(P) : P \in \mathrm{Syl}_p(G),\ Q \leq P \rangle.$$

Proof. Every finite group K is generated by its Sylow p-normalizers. Indeed, the subgroup J they generate is normal and contains a Sylow subgroup T; the Frattini argument gives $K = JN_K(T) = J$.

Let A be the group on the right. We show that $N_G(R) \leq A$ for every p-subgroup $R \geq Q$, by descending induction on $|R|$. The Sylow case is the definition of A. Otherwise, R is properly

contained in a Sylow subgroup of $N_G(R)$: if $R < P \in \mathrm{Syl}_p(G)$, the normalizer condition in P gives $R < N_P(R)$. Every $T \in \mathrm{Syl}_p(N_G(R))$ contains R, and hence Q, and has larger order. Induction gives $N_G(T) \le A$. Apply the preceding generation observation inside $N_G(R)$ to conclude $N_G(R) \le A$.

If $g \in S_G(Q)$, put $K = \langle Q, g\rangle$. A subnormal p-subgroup of K lies in $O_p(K)$, as follows along a subnormal series. Thus $Q \le O_p(K)$ and $g \in N_G(O_p(K)) \le A$. Conversely, if $Q \le P \in \mathrm{Syl}_p(G)$, then Q is subnormal in $P \trianglelefteq N_G(P)$, so $N_G(P) \subseteq S_G(Q)$. □

5.2. **The fixed-coset criterion.** We will repeatedly use the following elementary description of eligibility. Besides controlling induction values, it gives the centralizer and normalizer containments needed for block induction.

Lemma 5.2. *Let $P \in \mathrm{Syl}_p(G)$, $P \le H \le G$, and let $U \le H$ be a p-subgroup. Then $\mathrm{Sub}_G(U) \le H$ if and only if the base coset is the unique U-fixed coset in G/H. Consequently,*

$$U^g \le H \implies g \in H, \qquad C_G(U) \le H, \quad N_G(U) \le H. \tag{5.1}$$

For eligible $u \in H$ and $x = us$, $s \in C_G(u)_{p'}$, one has $x \in H$, its only fixed coset is the base coset, and

$$(\mathrm{Ind}_H^G \alpha)(x) = \alpha(x) \quad (\alpha \in \mathcal{C}(H)). \tag{5.2}$$

Proof. Suppose $\mathrm{Sub}_G(U) \le H$. Every Sylow containing U, and its normalizer, lies in H. If $U^g \le H$, choose $T \in \mathrm{Syl}_p(H)$ containing U^g. It is also a Sylow of G. The Sylow $T^{g^{-1}}$ contains U, so it lies in H. Its normalizer is contained in H, and H-conjugacy therefore gives $N_G(T) \le H$ as well. Choose $h \in H$ with $T^{g^{-1}h} = T$. Then $g^{-1}h \in H$, hence $g \in H$. A coset gH is U-fixed precisely when $U^g \le H$. This proves uniqueness and (5.1).

Conversely, suppose the base coset is the only U-fixed coset. A Sylow S containing U has a fixed point on G/H: a p-group acting on a set of size prime to p has a fixed point. That point is U-fixed, and hence is the base coset. Thus $S \le H$. If $g \in N_G(S)$, then $U^g \le S \le H$, so $g \in H$. All the normalizers defining $\mathrm{Sub}_G(U)$ therefore lie in H.

Finally (5.1) puts $C_G(u)$ in H, so $x \in H$. The p-part u is a power of x, since its commuting factors have coprime orders. Thus any x-fixed coset is u-fixed. The fixed-coset induction formula now has just the base term, giving (5.2). □

We will also use the following consequence. If $D \le H$ contains an eligible u, then $N_G(D) \le H$: a normalizer sends u into H.

5.3. **Induction quotients and p-sections.** Retain $1 \ne Q \le P \in \mathrm{Syl}_p(G)$, $H = \mathrm{Sub}_G(Q)$, and the union of eligible p-sections X_H from the introduction. Put

$$\mathscr{S}_H = \{U \le P : U \le P^g \text{ for some } g \notin H\}, \qquad \mathcal{C}(K) = \mathbb{Z}\,\mathrm{Irr}(K) \quad (K = G, H).$$

For a family $\mathscr{S}$, $I(K, P, \mathscr{S})$ is the integral span of $\mathrm{Ind}_L^K \alpha$, where $L \le K$, $\alpha \in \mathcal{C}(L)$, and $L \cap P \in \mathrm{Syl}_p(L) \cap \mathscr{S}$. Set

$$J_K = I(K, P, \mathscr{S}_H), \qquad \widehat{J}_K = \mathrm{sat}_{\mathcal{C}(K)} J_K = \{\xi \in \mathcal{C}(K) : n\xi \in J_K \text{ for some } n \ge 1\}.$$

For a block d of K, write

$$\mathcal{C}(K, d) = \mathbb{Z}\,\mathrm{Irr}(K, d), \qquad J_{K,d} = J_K \cap \mathcal{C}(K, d), \qquad \widehat{J}_{K,d} = \widehat{J}_K \cap \mathcal{C}(K, d),$$

and

$$\Lambda_{K,d} = \mathrm{ev}_{X_H}\,\mathcal{C}(K, d), \qquad C_+(K, d) = \mathbb{Z}\{\chi \in \mathrm{Irr}(K, d) : h_d(\chi) > 0\}.$$

Thus the lattice $\Lambda_{K,d,+}$ in the introduction is $\mathrm{ev}_{X_H} C_+(K, d)$.

Restriction followed by block projection induces the canonical map

$$\widehat{\rho}_b : \mathcal{C}(G, b)/\widehat{J}_{G,b} \longrightarrow \mathcal{C}(H, c)/\widehat{J}_{H,c}.$$

Lemma 5.3 identifies the saturated induction lattice with the kernel of evaluation on X_H. Theorem 5.4 and Proposition 5.7 show that the displayed map is the identity on the corresponding integral value lattices. Consequently the equality of positive-height lattices in Conjecture B(a) is exactly preservation of their images under this canonical map.

In the same identification, the congruence (1.5) is equivalent to

$$\chi_H - \varepsilon_\chi f_b(\chi) \in C_+(H,c) + \widehat{J}_H.$$

The quotient is by

$$C_+(K,d) + \operatorname{sat} J_{K,d}, \qquad \text{not by} \qquad \operatorname{sat}\big(C_+(K,d) + J_{K,d}\big).$$

Only invisible induction torsion is removed. Saturating the whole denominator could erase an integral height obstruction. Nor is a congruence modulo unsaturated J_H asserted. The matched height-zero degree p-parts both equal $|P:D|$.

5.4. **The integral kernel of p-section evaluation.** We next determine which virtual characters are invisible on the union of eligible p-sections. The kernel is the saturation of the induction lattice, with denominator bounded by the Sylow order. The proof first uses cyclic generator indicators in a p-group and then Brauer induction to pass to an arbitrary finite group.

For this subsection let G be any finite group, let $P \in \mathrm{Syl}_p(G)$, and suppose $N_G(P) \le H \le G$. Define

$$\mathscr{S}_H = \{U \le P : U \le P \cap P^g \text{ for some } g \notin H\}. \tag{5.3}$$

Lemma 5.3. *The family $\mathscr{S}_H$ is downward and G-fusion closed in P, and*

$$U \notin \mathscr{S}_H \quad \Longleftrightarrow \quad \mathrm{Sub}_G(U) \le H \qquad (U \le P). \tag{5.4}$$

Its visible mixed set in H is exactly X_H from Section 5.3. Two elements of X_H conjugate in G are conjugate in H.

Proof. If $U \le P^g$ with $g \notin H$, then $U^{g^{-1}} \le P \le H$; hence the base coset of G/H is not the only one fixed by U. Conversely, from $U^g \le H$, $g \notin H$, conjugate U^g into P by an element $h \in H$. Then $U \le P^{(gh)^{-1}}$, with $gh \notin H$. Lemma 5.2 proves (5.4). Downward closure is immediate.

If $U \notin \mathscr{S}_H$ and $U^g \le P$, the transporter assertion in Lemma 5.2 forces $g \in H$. Consequently $\mathrm{Sub}_G(U^g) = \mathrm{Sub}_G(U)^g \le H$. Applying this also with g^{-1} proves fusion closure of the complement and hence of $\mathscr{S}_H$.

For a cyclic subgroup generated by a p-part, (5.4) is precisely eligibility. Its centralizer is in H, so all its mixed elements lie there. If $x = us \in X_H$ and $x^g \in H$, taking p-parts gives $u^g \le H$, and 5.2 forces $g \in H$, proving the last assertion. □

For $K = G, H$, let $J_K = I(K, P, \mathscr{S}_H)$. On the G-side the visible set is the G-conjugacy closure of X_H. If $H = G$, include the eligible element $u = 1$, so $X_H = G$. This convention is needed for the kernel statement, rather than just for notation.

Theorem 5.4. *Let $P \in \mathrm{Syl}_p(K)$, and let $\mathscr{S}$ be a downward and K-fusion closed family in P. Put $J = I(K, P, \mathscr{S})$, and let X be the set of elements whose p-part generates a subgroup not conjugate into $\mathscr{S}$. Then*

$$\ker(\mathrm{ev}_X : \mathcal{C}(K) \to \mathbb{C}^X) = \operatorname{sat}_{\mathcal{C}(K)} J, \qquad |P| \operatorname{sat} J \subseteq J. \tag{5.5}$$

Proof. We first control the denominator for a prime-power group and then use Brauer induction.

An allowed inducing subgroup contains no conjugate of the p-part of an element of X. Thus every allowed induction vanishes on X, as does its saturation.

First consider a p-group T and a virtual character ξ supported where $\langle t \rangle \in \mathscr{T}$, for a downward fusion closed family $\mathscr{T}$. For each cyclic $C \in \mathscr{T}$, choose its generator indicator

$$\delta_C = 1_C - \frac{1}{p} \operatorname{Ind}_{C^p}^C 1_{C^p} \quad (C \ne 1), \qquad \delta_1 = 1_{\{1\}}. \tag{5.6}$$

The function $\operatorname{Ind}_C^T \delta_C$ has value $|N_T(C) : C|$ on generators of conjugates of C, and zero elsewhere. Partitioning the support gives

$$|T|\xi = \sum_{[C] \subseteq \mathscr{T}} |T : N_T(C)|\, |C| \operatorname{Ind}_C^T(\xi_C \delta_C). \tag{5.7}$$

For $C \neq 1$, its summand is

$$|T : N_T(C)| \left(|C| \operatorname{Ind}_C^T \xi_C - \frac{|C|}{p} \operatorname{Ind}_{C^p}^T \xi_{C^p} \right). \tag{5.8}$$

The coefficients are integers and both subgroups are allowed. For $C = 1$ use only the first term. Hence $|T|\xi \in I(T, T, \mathscr{T})$.

We now pass to an arbitrary group, keeping the denominator bound just obtained. Take $\xi \in \mathcal{C}(K)$ vanishing on X. Brauer induction of 1_K, followed by multiplication by ξ, expresses it as an integral sum of $\operatorname{Ind}_E^K(\xi_E \alpha_E)$, with E elementary and α_E a virtual character. Conjugate E so that its Sylow p-subgroup T lies in P. An elementary group splits as $E = T \times V_0$, with V_0 of order prime to p.

Expand in the product irreducible basis:

$$\xi_E \alpha_E = \sum_{\nu \in \operatorname{Irr}(V_0)} \xi_\nu \times \nu, \qquad \xi_\nu \in \mathcal{C}(T). \tag{5.9}$$

If $\langle t \rangle \notin \mathscr{S}$, the left side is zero on (t, v) for every $v \in V_0$. Linear independence of $\operatorname{Irr}(V_0)$ gives $\xi_\nu(t) = 0$ for every ν. The p-group calculation (5.7)–(5.8) puts $|T|(\xi_E \alpha_E)$ in the lattice induced from $C \times V_0$, $C \in \mathscr{S}$. These are allowed subgroups: $(C \times V_0) \cap P = C$, a full Sylow subgroup. Induce to K, and multiply by $|P|/|T|$. It follows that $|P|\xi \in J$.

This proves the reverse kernel inclusion and the bound in (5.5). □

Apply Theorem 5.4 with the family from Lemma 5.3, first to $K = G$ and then to $K = H$. The kernel on G may be tested just on X_H, since the missing points are its G-conjugates. Therefore

$$\ker(\operatorname{ev}_{X_H}|_{\mathcal{C}(K)}) = \widehat{J}_K, \qquad |P|\widehat{J}_K \subseteq J_K \quad (K = G, H). \tag{5.10}$$

Intersecting with a block gives its corresponding saturated kernel. In particular, a difference with zero mixed profile belongs to $\widehat{J}_K$; the proof does not put it in the unsaturated J_K.

We use Brauer induction in the form of Evseev [6, Theorem 2.2]. The denominator estimate above is separate from the quotient theorem.

5.5. **Visible blocks and heights.** An equality of nonzero mixed sections determines the relevant blocks. A nonzero generalized decomposition coefficient belongs to a block of the common centralizer. Block induction and the unique fixed coset identify the global and local blocks and give them a common defect group. Equality of degree p-parts then gives equality of heights. We establish this implication before constructing any correspondence.

Lemma 5.5. *Let $P \leq H \leq G$, $P \in \operatorname{Syl}_p(G)$, and let $u \in H$ be an eligible nonidentity p-element. Suppose $\chi \in \operatorname{Irr}(G, b)$, $\eta \in \operatorname{Irr}(H, c)$ have the same nonzero mixed section, up to a sign. Then $c^G = b$ is defined and b, c have a common defect group in H.*

Proof. A nonzero generalized decomposition coefficient identifies a block of the common centralizer.

By Lemma 5.2, $C = C_G(u) = C_H(u)$. On $C_{p'}$, write the generalized decomposition expansions

$$\chi(us) = \sum_{\varphi \in \operatorname{IBr}(C)} d^u_{\chi\varphi} \varphi(s), \qquad d^u_{\chi\varphi} = \epsilon d^u_{\eta\varphi}. \tag{5.11}$$

The equality follows from independence of the Brauer characters. Choose a nonzero coefficient, in a block d of C. Brauer's second main theorem gives

$$d^G = b, \qquad d^H = c. \tag{5.12}$$

These inductions are defined: $\langle u \rangle$ is a central p-subgroup of C, so it belongs to a defect group F of d, and $C_G(F) \leq C$. Defect containment under block induction gives a defect group E_0 of c containing F, and hence u.

If $g \in N_G(E_0)$, then $u^g \in E_0 \leq H$. The unique-coset criterion forces $g \in H$. Thus $N_G(E_0) \leq H$. Brauer's first main theorem now implies that c^G is defined with the same defect group E_0. Transitivity, using (5.12), gives $c^G = (d^H)^G = d^G = b$. □

The standard inputs are defect containment and transitivity of block induction, and Brauer's first and second main theorems. We use Sambale [27, Proposition 1.11 and Theorems 1.12–1.13]. The normalizer containment above is the additional coset argument.

This proof does not infer defects from character degrees. It applies independently of solvability and of the parity of p, whenever its hypotheses hold.

Now put $H = \mathrm{Sub}_G(Q)$, where $1 \neq Q \leq P \in \mathrm{Syl}_p(G)$. Let b be visible, with defect group D satisfying $Q \leq D \leq P$, and let c_b be its Brauer correspondent in H.

Lemma 5.6. *A signed equality of nonzero full mixed profiles pairs* $\mathrm{Irr}(G, b)$ *only with* $\mathrm{Irr}(H, c_b)$*, and conversely. If degree p-parts are preserved, all original heights are preserved. Also* $N_G(D) = N_H(D)$.

Proof. The unique fixed coset for Q gives $Q^g \leq H \Rightarrow g \in H$. In particular, $N_G(D) \leq H$, giving the last assertion and the existence of c_b by Brauer's first main theorem.

For a nonzero paired profile, choose a nonzero section. Lemma 5.5 gives $c^G = b$ with a common defect E_0. Choose g with $D^g = E_0 \leq H$. Then $Q^g \leq H$, so $g \in H$. Hence these defects are already H-conjugate. Uniqueness in Brauer's first main theorem gives $c = c_b$. Conversely, for a local row in c_b, Lemma 5.5 says that its partner belongs to $c_b^G = b$.

The two groups have the same order p-part, and the two blocks have the same defect order. Thus

$$\chi(1)_p = |P : D|p^{h_b(\chi)}, \qquad \eta(1)_p = |P : D|p^{h_{c_b}(\eta)}. \tag{5.13}$$

Equality of degree p-parts gives equality of heights. □

The order of the reasoning matters: generalized columns identify block induction; the normalizer condition gives equal defects; only then do degrees compare heights. No intermediate permutation of source parameters is required to preserve their parameter blocks.

The transporter assertion for Q concerns a subgroup, not necessarily a cyclic one. It remains valid when no individual nonidentity element of H is eligible.

Let $X = X_H$ be the full union of eligible p-sections, including $u = 1$ when $H = G$.

Proposition 5.7. *For a visible pair* (b, c_b)*,*

$$\begin{aligned} \mathrm{ev}_X \,\mathrm{pr}_{c_b} \,\mathrm{Res}_H^G \,\alpha &= \mathrm{ev}_X \,\alpha \quad (\alpha \in \mathcal{C}(G, b)), \\ \mathrm{ev}_X \,\mathrm{pr}_b \,\mathrm{Ind}_H^G \,\gamma &= \mathrm{ev}_X \,\gamma \quad (\gamma \in \mathcal{C}(H, c_b)). \end{aligned} \tag{5.14}$$

The canonical saturated block map is the identity under the identification with these integral value lattices.

Proof. It suffices to compare the generalized decomposition coefficients on each eligible section.

For $H = G$ the assertion is immediate. Otherwise fix an eligible u, and expand on the common centralizer as in (5.11). For each block d of that centralizer, Brauer's second main theorem allows coefficients only in the block d^G globally and d^H locally. Moreover $(d^H)^G = d^G$ with equal global and local defects: use the defect of d containing u and the normalizer argument in Lemma 5.5.

If $d^G = b$, Lemma 5.6's defect-conjugacy argument gives $d^H = c_b$. Conversely $d^H = c_b$ implies $d^G = b$. Thus projection of a restricted character to c_b retains exactly its possible nonzero Brauer coefficients in every eligible section. This proves the first equality, including for zero profiles.

For the second, 5.2 gives $\mathrm{Ind}_H^G \,\gamma(x) = \gamma(x)$ on X. Its section coefficients can occur only at $d^H = c_b$, hence only at $d^G = b$. Projection to b therefore retains them all. □

By (5.10), the kernels are $\widehat{J}_{G,b}$ and $\widehat{J}_{H,c_b}$. Hence (5.14) proves the canonical isomorphism on the saturated quotients directly. This is the value realization of the quotient and block maps of Evseev [6, Theorems 1.4 and 2.6; Propositions 2.4 and 2.11]. None of these identifications saturates the positive-height image.

5.6. **Clifford levels and ordinary heights.** The radical in (1.1) is chosen from the global group once. Locally we use $N_H = H \cap R_p(G)$. The constituent orbits and their levels use ordinary characters and restrictions.

Always $0 \le \delta^R_{K,d}(\chi) \le h_d(\chi)$. If $N_K = 1$, then $\delta = h_d$, since every block has a height-zero character. If $N_K = K$, each family is a singleton and $\delta = 0$. For abelian defect, Kessar–Malle [16, Theorem 1.1] gives height zero throughout, so again $\delta = 0$.

For G p-solvable and p odd, $R = G$, so $N_G = G$ and $N_H = H$. All Clifford levels vanish, although ordinary positive heights may occur. Part (b) then compares exact values at every ordinary height. In general the two indices differ; the weights in part (c) continue to use h_d.

5.7. **The three requirements.** Part (a) of Conjecture B compares entire p-sections. Its height-zero matching includes characters whose mixed profile is zero. Part (b) concerns pure p-elements and the quotient by the integral positive-Clifford-level lattice. A level-zero character whose profile belongs to that lattice is absent from A^R_0. Distinct characters with the same surviving profile remain distinct members of A^R_0.

Whenever $R_p(G) = 1$, part (b) follows from (a). Indeed $\delta = h$, and restriction of the common mixed lattice to any requested Y gives the equality of F^R_1. The height-zero bijection in (a) preserves degree p-parts, since the defect and the order p-part are common. Its congruences identify precisely the essential rows on the two sides. This applies at $p = 2$ and also to every nonabelian simple group when p divides its order. For such a simple group, B consists of (a) and, at odd primes, (c).

Part (c) compares integral values at a single eligible element with a weight for every ordinary height. It prescribes neither bijections of positive-height irreducibles nor a common coefficient matrix for different eligible elements. Proposition 6.5 makes its coefficient divisibility explicit. The matrices in the two directions need not be inverses.

For empty Y, the essential sets in (b) are empty. The height-zero counts in (a) remain part of the statement. This distinction is used in the counting consequence below.

Proposition 5.8. *Part (b) of Conjecture B is equivalent to the same assertion for every subset $Y \subseteq E_H$. One matching on E_H provides compatible matchings on all these subsets.*

Proof. Projection to the coordinates in Y sends $F^R_1(K, d; E_H)$ onto $F^R_1(K, d; Y)$, since the generating characters and their levels do not depend on the arguments. A level-zero profile belonging to the former lattice still belongs to the latter after projection. For every label in $A^R_0(G, b; E_H)$, the matching congruence gives

$$\mathrm{ev}_Y \chi \in F^R_1(G, b; Y) \quad \Longleftrightarrow \quad \mathrm{ev}_Y f_0(\chi) \in F^R_1(H, c; Y).$$

Consequently f_0 restricts to the required bijection of active labels on Y. Its signs, correction coefficients and degree p-parts are unchanged. The converse follows by taking $Y = E_H$. □

6. Consequences and exact recovery

We first prove the counting and exact-value consequences of the conjecture, and then describe its ordinary-height reconstruction. These deductions use only the stated clauses; the solvable theorem is proved in the following sections.

6.1. **Counting and exact recovery.** Taking the whole Sylow subgroup gives a height-zero counting statement. For values, the first useful observation is that the congruence becomes an equality when all Clifford levels are zero. The ordinary-height weights then follow from that exact correspondence.

Proposition 6.1. *Conjecture B implies Alperin–McKay for blocks of maximal defect, and implies McKay for every finite group.*

Proof. Take $Q = P \in \mathrm{Syl}_p(G)$. Then $H = \mathrm{Sub}_G(P) = N_G(P)$. The visible blocks of G are exactly those of defect P. Part (a) supplies a bijection of their height-zero characters with those of the corresponding blocks in H. These are the required maximal-defect Alperin–McKay bijections.

If $\chi \in \mathrm{Irr}(G, b)$, with defect group D, then

$$v_p(\chi(1)) = v_p(|G : D|) + h_b(\chi). \tag{6.1}$$

Both terms on the right are nonnegative. Thus χ has p'-degree precisely when $D = P$ and its height is zero. Every block of H has defect P, since its normal Sylow subgroup belongs to every

defect group. Brauer's first main theorem matches all these local blocks with the global blocks of defect P. Summing the height-zero bijections proves McKay. □

This argument does not require an eligible element or a nonempty family Y. If no elements are eligible, part (b) can be vacuous, while part (a) still supplies these height-zero bijections. This explains the separate role of part (a).

For a block with defect $D < P$, taking $Q = D$ gives $\mathrm{Sub}_G(D)$, which need not equal $N_G(D)$. Part (a) compares the block with its correspondent in the former subgroup. If the two subgroups coincide, this is directly an Alperin–McKay bijection. In general the implication to full Alperin–McKay would require an additional bridge; none is established by the preceding proof.

These results prove Theorem C(a).

The solvable proof 11.2 uses the published p-solvable Alperin–McKay equality to complete zero profiles. That use must not be recast as an independent proof of its input.

Lemma 6.2. *Suppose that $\delta^R = 0$ for every character in a visible pair (b, c). Part (b) of Conjecture B gives a bijection between all characters in these blocks with nonzero Y-profile, preserving degree p-parts and satisfying one exact signed equality on the entire eligible family Y.*

Proof. There are no positive Clifford levels, so

$$F_1^R = 0, \qquad A_0^R(K, d; Y) = \{\chi \in \mathrm{Irr}(K, d) : \mathrm{ev}_Y\, \chi \neq 0\}. \tag{6.2}$$

The correction in (1.7) is zero. Hence $\mathrm{ev}_Y\, \chi = \epsilon_\chi\, \mathrm{ev}_Y\, f_0(\chi)$, with one sign per vector. All nonzero rows occur in (6.2), including repetitions of the same vector from different characters. □

Corollary 6.3. *Conjecture B implies strong subnormalizer when G is p-solvable and p is odd. It also implies strong subnormalizer for groups with abelian Sylow p-subgroups, at any prime. Blockwise, the same recovery holds for a visible pair of abelian defect.*

Proof. In the first case $R_p(G) = G$. Thus $N_G = G$ and $N_H = H$, and each Clifford family consists of one character. Its degree has the same valuation as the minimum in that singleton, giving $\delta = 0$. Ordinary heights can still be positive; they do not affect this calculation.

For abelian defect, every ordinary character in either corresponding block has height zero, by Kessar–Malle [16, Theorem 1.1]. Since $0 \leq \delta \leq h$, again $\delta = 0$.

Now take $Q = \langle x \rangle$, $Y = \{x\}$, and $H = \mathrm{Sub}_G(x)$. A character nonzero at x has a defect group containing a conjugate of Q, so its block is visible after choosing that defect group. The same applies locally, with Brauer correspondence. Thus nonvisible blocks contribute zero, and the union of the bijections of Lemma 6.2 gives strong subnormalizer at x. At one use the identity. □

Corollary 6.3 proves Theorem C(b)–(c).

For solvable G, odd p, D supplies the premise Conjecture B. For a general p-solvable group this paragraph is an implication from the universal conjecture; it does not extend D beyond its proved hypotheses.

6.2. **Weighted reconstruction.** The weighted clause concerns ordinary heights at all levels. Fix an eligible u, and write

$$F_j^h(K, d; u) = \mathbb{Z}\{\chi(u) : \chi \in \mathrm{Irr}(K, d),\ h_d(\chi) \geq j\}.$$

These groups are separate from the Clifford-level lattice in (1.2).

Proposition 6.4. *For every $r \geq 0$,*

$$\mathcal{M}_r(K, d; u) = \sum_{j=0}^{r} p^{r-j} F_j^h(K, d; u), \qquad \mathcal{M}_{r+1} = F_{r+1}^h + p\mathcal{M}_r. \tag{6.3}$$

For a visible pair, $\mathcal{M}_0(G, b; u) = \mathcal{M}_0(H, c; u)$ holds unconditionally. Part (a) of Conjecture B gives the equality also for $r = 1$.

Proof. The terms contributed by a character of height a on the right of the first identity have coefficients p^{r-j}, with $0 \leq j \leq \min(r, a)$. They generate $p^{\max(0, r-a)}\mathbb{Z}\chi(u)$, which is its contribution to the left. The second identity follows by separating the term with $j = r + 1$.

Proposition 5.7, restricted to u, gives equality of the integral unweighted value lattices, which are $\mathcal{M}_0$. Part (a) gives equality of F_1^h after restriction from X_H. Use $\mathcal{M}_1 = p\mathcal{M}_0 + F_1^h$. □

In particular,

$$p\mathcal{M}_r \subseteq \mathcal{M}_{r+1} \subseteq \mathcal{M}_r, \qquad p^r\mathcal{M}_0 \subseteq \mathcal{M}_r \subseteq \mathcal{M}_0.$$

For a fixed pair and element, let a bound the heights of all characters with nonzero value on either side. For $r \geq a$, $\mathcal{M}_{r+1} = p\mathcal{M}_r$. The height bound depends on the blocks; the conjecture has no uniform height cutoff.

Proposition 6.5. *For a visible pair and an eligible u, the equalities $\mathcal{M}_r(G, b; u) = \mathcal{M}_r(H, c; u)$ for every $r \geq 0$ are equivalent to integral reconstructions in both directions such that*

$$\chi(u) = \sum_{\psi \in \mathrm{Irr}(H,c)} a_{\psi\chi}\psi(u), \qquad p^{\max\{0, h_b(\chi) - h_c(\psi)\}} \mid a_{\psi\chi}, \tag{6.4}$$

and the analogous condition holds from local to global values. The divisibility condition is equivalently

$$\chi(1)_p \mid a_{\psi\chi}\psi(1)_p. \tag{6.5}$$

Thus, when p is odd and in the presence of (a), part (c) of Conjecture B is equivalent to these reconstructions.

Proof. Take $r = h_b(\chi)$ in the inclusion from global to local weighted lattices. The generator $\chi(u)$ then belongs to the local lattice, which gives (6.4). Conversely multiply that identity by $p^{\max(0, r - h_b(\chi))}$. The inequality

$$\max(0, r - h_b(\chi)) + \max(0, h_b(\chi) - h_c(\psi)) \geq \max(0, r - h_c(\psi))$$

puts each summand in $\mathcal{M}_r(H, c; u)$. Do the same in the other direction. The common defect and equality $|G|_p = |H|_p$ give (6.5) from (5.13). Proposition 6.4 supplies levels zero and one for the final assertion. □

The coefficients may depend on u. Applying a Galois automorphism to (6.4) transports the same coefficients to all generators of $\langle u \rangle$. It does not give a common reconstruction on an arbitrary family of eligible elements. Inverting p would identify every weighted lattice with $\mathbb{Z}[1/p]\mathcal{M}_0$, removing the divisibility in (6.5). The integral formulation retains it.

Lemma 6.6. *For a visible pair, suppose that on each requested finite pure eligible family the characters with nonzero profile admit an exact signed bijection preserving degree p-parts. Then part (c) of Conjecture B holds for this pair. In particular, if every Clifford level is zero, part (b) implies (c).*

Proof. Use the family $\{u\}$. The common defect implies that corresponding characters have the same ordinary height. Their weighted generators are equal up to sign at every r, and zero values contribute nothing. For the final assertion, use Lemma 6.2. □

Proposition 6.7. *Fix a visible pair and an eligible u. Suppose that every positive-height character with nonzero value at u, on either side, has the same height $m > 0$. Then part (a) of Conjecture B implies its weighted clause at u. If all positive-height values vanish on both sides, the same conclusion holds.*

Proof. Under the first hypothesis,

$$\mathcal{M}_r = p^r F_0^h + p^{\max(0, r-m)} F_1^h.$$

The two lattices on the right agree by Proposition 5.7 and part (a), respectively. In the vanishing case $\mathcal{M}_r = p^r F_0^h$. □

6.3. **The relation with Alperin's problem.** The connection between subnormalizers and Alperin's Main Problem is the starting point of Moretó [24]. Conjecture B retains irreducible multiplicities at height zero and Clifford level zero, while (6.4) gives integral reconstruction at every ordinary height. The correction lattices need not have the matched irreducibles as a basis, and a quotient class need not determine the value field of its representatives.

The saturated induction kernel has a separate role: it removes exactly the invisible values by Theorem 5.4. It does not saturate Λ_+, F_1^R, or the weighted lattices. Exact signed values follow in the cases of Corollary 6.3; in general the weighted clause asserts the coefficient divisibility in (6.5).

7. The solvable theorem and its Clifford reductions

The proof occupies Sections 7–11. We first explain the relative form needed for the Clifford reductions.

We prove the solvable assertion of Theorem D(a). Throughout this proof G is solvable and p is odd.

The main task is a simultaneous comparison of mixed values over an invariant lower character. Three features of this comparison will be used later. It includes every irreducible character whose selected values are not all zero. It preserves the p-part of the degree. Each matched pair has one sign on the entire requested family. These requirements permit Clifford induction and allow the block comparison to be made after the character correspondence has been constructed.

There are two inductions in the proof, and they have different purposes. The induction on the group order constructs the relative character correspondence. The induction on the dimension of a finite vector space constructs the source values needed in its fully ramified step. The source argument does not use the relative character theorem. This separation is what makes the group-order induction legitimate.

In this section we reduce the character problem to full ramification. Section 8 constructs the source inside the original character algebra and states the source localization theorem. Sections 9 and 10 prove the geometric part of that theorem. Section 11 then completes the relative induction and uses the block preliminaries to pass from mixed values to heights and the three parts of the conjecture. This proves the solvable assertion of Theorem D(a).

The mechanism in the ramified case can be described before its construction. An ordinary source correspondence writes

$$\chi_B = \psi\eta,$$

where η ranges over all irreducibles above the chosen lower character. On one subgroup $K \le B$ containing a Sylow subgroup, we arrange $\psi(x) = \epsilon\Gamma(x)$ on all selected mixed sections, with Γ linear. On the parameters with a nonzero selected profile, a smaller relative correspondence f from B to K transports tensoring by Γ to

$$F = f^{-1} \circ (\Gamma \otimes -) \circ f.$$

Thus $\psi\eta$ has, up to one sign, the same selected values as the irreducible parameter $F\eta$. The source degree is prime to p, and tensoring by a linear character leaves degrees unchanged. These operations are therefore compatible with the required degree p-parts. The algebra in Section 8 supplies the factorization; the geometry in Sections 9 and 10 supplies the common K, Γ, ϵ. This is the only branch that requires source localization. It also explains why a separate twist on each centralizer would not suffice.

7.1. **The relative assertion.** The relative form allows an arbitrary invariant lower character. Here p is odd, G is solvable, and

$$P \in \mathrm{Syl}_p(G), \quad P \le H \le G, \quad Z_0 \trianglelefteq G, \quad Z_0 \le H, \quad \lambda_0 \in \mathrm{Irr}(Z_0) \text{ is } G\text{-invariant.} \tag{7.1}$$

Let Y be a finite family of nonidentity p-elements of H, with $\mathrm{Sub}_G(u) \le H$ for every $u \in Y$. Write

$$X_Y = \{us : u \in Y,\ s \in C_G(u)_{p'}\}, \qquad \mathcal{I}_K(X_Y) = \{\chi \in \mathrm{Irr}(K \mid \lambda_0) : \chi|_{X_Y} \neq 0\}. \tag{7.2}$$

Here $K = G, H$. Lemma 5.2 gives $C_G(u) \le H$, so $X_Y \subseteq H$.

Theorem 7.1. *For the solvable odd-prime data* (7.1)–(7.2)*, there are a bijection* $f : \mathcal{I}_G(X_Y) \to \mathcal{I}_H(X_Y)$ *and signs* $\epsilon_\chi \in \{1, -1\}$ *such that*

$$\chi(x) = \epsilon_\chi f(\chi)(x) \quad (x \in X_Y), \qquad \chi(1)_p = f(\chi)(1)_p. \tag{7.3}$$

The lower character is arbitrary subject to (7.1)*.*

We prove this relative assertion by induction on the group order, simultaneously for all the data in (7.1)–(7.2). An inductive call may enlarge the lower normal subgroup and replace its character by an invariant constituent. We assume neither centrality nor linearity of that character.

If Y is empty there is nothing to prove, and if $H = G$ we use the identity. Otherwise choose a maximal subgroup B containing H, and put $L = \mathrm{core}_G(B)$. The following lemmas construct a correspondence from G to B. Proper inertia uses a smaller inertia group; extension uses restriction; full ramification uses a smaller relative correspondence on the original subgroup B. A final smaller call passes from B to H. Every character-theoretic inductive call therefore decreases the original group order.

The full-ramification step uses Propositions 8.4 and 8.11, Corollary 9.30 and Theorem 10.1. Their geometric proofs do not use the relative theorem. The only character induction used in the compensation argument is on the original smaller subgroup B.

7.2. **Reduction to an invariant constituent.** Choose a maximal subgroup containing the local group. Its primitive quotient is affine. Clifford theory then partitions the characters according to constituents of its core. We first show that only one local orbit can contribute, and that proper inertia is handled by the inductive hypothesis with exactly the same fixed cosets.

Lemma 7.2. *In the nonempty proper case above,*

$$G/L = (N/L) \rtimes (B/L) = V \rtimes A, \quad V \cong \mathbb{F}_\ell^d, \quad \ell \neq p, \tag{7.4}$$

where $N \trianglelefteq G$*,* $A \neq 1$ *is faithful irreducible on* V*, and* $Z_0 \leq L$*. Every* $u \in Y$ *acts fixed point freely on* V*.*

Proof. The faithful action of G/L on its cosets of B/L is primitive. A minimal normal subgroup of this solvable group is elementary abelian. Its orbits form a block system, so it is transitive. An abelian transitive group has equal point stabilizers; faithfulness makes these trivial. It is therefore regular, giving (7.4). The point stabilizer A acts faithfully on V: an element centralizing the regular abelian group and fixing its base point fixes every point. A proper nonzero A-invariant subspace would give a nontrivial block system, so the action is irreducible.

The index $[G : B] = |V|$ is prime to p, because B contains a full Sylow subgroup. Thus $\ell \neq p$. Normality of Z_0 and its containment in B give $Z_0 \leq L$.

Lemma 5.2 says that the base coset of G/B is the unique one fixed by each $u \in Y$. In the affine description this is precisely $C_V(u) = 0$. If $A = 1$, every p-element of B acts trivially on V, contradicting this conclusion. Hence $A \neq 1$. □

We partition the characters over λ_0 according to the G-orbits on $\mathrm{Irr}(L \mid \lambda_0)$. An orbit can contribute a nonzero mixed profile only if one of its members is fixed by an element of Y. Indeed Clifford induction at us can be nonzero only if us, and hence its p-part u, fixes an inertia coset. The same observation applies to local characters and their B-orbits.

We call these orbits potentially contributing. This term does not assert that an induced sum is nonzero: such a sum may still cancel, and that possibility is retained.

Lemma 7.3. *Choose* ξ *in a potentially contributing orbit, fixed by* $u_0 \in Y$*, and put* $T = G_\xi$*,* $J = T \cap B$*. Then* J *contains a full Sylow subgroup of* T*. For* $x \in X_Y$*, every* x*-fixed coset in* G/T *lies in the base* B*-orbit, and the fixed cosets identify with those in* B/J*. There is exactly one potentially contributing* B*-orbit in the chosen* G*-orbit.*

Proof. Choose $R \in \mathrm{Syl}_p(T)$ containing u_0, and extend it to a Sylow subgroup of G. That Sylow contains u_0 and hence lies in B. Thus $R \leq J$.

Use left cosets, with $x^g = g^{-1}xg$. If gT is fixed by $u \in Y$, conjugate $u^g \in T$ into R by $t \in T$. Then $u^{gt} \in R \leq B$. The unique fixed coset of G/B forces $gt \in B$. Consequently $gT = bT$ for some $b \in B$. The same applies to an $x = us$-fixed coset, which is fixed by the p-part u.

The equivariant injection $bJ \mapsto bT$ identifies B/J with this base orbit. For $b \in B$, $x^b \in T$ is equivalent to $x^b \in J$. Hence the fixed cosets agree exactly. Applying the same argument to the orbit of constituents shows that every constituent fixed by some member of Y lies in the base B-orbit, proving uniqueness. □

Define the entire family needed for the smaller call by

$$Y_T = \{u^b \in J : u \in Y,\ b \in B\}. \tag{7.5}$$

For $v \in Y_T$, the base coset is the unique fixed coset of G/B. The injection $tJ \mapsto tB$ from T/J therefore gives a unique fixed coset there. Since J contains R, Lemma 5.2 gives $\mathrm{Sub}_T(v) \le J$ and $C_T(v) \le J$.

In particular the smaller relative problem has legitimate data $(T, J; L, \xi, Y_T)$. The normal L is contained in J, and ξ is T-invariant. No claim about normalizing an arbitrary larger Sylow is required.

Proposition 7.4. *Assume the relative theorem for groups smaller than G. If $T < G$ in Lemma 7.3, there is an exhaustive signed matching of the nonzero original profiles in this orbit, from G to B, preserving degree p-parts.*

Proof. The same fixed cosets will make induction respect the smaller correspondence.

Apply the smaller theorem to $(T, J; L, \xi, Y_T)$. Write its pairs as α, β, with one sign a_α on all the required mixed sections. Clifford induction gives the irreducible characters α^G and β^B, and gives bijections on the full Clifford families over the corresponding constituent orbits. By Lemma 7.3, for every $x = us \in X_Y$,

$$\begin{aligned} \alpha^G(x) &= \sum_{bJ \in (B/J)^x} \alpha(x^b) \\ &= a_\alpha \sum_{bJ \in (B/J)^x} \beta(x^b) = a_\alpha \beta^B(x). \end{aligned} \tag{7.6}$$

Indeed $x^b \in J$ has p-part $u^b \in Y_T$ and commuting p'-part in $C_T(u^b)$. Thus the same inductive sign applies to every summand, for every u at once.

The indices satisfy

$$[G : T]_p = [B : J]_p, \tag{7.7}$$

since $[G : B]$ and $[T : J]$ are prime to p. Together with the inductive degree equality this proves the asserted preservation.

If a paired induced sum is zero on the whole original family, (7.6) makes its partner zero too. Remove precisely these pairs. Exhaustivity is not lost: an active global row must have an active smaller profile by the displayed sum, and every active local row lies in the unique local orbit of Lemma 7.3 and likewise has an active smaller profile. Clifford bijectivity then accounts for all of them. Thus the same equality removes every cancelled pair and retains every nonzero induced character. □

The lower data remain the original character ξ of L. The only induction call is on $T < G$. No projective cover is substituted for that smaller group.

7.3. **The invariant cases.** Suppose now that the constituent of the core is invariant. We first record a coprime averaging lemma. It will give a distinguished extension in the present section and will later identify the action on a source with the action constructed in a model. We then show that extension and full ramification are the only possibilities.

Lemma 7.5. *Let $A \neq 1$ be solvable, acting faithfully and irreducibly on a nonzero $\mathbb{F}_\ell$-space V, and put $D = F(A)$. Then*

$$D \neq 1, \quad (|D|, \ell) = 1, \quad C_V(D) = 0, \quad H^1(A, V) = 0. \tag{7.8}$$

Proof. An ℓ-group acting in characteristic ℓ has nonzero fixed space. Thus $C_V(O_\ell(A)) \neq 0$; its A-invariance and irreducibility make it all of V. Faithfulness gives $O_\ell(A) = 1$.

A nontrivial finite solvable group has nontrivial Fitting subgroup. Its Sylow ℓ-subgroup is characteristic in the nilpotent group $F(A)$, hence normal in A, so is trivial. The space $C_V(D)$ is A-invariant. It cannot be V, by faithfulness and $D \neq 1$; hence it is zero.

Let $z : A \to V$ be a cocycle, so $z(ab) = z(a) + az(b)$. Division by $|D|$ is defined in $\mathbb{F}_\ell$. Set

$$t = |D|^{-1} \sum_{h \in D} z(h).$$

For $k \in D$, sum $z(kh) = z(k) + kz(h)$ over $h \in D$. It follows that $z(k) = t - kt$. Subtract this principal cocycle and assume $z(D) = 0$. For $a \in A$, $k \in D$, normality gives

$$z(ka) = kz(a), \qquad z(a(a^{-1}ka)) = z(a).$$

Their arguments are equal, so $z(a) \in C_V(D) = 0$. The original cocycle is therefore $z(a) = t - at$. □

We also use averaging on a finite abelian ℓ-group M, written additively, with an action of an ℓ'-group D. The integer $|D|$ is invertible modulo the exponent of M. Thus

$$\pi = |D|^{-1} \sum_{h \in D} h$$

is an endomorphism which is the identity on $C_M(D)$ and zero on each difference $hm - m$. In particular,

$$M = [M, D] \implies C_M(D) = 0. \tag{7.9}$$

This applies even if M has exponent larger than ℓ. No vector-space structure on an abelianization is being assumed.

Lemma 7.6. *Suppose $\xi \in \mathrm{Irr}(L)$ is G-invariant in* (7.4)*. Then either ξ extends to N, or it is fully ramified in N/L.*

Proof. Let $e = e_\xi$, $d = \xi(1)$, and $\mathcal{R} = \mathbb{C}Le \cong M_d(\mathbb{C})$. Conjugation by a coset representative is an automorphism of the lower full matrix algebra. Implement it by a matrix in that algebra and multiply the representative by the inverse implementing matrix. Invariance therefore gives a unit centralizing $\mathcal{R}$ in every homogeneous component indexed by $v \in V$. Consequently the double-centralizer construction gives

$$\mathbb{C}Ne \cong \mathcal{R} \otimes \mathcal{S}, \qquad \mathcal{S} = C_{\mathbb{C}Ne}(\mathcal{R}) = \bigoplus_{v \in V} \mathbb{C}t_v, \quad t_v t_w = \alpha(v, w) t_{v+w}. \tag{7.10}$$

This part of the construction uses only invariance, not full ramification. The commutators give an alternating bicharacter $\beta(v, w) = \alpha(v, w)/\alpha(w, v)$. Its radical is A-invariant. Irreducibility of V makes it either zero or all of V.

The algebra $\mathcal{S}$ is semisimple, as is clear from (7.10) and semisimplicity of $\mathbb{C}Ne$. If the radical is zero, its center is $\mathbb{C}$, so it is a matrix algebra of dimension $|V|$. There is a unique irreducible θ over ξ, with $\theta_L = \sqrt{|V|}\,\xi$; this is full ramification.

If the radical is all of V, $\mathcal{S}$ is commutative semisimple. Every simple summand of $\mathbb{C}Ne$ is then $M_d(\mathbb{C})$. Thus every character over ξ has degree d and restricts to ξ. In particular ξ extends to N. □

This proof avoids a separate assertion that a symmetric projective factor splits. It also keeps ξ in the original algebra when its degree is divisible by p.

In the extension case the extensions form a torsor over $\mathrm{Lin}(N/L)$, by multiplication with linear characters. This is the affine space used next. An extension to N does not in itself give an extension to G.

Lemma 7.7. *In the extension case of Lemma 7.6 there is a G-invariant extension $\theta \in \mathrm{Irr}(N)$ of ξ. For every eligible $u \in B$, it is the unique extension fixed by u. All other extension orbits have zero mixed profile on X_Y.*

Proof. The extension torsor is an affine space over V^*. The action on its points factors through $A = B/L$. By Lemma 7.5, $D = F(A)$ is a nontrivial ℓ'-group with zero fixed space on V, and also on V^*. Average the orbit of any affine point over D, using the inverse of $|D|$ in $\mathbb{F}_\ell$. The barycenter

is independent of an origin, because its coefficients sum to one. This gives a fixed point. It is unique because $C_{V^*}(D) = 0$. Normality of D makes that point A-invariant. It is the required θ; inner automorphisms from N already fix its character.

If $C_V(u) = 0$, the action of u on V^* has no fixed vector. Since θ is fixed, it is the only fixed point of the torsor. The same conclusion holds for every conjugate of u, and θ is fixed by all of G.

An irreducible character above another extension orbit is induced from its inertia group. A nonzero value at us would require an inertia coset fixed by us, and therefore an extension in that orbit fixed by u. Only θ can be fixed, a contradiction. □

It follows that every nonzero global profile over this ξ lies above θ. The local characters are still all the characters in $\mathrm{Irr}(B \mid \xi)$. The next algebra isomorphism identifies their complete parameter sets, preserving every degree and restriction.

The argument has not required θ to extend to G. That possible obstruction belongs to the common projective parameter factor, and need not be removed.

Proposition 7.8. *For θ in Lemma 7.7, restriction gives a complete bijection*

$$\mathrm{Irr}(G \mid \theta) \longrightarrow \mathrm{Irr}(B \mid \xi), \qquad \chi \longmapsto \chi_B. \tag{7.11}$$

All degrees are preserved. Together with Lemma 7.7 this gives the required matching of active profiles in the extension branch, with sign positive.

Proof. Restriction is realized by an isomorphism of the two character algebras.

Since $\theta_L = \xi$ is irreducible, the map $\mathbb{C}Le_\xi \to \mathbb{C}Ne_\theta$, $a \mapsto ae_\theta$, is an isomorphism of matrix algebras of dimension $\xi(1)^2$. Now θ is G-invariant, so e_θ is central in $\mathbb{C}G$. Thus

$$\Phi : \mathbb{C}Be_\xi \longrightarrow \mathbb{C}Ge_\theta, \qquad a \longmapsto ae_\theta \tag{7.12}$$

is an algebra homomorphism, sending its identity e_ξ to e_θ.

Grade the source by B/L and the target by $G/N \cong B/L$. On the component indexed by bL, Φ is exactly the preceding matrix isomorphism followed by multiplication by b. Hence it is a linear isomorphism on each component, and therefore an algebra isomorphism.

Simple modules on the two sides correspond exhaustively. For a G-module on which e_θ acts as the identity, the element be_ξ acts under (7.12) as be_θ, namely as the original element b. The corresponding simple B-module is precisely its restriction. Its dimension and all its traces on B are unchanged, proving (7.11). □

The extension branch therefore works over the original lower character. In (7.12) a projective obstruction may still occur in a description by quotient representations, but it is the same on both sides of the actual algebra isomorphism.

No lower trace is divided out. In particular, the correspondence remains valid at elements on which such a trace is zero.

7.4. **The remaining fully ramified case.** The remaining case explains why sources enter the proof. In a fully ramified section, the irreducible characters on the two sides have the same parameters, but restriction multiplies a parameter by a source character. We must account for this extra factor simultaneously on all the selected mixed sections.

The source localization theorem, Theorem 8.1, will make that factor a signed linear character on a subgroup containing a Sylow subgroup. A smaller relative correspondence then transports tensoring by the linear character back to a permutation of the original parameters. Proposition 8.11 writes out this compensation, including the removal of zero profiles. We defer the proof of the following assertion until Section 11, after establishing those source results.

Proposition 7.9. *Assume the relative theorem for groups of order less than $|G|$. If the invariant ξ is fully ramified in N/L, the active profiles over ξ admit the required signed matching from G to B.*

8. Sources and relative compensation

We now supply the source result needed in the fully ramified case. There are two tasks. The algebraic task is to separate a source factor from the original lower character, keeping the complete set of parameters. The geometric task is to choose the source action so that its mixed values are controlled by one linear character on a suitable subgroup. We state the common goal first. The algebraic construction and compensation occupy this section; the geometric proof follows in Sections 9 and 10.

8.1. **The source theorem.** We give the algebraic construction used at the fully ramified step. All algebras and representations are over $\mathbb{C}$. Source traces are taken in the source dimension, not the dimension of the lower character.

Fix a fully ramified primitive section

$$B < X, \quad L = \mathrm{core}_X(B), \quad X/L = (N/L) \rtimes (B/L) = V \rtimes A, \tag{8.1}$$

where X is solvable, V is elementary abelian of characteristic ℓ, and $A \neq 1$ acts faithfully and irreducibly on V. Let $\xi \in \mathrm{Irr}(L)$ be X-invariant and fully ramified in N/L. There is a unique $\vartheta \in \mathrm{Irr}(N \mid \xi)$, and

$$\vartheta_L = n\xi, \quad |V| = n^2 = \ell^{2m}, \quad d = \xi(1), \quad n = \ell^m.$$

The character ξ need not be linear, central, or extendible. Let $e = e_\xi$ be its central primitive idempotent in $\mathbb{C}L$. It is central in $\mathbb{C}X$. Put

$$\mathfrak{R} = \mathbb{C}Le, \quad \mathfrak{N} = \mathbb{C}Ne, \quad \mathfrak{G} = \mathbb{C}Xe, \quad \mathcal{S} = C_{\mathfrak{N}}(\mathfrak{R}). \tag{8.2}$$

We construct an A-invariant extraspecial $E \leq \mathcal{S}^\times$ whose complex span is $\mathcal{S}$. An ordinary implementer of its actual action then gives a complete correspondence over the original ξ. No central cover is used.

The final prime p enters when localization is applied. Then $\ell \neq p$, since the maximal subgroup contains a Sylow p-subgroup of X.

For group conjugacy use $U^g = g^{-1}Ug$. On the source algebra write

$${}^a s = bsb^{-1} \qquad (a = bL \in A,\ s \in \mathcal{S}).$$

This action is well defined because L centralizes $\mathcal{S}$. For a subgroup of algebra units, its complex span means its image in that algebra, not its abstract group algebra.

The complete correspondence is the fully ramified complex case of Ladisch's magic-representation construction [18]. The following finite-group construction realizes the source inside the original algebra, with its actual action.

In (8.1), the space X/B identifies with the affine space V. For $u \in B$ its fixed points are $C_V(u)$; eligibility therefore gives $C_V(u) = 0$. Neither consequence requires L to be a p'-group.

Three spaces occur in this construction, and it is useful to distinguish them before discussing their actions (Table 3).

Table 3. The three modules in the fully ramified construction

Space or module	Dimension	Role
$V = N/L$	$2m$ over $\mathbb{F}_\ell$	The finite space on which the complement acts.
A simple $\mathcal{S}$-module W	$n = \ell^m$ over $\mathbb{C}$	The source; its trace is the extra restriction factor.
The simple $\mathfrak{N}$-module	nd over $\mathbb{C}$	The original module above the lower character of degree d.

The source character is a trace on W. Taking the trace on the last module instead would multiply it by the lower degree. Moreover, a zero lower trace cannot be cancelled from a character identity. Keeping the whole matrix algebra will avoid both problems.

An *ordinary implementer* is a homomorphism $\sigma : A \to \mathcal{S}^\times$ inducing the given conjugation action on $\mathcal{S}$. Its character is the trace ψ on W. The ordinary homomorphism condition is needed

for the tensor decomposition of characters; a projective implementer alone would not give that decomposition.

Let $p \neq \ell$ be odd. For an ordinary source character ω of an action A on V, the proof requires

$$P_A \in \mathrm{Syl}_p(A), \quad P_A \leq K_A \leq A, \quad \Gamma \in \mathrm{Irr}(K_A) \text{ linear of 2-power order}, \quad \epsilon \in \{1, -1\},$$

such that every $u \in P_A$ with $C_V(u) = 0$ fixes just the base coset of A/K_A, and

$$\omega(us) = \epsilon\Gamma(us) \qquad (s \in C_A(u)_{p'}). \tag{8.3}$$

The same data work for the whole family. In particular $C_A(u) \leq K_A$. This is a sufficient geometric property, not a reformulation of Conjecture B.

Theorem 8.1 (Source localization)**.** *In the fully ramified primitive setting* (8.1)–(8.2)*, let $p \neq \ell$ be odd and fix $P_A \in \mathrm{Syl}_p(A)$. The given action of A on $\mathcal{S}$ has an ordinary implementer whose source character ψ satisfies* (8.3)*, with $\omega = \psi$, for one choice of K_A, Γ, ϵ on the entire fixed-point-free family in P_A. If $\ell = 2$, one may take $K_A = A$.*

The subgroup condition and the character identity have separate uses. The unique fixed coset puts every required centralizer in K_A and supplies the eligibility condition for the relative problem on B. That problem is available in the group-order induction; when $K_A = A$, the identity suffices. The character identity absorbs the source factor by tensoring with Γ. A single choice on the whole family is required for both uses. In particular, separate linear characters on the different centralizers would not suffice.

The proof of Theorem 8.1 concludes at the end of Section 10. Its geometric arguments contain no appeal to the relative character theorem. Until then, the compensation argument below uses (8.3) as an explicit hypothesis.

8.2. **An extraspecial source inside the algebra.** The homogeneous components of the centralizer algebra contain a finite phase group. Taking commutators with the coprime Fitting subgroup extracts an invariant extraspecial group. We then obtain an ordinary implementer for the original subgroup, rather than working on an auxiliary central extension.

Lemma 8.2. *In* (8.1)–(8.2)*,*

$$\mathfrak{R} \simeq M_d(\mathbb{C}), \quad \mathfrak{N} \simeq M_{nd}(\mathbb{C}), \quad \mathcal{S} \simeq M_n(\mathbb{C}), \quad \mathfrak{R} \otimes_{\mathbb{C}} \mathcal{S} \xrightarrow{\sim} \mathfrak{N}. \tag{8.4}$$

Moreover $\mathcal{S} = \bigoplus_{v \in V} \mathcal{S}_v$, where each $\mathcal{S}_v$ is a line containing units, $\mathcal{S}_0 = \mathbb{C}e$, and $\mathcal{S}_v\mathcal{S}_w = \mathcal{S}_{v+w}$. Commutators define a nondegenerate alternating bicharacter

$$\beta : V \times V \longrightarrow \mu_\ell. \tag{8.5}$$

The actual A-action permutes these lines according to its action on V, and preserves β.

Proof. The first algebra is the simple component of ξ. Full ramification gives $e = e_\vartheta$ in $\mathbb{C}N$, so $\mathfrak{N} \simeq M_{nd}(\mathbb{C})$. The double centralizer theorem gives the remaining claims in (8.4).

Choose $y_v \in N$ representing $v \in N/L$. Then $\mathfrak{N} = \bigoplus_v \mathfrak{R}y_v e$. Conjugation by y_v is an automorphism of the full matrix algebra $\mathfrak{R}$; choose $r_v \in \mathfrak{R}^\times$ implementing it. The unit $r_v^{-1}y_v e$ centralizes $\mathfrak{R}$. Every element of $\mathfrak{R}y_v e$ centralizing it is a scalar multiple of this unit, since $Z(\mathfrak{R}) = \mathbb{C}e$. These lines are $\mathcal{S}_v$. Products of their units remain units and have the stated degree.

Multiplicative commutators of homogeneous units are scalars. Their centrality makes β multiplicative in each variable; the exponent of V puts its values in μ_ℓ. It is alternating. If v is in its radical, a unit in $\mathcal{S}_v$ commutes with every homogeneous line and hence is central in $\mathcal{S}$. But $Z(\mathcal{S}) = \mathbb{C}e = \mathcal{S}_0$, so $v = 0$.

Conjugation by B preserves the grading and fixes its scalar commutators. Its action on $\mathcal{S}$ factors through B/L. This proves the last assertion. □

Here μ_j denotes the complex j-th roots of unity. Multiplication is the tensor isomorphism in (8.4). In particular, it retains the original algebra $\mathbb{C}Le$; the lower character has not been replaced by a linear character of an auxiliary group.

Lemma 8.3. *There is a finite A-invariant $T \le \mathcal{S}^\times$ with*

$$T/Z(T) \simeq V, \quad \operatorname{span}_{\mathbb{C}} T = \mathcal{S}, \quad T' = \mu_\ell e, \quad Z(T) = \begin{cases} \mu_\ell e, & \ell \text{ odd}, \\ \mu_4 e, & \ell = 2. \end{cases} \tag{8.6}$$

For odd ℓ, T has exponent ℓ.

Proof. Choose a unit $t_v \in \mathcal{S}_v$ with $t_v^\ell = e$, and take $t_0 = e$. Rescaling a unit achieves this, since its ℓ-th power is a nonzero complex scalar. Let $j = \ell$ if ℓ is odd and $j = 4$ otherwise, and put

$$T = \{\zeta t_v : \zeta \in \mu_j,\ v \in V\}.$$

Write $t_v t_w = c_{v,w} t_{v+w}$. Since the commutators are central of exponent ℓ, the class-two power formula gives

$$(t_v t_w)^\ell = e \ (\ell \text{ odd}), \qquad (t_v t_w)^2 \in \mu_2 e \ (\ell = 2).$$

Thus $c_{v,w} \in \mu_\ell$ in the odd case and $c_{v,w} \in \mu_4$ in the binary case. The finite set T is closed under multiplication, contains e, and consists of units, so it is a group. Its grading map onto V has kernel the stated scalars. Nondegeneracy of β shows that these are exactly $Z(T)$.

The commutators generate $\mu_\ell e$, since β is nontrivial and ℓ is prime. The units t_v span $\mathcal{S}$. For $a \in A$, write ${}^a t_v = c\, t_{av}$. Taking ℓ-th powers gives $c^\ell = 1$, so A preserves T. For odd ℓ, every ζt_v has ℓ-th power e. □

Fourth roots in the binary construction are essential: products of normalized involutions can square to $-e$. We have not asserted that the involutions form a subgroup.

This is a genuine action on a finite group: it is the restriction of the original A-action on $\mathcal{S}$. It fixes the scalar center pointwise and induces the original faithful irreducible action on V.

Proposition 8.4. *With $D = F(A)$ and T as in* (8.6)*, the subgroup $E = [T, D]$ is A-invariant and extraspecial, and*

$$T = Z(T)E, \quad E \cap Z(T) = Z(E) = E' = \mu_\ell e, \quad |E| = \ell^{1+2m}, \quad \operatorname{span}_{\mathbb{C}} E = \mathcal{S}. \tag{8.7}$$

It has exponent ℓ if ℓ is odd. A simple $\mathcal{S}$-module restricts to a faithful irreducible representation of E, of degree n.

Proof. By Lemma 7.5, $(|D|, \ell) = 1$ and $C_V(D) = 0$. Averaging on V gives $[V, D] = V$. Thus E maps onto $T/Z(T)$, and $T = Z(T)E$. Normality of D in A makes E A-invariant.

Since $Z(T)$ is central, $E' = T' = \mu_\ell e$. Also D fixes $Z(T)$, so $E = [T, D] = [E, D]$. The finite abelian ℓ-group $M = E/E'$ therefore satisfies $M = [M, D]$. Equation (7.9) gives $C_M(D) = 0$. But $(E \cap Z(T))/E' \le M$ is D-fixed, so $E \cap Z(T) = E'$.

An element of E centralizes E if and only if it centralizes $Z(T)E = T$. Hence $Z(E) = E \cap Z(T) = E'$, of order ℓ. Since $E/Z(E) \simeq V$ is elementary abelian, $\Phi(E) = E'E^\ell = E'$. This proves extraspeciality and the order. Moreover $T = Z(T)E$ implies that E spans $\mathcal{S}$. Its action on a simple $\mathcal{S}$-module is irreducible, and faithful because $\mathcal{S} \simeq M_n(\mathbb{C})$ acts faithfully. For odd ℓ, the exponent assertion follows from $E \le T$. □

This proof allows A of even order. For the binary source define $Q_E : V \to \mathbb{F}_2$ by

$$z^2 = (-e)^{Q_E(v)} \quad (z \in E \text{ mapping to } v).$$

It is well defined because $Z(E) = \{e, -e\}$. The square formula gives its polar form from β; it is nondegenerate and A-invariant. Thus the later orthogonal form belongs to this particular extraspecial source, not just an abstract quotient.

Proposition 8.5. *In* (8.1)*, the original A-action on $\mathcal{S}$ has an ordinary implementer*

$$\sigma : A \longrightarrow \mathcal{S}^\times, \qquad {}^a s = \sigma(a) s \sigma(a)^{-1}. \tag{8.8}$$

No triviality assumption on the projective factor of ξ is needed.

Proof. Apply the strong-section theorem of Dade–Isaacs in Ladisch's formulation [17, Definition 4.36 and Theorem 4.38] to $(X, N, L, \vartheta, \xi)$. Both characters are X-invariant; for ϑ this follows from its uniqueness over ξ.

Let $D = F(A)$ and let M be its inverse image in B. Lemma 7.5 gives

$$M \trianglelefteq B, \quad NM \trianglelefteq X, \quad M \cap N = L, \quad C_{N/L}(M) = 0, \quad C_M(N/L) = L.$$

The normality of NM follows from normality of $V \rtimes D$ in $V \rtimes A$. The last equality uses faithfulness on V. Thus the operator group $M/C_M(N/L) = D$ is coprime to N/L. The theorem's goodness condition holds: its required centralizer is L, which acts trivially on $\mathcal{S} = (\mathbb{C}Ne)^L$. These are exactly the strong-section hypotheses.

The theorem constructs the complement as $N_X(M)$. This is the original B. Indeed $B \le N_X(M)$. If $y \in N$ normalizes M, then $[y, M] \le N \cap M = L$, so $yL \in C_V(D) = 0$. Since $X = NB$, this proves $N_X(M) = B$. The magic representation supplied by the theorem is therefore defined on B/L, inside the original $\mathcal{S}$, and gives (8.8). □

This is an application of a published existence theorem, not a claim to have reproved it. It does not prescribe the mixed values needed for localization. The geometric proof will choose an implementer with those trace identities.

For a simple $\mathcal{S}$-module W, of dimension n, put

$$\psi(a) = \mathrm{Tr}_W(\sigma(a)).$$

This is an n-dimensional trace. Taking the trace on the nd-dimensional N-module instead would introduce an incorrect factor $d = \xi(1)$.

8.3. **The complete correspondence and its parameters.** An ordinary implementer separates the source factor from the lower character factor. This gives a correspondence for every lower parameter, including those with zero selected trace. We record how the correspondence changes when the implementer is multiplied by a linear character, and then identify the actual source actions.

Theorem 8.6. *For any ordinary implementer σ in* (8.8)*, with character ψ inflated to B, there is a bijection of all irreducibles*

$$\iota_\sigma : \mathrm{Irr}(X \mid \xi) \longrightarrow \mathrm{Irr}(B \mid \xi)$$

such that, for $\eta = \iota_\sigma(\chi)$,

$$\chi_B = \psi\eta, \qquad \chi(1) = n\eta(1). \tag{8.9}$$

No degree or height restrictions are made on the parameters.

Proof. We separate the source action from a commuting copy of the lower character algebra.

Let $\sigma_b = \sigma(bL)$, and put $c_b = \sigma_b^{-1} be \in \mathfrak{G}$. It centralizes $\mathcal{S}$ by (8.8), and

$$\begin{aligned} c_b c_c &= \sigma_b^{-1}(b\sigma_c^{-1}b^{-1})bce \\ &= \sigma_c^{-1}\sigma_b^{-1}bce = \sigma_{bc}^{-1}bce = c_{bc}. \end{aligned} \tag{8.10}$$

For $l \in L$, $c_l = le$. Thus $b \mapsto c_b$ induces a unital algebra homomorphism $\kappa : \mathbb{C}Be \to C_{\mathfrak{G}}(\mathcal{S})$, fixing $\mathfrak{R}$.

For injectivity, choose representatives b_a for B/L. A relation $\sum_a r_a c_{b_a} = 0$, with $r_a \in \mathfrak{R}$, has terms in distinct N-coset components of $\mathfrak{G}$. Thus $r_a \sigma_{b_a}^{-1} = 0$, so every $r_a = 0$.

Multiplication gives an algebra homomorphism

$$\mathcal{S} \otimes_{\mathbb{C}} \mathbb{C}Be \longrightarrow \mathbb{C}Xe, \qquad s \otimes y \longmapsto s\kappa(y). \tag{8.11}$$

Its image contains $\mathfrak{R}$, $\mathcal{S}$, and $be = \sigma_b c_b$ for every $b \in B$. These elements generate $\mathbb{C}Xe$, by (8.4) and $X = NB$, so it is surjective. Domain and codomain have equal dimension,

$$n^2 |B : L| d^2 = |X : L| d^2,$$

and hence (8.11) is an isomorphism.

The simple modules of this tensor product are precisely $W \otimes U$, where U runs over all simple $\mathbb{C}Be$-modules. On them b acts as $\sigma_b \otimes \rho_U(b)$. Taking traces gives $\psi(b)\eta(b)$, and degrees give $n\eta(1)$.

As e is central, these are precisely all irreducibles over ξ on both sides. This proves (8.9) and completeness. □

Corollary 8.7. *The correspondence* (8.9) *preserves the projective parameter factor over* (B, L, ξ)*, even where the lower trace is zero. If* σ' *is another ordinary implementer of the same action, then* $\sigma' = \lambda\sigma$ *for a linear character* λ *of* A*, and the parameters change by* $\eta' = \lambda^{-1}\eta$*.*

Proof. Choose $\rho : L \to \mathrm{GL}_d(\mathbb{C})$ affording ξ. Invariance gives a projective extension T_b to B, with $T_l = \rho(l)$ and factor inflated from A:

$$T_b T_c = \alpha(bL, cL) T_{bc}. \tag{8.12}$$

To construct it, choose an intertwiner on each L-coset and set $T_{lb} = \rho(l)T_b$. Schur's lemma makes the product discrepancy scalar, depending only on the two quotient cosets.

A B-module U over ξ is L-isomorphic to $\mathbb{C}^d \otimes M$. After removing T_b, its remaining action commutes with $\rho(L)$, and hence acts on M. Thus

$$\rho_U(b) = T_b \otimes R(bL), \qquad R(a)R(c) = \alpha(a, c)^{-1} R(ac). \tag{8.13}$$

Conversely, every irreducible projective parameter with this factor gives an irreducible U, by the same double centralizer. The B-action in Theorem 8.6 is

$$\sigma(bL) \otimes T_b \otimes R(bL).$$

Its first factor is ordinary, so the parameter factor remains α^{-1}; it has not been trivialized.

Writing $r_b = \mathrm{Tr}(T_b)$, for every $b \in B$ we obtain

$$\eta(b) = r_b \,\mathrm{Tr}(R(bL)), \qquad \chi(b) = \psi(bL)\, r_b \,\mathrm{Tr}(R(bL)). \tag{8.14}$$

These identities remain valid when $r_b = 0$. No cancellation of r_b was used.

Finally $\sigma'(a)$ and $\sigma(a)$ induce the same inner automorphism of the full matrix algebra $\mathcal{S}$; their quotient is a scalar $\lambda(a)$. Their homomorphism identities give $\lambda(ac) = \lambda(a)\lambda(c)$. In (8.11), replacing σ by $\lambda\sigma$ changes its parameters by the inverse tensor twist. This preserves their degrees and the factor in (8.13). The inverse twist follows from the algebra maps, not from cancelling possibly zero values of ψ. □

Working with the whole source is essential here. A calculation involving only a cap, or only parameters of degree prime to p, would not prove completeness in (8.9).

Lemma 8.8. *Two genuine actions* $\alpha, \beta : A \to \mathrm{Aut}(E)$ *on the source* (8.7)*, fixing* $Z(E)$ *and inducing the same action on* V*, are conjugate by an inner automorphism of* E*. An ordinary implementer for one transports to an ordinary implementer for the other with identical traces on all of* A*.*

Proof. Here $\mathrm{Inn}(q)$ means $x \mapsto qxq^{-1}$. The kernel of the induced action on V, among automorphisms fixing $Z(E)$, is $\mathrm{Inn}(E) \simeq V$. Indeed any such automorphism is $x \mapsto x\varphi(xZ(E))$, with a homomorphism $\varphi : V \to Z(E)$. Nondegeneracy of the commutator form identifies every φ with conjugation by an element of $E/Z(E)$.

Consequently write

$$\beta(a) = \mathrm{Inn}(z(a))\alpha(a), \qquad z(a) \in V.$$

The two homomorphism identities give $z(ac) = z(a) + az(c)$. Lemma 7.5 gives $z(a) = t - at$ for some $t \in V$. If $q \in E$ represents t, then

$$\beta(a) = \mathrm{Inn}(q)\alpha(a)\,\mathrm{Inn}(q)^{-1}. \tag{8.15}$$

For the source representation ρ_E, an implementer $\sigma_0(a)$ for α transports to

$$\rho_E(q)\sigma_0(a)\rho_E(q)^{-1}. \tag{8.16}$$

This implements β, is ordinary, and has exactly the same trace as $\sigma_0(a)$. □

For odd ℓ, the exponent-ℓ source can be identified with the Heisenberg model

$$(v, t)(w, s) = (v + w, t + s + \tfrac{1}{2}\langle v, w\rangle).$$

A symplectic transformation acts genuinely by $(v, t) \mapsto (av, t)$. After choosing its ordinary Weil implementer, (8.15)–(8.16) identify it with the original action.

For $\ell = 2$, use the square form Q_E from Proposition 8.4. An isometric model must first supply a genuine action and an ordinary implementer. The lemma then identifies its lift. It does not construct a lift from $H^1 = 0$, or turn a merely projective equivalence into an ordinary representation. Since E spans $\mathcal{S}$, implementing E implements the full algebra action. The trace equality in (8.16) retains mixed identities on the entire corresponding family of A-elements.

8.4. **Compensating a linear character.** We next explain how the source theorem will be used. The algebraic correspondence in Theorem 8.6 gives every restriction as a source character times an irreducible parameter. If that source is a signed linear character on a local subgroup, tensoring permutes the local parameters. Transporting this permutation through one smaller relative correspondence gives the required permutation of the original parameters. The following lemma makes this operation explicit; its proof uses the same correspondence on the entire requested family.

For $L \trianglelefteq B$, $\xi \in \mathrm{Irr}(L)$ invariant in B, and $P_B \le K \le B$ containing L, write $\mathcal{R}(B, K; L, \xi, Y)$ for the relative conclusion 7.1 on

$$X_Y = \{us : u \in Y,\ s \in C_B(u)_{p'}\} \subseteq K.$$

Here $P_B \in \mathrm{Syl}_p(B)$, $Y \subseteq P_B$ is a finite eligible family, and the inclusion follows from Lemma 5.2. For a subset Z of the relevant group, let $\mathcal{I}_J(Z)$ be the characters in $\mathrm{Irr}(J \mid \xi)$ with nonzero profile on Z.

Lemma 8.9. *Assume $\mathcal{R}(B, K; L, \xi, Y)$, with bijection f and signs a_η, so $\eta(x) = a_\eta f(\eta)(x)$ on X_Y. Let Γ be linear on K/L. If $\omega(x) = \epsilon\Gamma(x)$ on X_Y, with one sign ϵ, there is a permutation F of $\mathcal{I}_B(X_Y)$ preserving degree p-parts, such that*

$$\omega(x)\eta(x) = \epsilon a_\eta a_{F\eta}(F\eta)(x) \quad (x \in X_Y). \tag{8.17}$$

Proof. Tensoring by Γ preserves the fiber over ξ, since $\Gamma_L = 1$. It preserves degrees and nonzero profiles. Define

$$F = f^{-1} \circ (\Gamma \otimes -) \circ f.$$

Put $\eta' = F\eta$. Then $f(\eta') = \Gamma f(\eta)$, and therefore, on the whole X_Y,

$$\omega(x)\eta(x) = \epsilon\Gamma(x)a_\eta f(\eta)(x) = \epsilon a_\eta f(\eta')(x) = \epsilon a_\eta a_{\eta'}\eta'(x).$$

Its sign is common to the full family, and degree p-parts are preserved through the three maps defining F. □

No order condition on the linear character is needed in this lemma; the applied geometric characters have 2-power order.

Lemma 8.10. *Suppose $\chi_B = \psi\iota(\chi)$ is the complete correspondence* (8.9)*. Let $X_* \subseteq B$, suppose $\psi(x) \neq 0$ for every $x \in X_*$, and let F be a permutation of $\mathcal{I}_B(X_*)$ satisfying $\psi(x)\eta(x) = b_\eta(F\eta)(x)$ on X_*. For any subset X_0 of the B-conjugacy closure of X_*, the map $\chi \mapsto F\iota(\chi)$ restricts to a signed bijection of all nonzero X_0-profiles.*

Proof. The identity and nonvanishing of ψ extend to that closure because all three functions are class functions. Thus $\eta|_{X_0} = 0$ if and only if $(F\eta)|_{X_0} = 0$. Also $\chi|_{X_0} = 0$ if and only if $\iota(\chi)|_{X_0} = 0$. Every nonzero X_0-profile belongs to the original selected set, by conjugacy invariance. Hence the restrictions are exhaustive bijections, with the same signs. □

Proposition 8.11. *In* (8.1)*, let $p \neq \ell$ be odd, let $P_B \in \mathrm{Syl}_p(X)$ lie in B, and let Y be a finite family of nonidentity p-elements in B with $\mathrm{Sub}_X(u) \le B$. Assume the source has an ordinary implementer with localization data* (8.3)*, for the Sylow $P_A = P_B L/L$ of A. Put $K/L = K_A$ inside B/L, and*

$$Y_* = \{v \in P_B : C_V(v) = 0\}.$$

If $\mathcal{R}(B, K; L, \xi, Y_*)$ is available, or $K = B$, there is a signed bijection of the nonzero profiles over ξ in X, B, on every us with $u \in Y$, $s \in C_X(u)_{p'}$, preserving degree p-parts.

Proof. We first enlarge the family so that the compensation lemma applies with one permutation of parameters.

The empty family is immediate, so assume $Y \neq \varnothing$. Lemma 5.2 on X/B gives $C_V(u) = 0$ for all $u \in Y$. Choose a B-conjugate in P_B for each u; it belongs to Y_*. Notice that L may have a nontrivial p-part: the image of P_B is still a full Sylow of A.

For $v \in Y_*$, the coset action B/K is the action A/K_A, so (8.3) gives a unique v-fixed coset. Since K contains P_B, Lemma 5.2 gives $\mathrm{Sub}_B(v) \leq K$, and $C_B(v) \leq K$. This verifies the subgroup and eligibility hypotheses for the smaller relative correspondence.

Inflate the geometric linear character to Γ on K/L. If $s \in C_B(v)_{p'}$, its image is a p'-element centralizing the image of v. Therefore (8.3) gives

$$\psi(vs) = \epsilon\Gamma(vs).$$

Lemma 8.9 supplies a single permutation F of the nonzero X_{Y_*}-profiles over ξ, preserving degree p-parts, with $\psi\eta = b_\eta F\eta$ throughout X_{Y_*}. If $K = B$, take the identity relative map in that lemma.

By Theorem 8.6, $\chi_B = \psi\iota_\sigma(\chi)$ for all characters over ξ, and ψ is nonzero on X_{Y_*}. For the original family, Lemma 5.2 gives $C_X(u) = C_B(u)$. Conjugating u into P_B also conjugates each commuting p'-element into $C_B(v)_{p'}$. Consequently X_Y is contained in the B-conjugacy closure of X_{Y_*}. Lemma 8.10 restricts the matching to precisely its nonzero original profiles, removing zero profiles on both sides.

Finally $n = \ell^m$ is prime to p, so $\chi(1)_p = \iota_\sigma(\chi)(1)_p$. Composition with F preserves that equality. The resulting sign for a character is independent of the choice of u. □

In the induction for 7.1 the relative call is on the original $B < X$, so its group order decreases. No call is made on a cover. Block identification is a later step; this proposition does not assert that intermediate linear twists preserve parameter blocks.

9. Localization in odd source characteristic

We prove the odd-source part of Theorem 8.1. Its required conclusion is (8.3), with the canonical ordinary Weil character. The fixed-point-free Weil formula turns the problem into determinant calculations. Our first task is to make the resulting local characters restrictions of one character on a suitable subgroup. Our second task is to retain that assertion under the decompositions of a solvable irreducible linear group.

The calculations follow that order. Linear hyperbolic and unitary factors give the basic characters. Semilinear normalizers require a common field phase. Tensor products and permuted sums then reduce the problem by dimension. Only an absolutely irreducible extraspecial two-group remains; its cyclic spectra and small fixed factors are treated at the end of the section.

9.1. **Localization and restriction.** Throughout this section, q is an odd prime power and p is an odd prime different from its characteristic. All vector spaces in this section are over finite fields; their Weil characters are complex characters. Write χ_q for the quadratic character of $\mathbb{F}_q^\times$. If V is symplectic of dimension $2a$, write ω_V for the canonical Weil character. We use the following published formula:

$$\omega_V(x) = \chi_q\big((-1)^a \det(1-x)\big) \qquad (C_V(x) = 0). \tag{9.1}$$

This is Ladisch [19, Theorem B and Corollary 11.5]. In particular, (9.1) does not require that the order of x be prime to the source characteristic.

If u is a p-element and $s \in C_A(u)$ has p'-order, then u is the p-part of us. Consequently

$$C_V(u) = 0 \quad \Longrightarrow \quad C_V(us) = 0. \tag{9.2}$$

Indeed, u is a power of us, so every vector fixed by us is fixed by u. We call such a u *fixed point free*. This is the geometric eligibility condition; the subnormalizer condition in the character theorem enters through unique fixed cosets.

The modulus of each fixed-point-free Weil value is one, but its local description must carry more information than that. Proposition 9.5 below will give a quadratic character on each centralizer. Those characters are not yet a solution to (8.3): the subgroup and its character must be chosen for

the whole Sylow family. The semilinear phase and the later small-factor localizers provide precisely that common choice.

There are two useful elementary consequences of (9.1). For an orthogonal direct sum of invariant symplectic spaces on which x is fixed point free, its value is the product of their values. Also, if a symplectic form over $\mathbb{F}_{q^j}$ is traced to $\mathbb{F}_q$, the fixed point free values calculated over the two fields agree. To see the latter, use

$$\chi_q(N_{\mathbb{F}_{q^j}/\mathbb{F}_q} z) = \chi_{q^j}(z), \qquad \chi_q(-1)^{ja} = \chi_{q^j}(-1)^a. \tag{9.3}$$

Only this fixed point free assertion about restriction of scalars is needed below.

We prove localization for the full semilinear unitary normalizers, tensor actions with an even orthogonal factor, and the operations needed to assemble and restrict these data. The final reduction isolates the irreducible extraspecial case. Its exceptional configurations are not proved by the semilinear calculation.

For $A \le \mathrm{Sp}(V,q)$ and $P \in \mathrm{Syl}_p(A)$, property $\mathcal{L}_p(A,V)$ means that there exist

$$P \le K \le A, \quad \Gamma \in \mathrm{Lin}(K), \quad |\Gamma| \text{ a power of 2}, \quad \varepsilon \in \{1,-1\}, \tag{9.4}$$

such that every fixed point free $u \in P$ fixes just the base coset in A/K, and

$$\omega_V(us) = \varepsilon\Gamma(us) \qquad (s \in C_A(u)_{p'}). \tag{9.5}$$

The value on the right is defined: uniqueness of the fixed coset gives $C_A(u) \le K$. By Lemma 5.2 it also gives $\mathrm{Sub}_A(u) \le K$. The same K, Γ, ε must work for all these u. If the set is empty, take $K = A, \Gamma = 1, \varepsilon = 1$.

Lemma 9.1. *Property $\mathcal{L}_p$ restricts to subgroups. It also pulls back along a surjection $\rho : A \to \bar{A}$ between two actions if every fixed point free p-element u of the first action has fixed point free image and*

$$\omega_V(us) = \omega_{\bar{V}}(\rho(us)) \qquad (s \in C_A(u)_{p'}). \tag{9.6}$$

Proof. Let $D \le A$. Conjugating the data, we may assume that a Sylow subgroup Q of D lies in P. Take $K_D = D \cap K$, restrict Γ, and keep ε. The equivariant injection

$$D/(D \cap K) \longrightarrow A/K, \qquad d(D \cap K) \longmapsto dK$$

contains the base coset. An eligible $u \in Q$ can fix no other coset in its image. Both the coset assertion and (9.5) therefore restrict to D.

For a surjection, $\rho(P)$ is a Sylow subgroup of $\bar{A}$. Pull back its localizer $\bar{K}$ and its character, and use the equivariant bijection $A/\rho^{-1}(\bar{K}) \simeq \bar{A}/\bar{K}$. The hypothesis on eligibility gives uniqueness; (9.6) gives the value identity. □

These assertions concern geometric data. They do not restrict a permutation of irreducible characters of an overgroup. The character compensation of Proposition 8.11 is applied in the original smaller group over its original lower character.

9.2. **The linear factors.** We first deal with linear actions. The fixed-point-free hypothesis makes their Weil values signs. The determinant on a hyperbolic pair and the quadratic determinant character on a unitary factor give ordinary characters on their normalizers.

Put $d = \mathrm{ord}_p(q)$. Define

$$\sigma_{p,q}(r) = \begin{cases} (-1)^{r(q-1)/2}, & d \text{ odd}, \\ (-1)^{r(q-1)/2 + r/c}, & d = 2c \text{ and } c \mid r. \end{cases} \tag{9.7}$$

In the remaining case $d = 2c$, $c \nmid r$, put $\sigma_{p,q}(r) = 1$. This is a convention for an empty case, not the value of a determinant of an existing element.

Lemma 9.2. *If $T \in \mathrm{Sp}_{2r}(q)$ is a fixed point free p-element, then*

$$\chi_q(\det(1 - T^2)) = \sigma_{p,q}(r). \tag{9.8}$$

If $d = 2c$, such an element requires $c \mid r$.

Proof. The nontrivial eigenvalues have orders p^j. Their Frobenius orbits have lengths $d_j = \mathrm{ord}_{p^j}(q) = dp^{a_j}$, $a_j \geq 0$. This follows because the kernel of reduction of units modulo p^j to units modulo p is a p-group. An orbit contains the inverse of one of its members exactly when d_j is even: the only involution in $(\mathbb{Z}/p^j\mathbb{Z})^\times$ is -1.

Choose one eigenvalue from each inverse pair, with multiplicity, and let $L = \prod(\lambda - \lambda^{-1})$. Then

$$\det(1 - T^2) = (-1)^r L^2.$$

Frobenius changes L by a minus sign for each self-dual orbit, counted with multiplicity. Two orbits interchanged by inversion contribute no sign. Thus, if N counts the self-dual orbits with multiplicity,

$$\chi_q(\det(1 - T^2)) = (-1)^{r(q-1)/2+N}.$$

If d is odd, $N = 0$. If $d = 2c$, all orbits are self-dual and have half-length cp^{a_j}. Hence $c \mid r$ and $N \equiv r/c \pmod 2$. This proves (9.8). □

The proof uses the full element T. Raising an arbitrary fixed point free p-element to a power of order p need not preserve fixed point freeness.

Let $V = Y \oplus Y^*$, $\dim Y = n$, with its hyperbolic alternating form. Set

$$h_B = \mathrm{diag}(B, B^{-T}), \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad N_h = \langle h_B, J : B \in \mathrm{GL}_n(q) \rangle.$$

Its two cosets are h_B and $h_B J$. Define $\gamma(h_B) = \chi_q(\det B)$. If $n = 2r$, set $\gamma(J) = \sigma_{p,q}(r)$. If n is odd, choose $\gamma(J)^2 = \chi_q((-1)^n)$. These rules define an ordinary character of order dividing four: inverse transpose preserves $\chi_q \det$, and $J^2 = h_{-I}$.

Proposition 9.3. *Property $\mathcal{L}_p(N_h, V)$ holds with $K = N_h$, $\varepsilon = 1$, and $\Gamma = \gamma$.*

Proof. As p is odd, an eligible u has the form h_T. For a fixed point free h_C, block determinants give

$$\det(1 - h_C) = (-1)^n \det(C)^{-1} \det(1 - C)^2,$$

so (9.1) gives $\omega_V(h_C) = \chi_q(\det C)$. This proves the assertion when s preserves both summands.

Suppose $s = h_B J$ centralizes u. Then $TBT^T = B$. Thus T is similar to T^{-T}; since its nontrivial odd-order eigenvalues occur in inverse pairs, $n = 2r$ and $\det T = 1$. Put

$$H = TB + T^{-1}B^T, \qquad Z = (T - T^{-1})H.$$

The first matrix is symmetric and $THT^T = H$. Block elimination gives

$$\det(1 - us) = \frac{\det H}{\det B}. \tag{9.9}$$

Consequently H is nonsingular by (9.2). The matrix Z is nonsingular alternating and $TZT^T = Z$. Its determinant is a square. Also $\det(T - T^{-1}) = \det(1 - T^2)$. Applying Lemma 9.2 to the symplectic action defined by Z (equivalently to T^T with Gram matrix Z) yields

$$\chi_q(\det H) = \sigma_{p,q}(r).$$

Since the semidimension of V is the even integer n, (9.1) and (9.9) give $\omega_V(us) = \sigma_{p,q}(r)\chi_q(\det B) = \gamma(us)$. In odd dimension no such exterior s exists. □

Let $K = \mathbb{F}_{Q^2}$, with $\bar{z} = z^Q$, and write $\mathrm{GU}_m(Q)$ for the isometry group of a nondegenerate hermitian form. For $z^{Q+1} = 1$, put $\delta_Q(z) = z^{(Q+1)/2} \in \{1, -1\}$. Choose $\beta \neq 0$ with $\bar{\beta} = -\beta$ and trace β times the hermitian form to an alternating form.

Lemma 9.4. *If $g \in \mathrm{GU}_m(Q)$ is fixed point free, then*

$$\omega(g) = (-1)^m \delta_Q(\det_K g). \tag{9.10}$$

This remains its fixed point free value after restriction of scalars from $\mathbb{F}_Q$ to a subfield of odd order.

Proof. Let $z = \det_K(1-g)$ and $d = \det_K g$. Unitarity gives $\bar{z} = (-1)^m d^{-1} z$, whence

$$\chi_{Q^2}(z) = (-1)^{m(Q+1)/2}\delta_Q(d).$$

Now $\det_{\mathbb{F}_Q}(1-g) = N_{K/\mathbb{F}_Q}(z)$. Multiplying the last expression by $\chi_Q(-1)^m = (-1)^{m(Q-1)/2}$ proves (9.10) by (9.1). The last assertion is (9.3). □

Proposition 9.5. *Let $u \in \mathrm{Sp}(V,q)$ be a fixed point free p-element. There is a quadratic character $\eta_u \in \mathrm{Lin}(C_{\mathrm{Sp}(V,q)}(u))$ such that*

$$\omega_V(us) = \omega_V(u)\eta_u(s) \qquad (s \in C_{\mathrm{Sp}(V,q)}(u)_{p'}). \tag{9.11}$$

For $u \in \mathrm{GU}_m(Q)$ fixed point free, with its traced symplectic action,

$$\omega_V(u) = (-1)^m, \qquad \eta_u|_{C_{\mathrm{GU}_m(Q)}(u)} = \delta_Q \circ \det_K. \tag{9.12}$$

Proof. The element u is semisimple. Its primary spaces are paired by reciprocal polynomials; neither $X-1$ nor $X+1$ occurs. A pair of distinct reciprocal irreducibles gives a hyperbolic space with centralizer $\mathrm{GL}_b(q^e)$. A self-reciprocal irreducible of degree $2e$ gives a hermitian space with centralizer $\mathrm{GU}_b(q^e)$. This describes the whole centralizer, since a commuting operator preserves each primary space and its pairing.

On a general linear factor use the quadratic determinant character on one lagrangian, as in the first part of Proposition 9.3. On a unitary factor use $\delta\det$, as in Lemma 9.4. Their product is an ordinary quadratic character on the full centralizer. In each factor the p-part contributes trivially to that character; its remaining constant is its Weil value at u. Equations (9.1)–(9.3) therefore prove (9.11), with all centralizer p'-elements included.

Lemma 9.4 gives (9.12) first on p'-elements of the unitary centralizer. Both characters there are quadratic and kill every p-element, so equality extends to every element by its commuting p- and p'-parts. □

9.3. **Semilinear unitary actions.** For a semilinear element, an odd power gives a linear norm. We must compare its determinant with the original determinant over the smaller field, and choose one field phase for the entire normalizer. The next calculations establish that compatibility before applying it to the mixed values.

Lemma 9.6. *If $x \in \mathrm{Sp}(V,q)$, k is positive and odd, and both x and x^k are fixed point free, then $\omega_V(x^k) = \omega_V(x)$.*

Proof. Write $k = 2a+1$ and put

$$R = x^{-a}(1 + x + \cdots + x^{2a}).$$

It is invertible by the two hypotheses. The adjoint of x is x^{-1}, so R is self-adjoint. If B is a Gram matrix for the alternating form, BR is again nonsingular alternating. Thus $\det R$ is a square. Since $\det x = 1$,

$$\frac{\det(1-x^k)}{\det(1-x)} = \det R.$$

Now apply (9.1). □

In particular, suppose u is a p-element, l is odd, and both u and u^l are fixed point free. For $s \in C_A(u)_{p'}$, choose a positive odd integer k with $k \equiv l \pmod{|u|}$ and $k \equiv 1 \pmod{|s|}$. The Chinese remainder theorem gives such an integer. By (9.2) and Lemma 9.6,

$$\omega_V(us) = \omega_V(u^l s). \tag{9.13}$$

The centralizer element s is unchanged.

Lemma 9.7. *The field characteristic is arbitrary. Let x be an invertible semilinear map on K^m, with field automorphism τ of order l, and let $K_0 = K^\tau$ contain $\mathbb{F}_q$. Put $D = \det_K(1-x^l)$. Then $D \in K_0$ and*

$$\det_{\mathbb{F}_q}(1-x) = N_{K_0/\mathbb{F}_q}(D), \qquad \det_{\mathbb{F}_q}(1-x^l) = N_{K_0/\mathbb{F}_q}(D)^l. \tag{9.14}$$

In particular, x is fixed point free if and only if x^l is fixed point free.

Proof. The operator x commutes semilinearly with x^l, so the characteristic polynomial of x^l is τ-fixed. This proves $D \in K_0$. Extend scalars from K_0 to an algebraic closure. Its l embeddings of K give l summands cyclically permuted by x. Block elimination identifies the determinant of $1-x$ on their sum with $\det_K(1-x^l)$. Descending first to K_0 and then by the norm proves the first formula. The second follows by the determinant formula for restriction of scalars, since $D \in K_0$. □

Fix $c, m \geq 1$, and set

$$K = \mathbb{F}_{q^{2c}}, \quad Q = q^c, \quad \bar{z} = z^Q, \quad V = K^m.$$

Choose a hermitian orthonormal basis. If g_0 generates $K^\times$, choose

$$\beta = g_0^{(Q+1)/2}, \qquad a = g_0^{(q-1)/2}, \qquad b(v,w) = \mathrm{Tr}_{K/\mathbb{F}_q}(\beta\, v^T \bar{w}). \tag{9.15}$$

Then $\bar{\beta} = -\beta$. The map $r(v) = av^{(q)}$ preserves b, since $a\bar{a} = \beta^{q-1}$. Direct multiplication gives

$$rhr^{-1} = h^{(q)} \quad (h \in \mathrm{GU}_m(Q)), \qquad r^{2c} = -I. \tag{9.16}$$

The normalizer of the scalar field in $\mathrm{Sp}(V,b)$ is

$$N = N_{m,c}(q) = \langle \mathrm{GU}_m(Q), r \rangle, \qquad N/\mathrm{GU}_m(Q) \simeq C_{2c}. \tag{9.17}$$

Indeed, a field-normalizing linear map is semilinear for a power of Frobenius; after multiplication by the corresponding power of r^{-1}, it is K-linear. Preservation of b, tested also on all scalar multiples of a vector, then forces preservation of the hermitian form.

The character $\delta_Q \det_K$ of $\mathrm{GU}_m(Q)$ is invariant under Frobenius. A value λ on r extends it to an ordinary linear character of N exactly when

$$\lambda^{2c} = (-1)^{m(Q+1)/2}. \tag{9.18}$$

Such an extension of 2-power order always exists. If there are no fixed point free p-elements, any one suffices. In the nonempty case we shall specify λ and prove its value on every active field component.

If $u \in N$ is a fixed point free p-element, its image in the field group has odd order l. Lemma 9.7 shows that $v = u^l$ is a fixed point free element of $\mathrm{GU}_m(Q)$. Equation (9.13) reduces the required mixed values to this linear v, without changing s.

Let $u \in \mathrm{GU}_m(Q)$ be a fixed point free p-element. Suppose the field image of $C_N(u)$ has Sylow 2-subgroup of order $\ell = 2^b > 1$. Put

$$n = 2c, \quad j = n/\ell, \quad q_0 = q^j, \quad K_0 = \mathbb{F}_{q_0}. \tag{9.19}$$

The image contains the automorphism $z \mapsto z^{q_0}$. Thus the characteristic polynomial of u over K lies in $K_0[X]$. Since $j \mid c$, the involution $^-$ fixes K_0. Unitarity therefore makes this polynomial reciprocal. It has no roots $1, -1$, so m is even.

Lemma 9.8. *There exist $u_0, H \in \mathrm{GL}_m(K_0)$, with H symmetric, and $h \in \mathrm{GL}_m(K)$, such that*

$$u_0^T H u_0 = H, \qquad \bar{h}^T h = H, \qquad u = h u_0 h^{-1}. \tag{9.20}$$

The element u_0 is a fixed point free p-element, and

$$\chi_{q_0}(\det H) = \sigma_{p,q_0}(m/2). \tag{9.21}$$

Proof. Realize each pair of distinct reciprocal irreducible factors of the prescribed polynomial over K_0 on a module and its dual, with their symmetric hyperbolic pairing. For each self-reciprocal factor use its field of roots with the involution sending a root to its inverse. The trace of a nondegenerate hermitian form over that field is symmetric over K_0. Taking the prescribed multiplicities constructs u_0, H. All eigenvalues have p-power order and are different from one, as for u.

After extending to K, the matrix H is hermitian. Two semisimple unitary isometries with the same characteristic polynomial over a finite field are conjugate by an isometry of their hermitian spaces. Here this can be checked on the primary spaces: a reciprocal-conjugate pair has a perfect pairing between the two modules; a self-dual primary space has a hermitian form over its endomorphism field. Such pairings, and finite-field hermitian forms of a given dimension, are unique up to isometry. This gives h with both equalities required in (9.20).

The matrix $H(u_0 - u_0^{-1})$ is nonsingular alternating. Its determinant is a square, and u_0 preserves it. Apply Lemma 9.2 to this symplectic form to obtain (9.21). □

This descent retains the discriminant. It does not assert that H has square determinant.

Let U be symplectic of dimension $2r$, and W orthogonal of dimension $2m$, over $\mathbb{F}_q$. Let R, S be similitudes of multipliers ν, ν^{-1}, respectively, and put

$$x = R \otimes S, \qquad D = \det(S)\nu^m \in \{1, -1\}, \qquad \Lambda = \chi_q(\nu).$$

Assume that x is fixed point free.

Lemma 9.9. *In this situation,*

$$\omega_{U\otimes W}(x) = \begin{cases} \Lambda^{mr}, & D = 1, \\ \Lambda^{(m-1)r}\chi_q(\det(1 - \nu^{-1}R^2)), & D = -1. \end{cases} \tag{9.22}$$

In the second case the displayed determinant is nonzero. No semisimplicity hypothesis on R, S is needed.

Proof. Choose $z^2 = \nu$ in the quadratic extension and normalize $R' = z^{-1}R$, $S' = zS$. Their product tensor is still x, and they are isometries. Write $\operatorname{charpol}_{R'}(X) = X^r Q(X + X^{-1})$. If $D = 1$, write $\operatorname{charpol}_{S'}(X) = X^m P(X + X^{-1})$. These are monic polynomials. Pairing inverse roots gives

$$\prod_{\epsilon,\eta=\pm 1} (1 - \alpha^\epsilon \beta^\eta) = \left((\alpha + \alpha^{-1}) - (\beta + \beta^{-1})\right)^2, \qquad \det(1 - x) = \operatorname{Res}(P, Q)^2. \tag{9.23}$$

The resultant is nonzero. Frobenius sends both normalized isometries to Λ times themselves and hence multiplies every compressed root by Λ. Therefore $\operatorname{Res}(P, Q)^q = \Lambda^{mr} \operatorname{Res}(P, Q)$. The quadratic character of its square is Λ^{mr}.

If $D = -1$, the multiplicities of both 1 and -1 as eigenvalues of S' are odd. Remove one of each and write

$$\operatorname{charpol}_{S'}(X) = (X^2 - 1)X^{m-1}P(X + X^{-1}).$$

The same calculation gives

$$\det(1 - x) = \det(1 - \nu^{-1}R^2) \operatorname{Res}(P, Q)^2, \qquad \operatorname{Res}(P, Q)^q = \Lambda^{(m-1)r} \operatorname{Res}(P, Q). \tag{9.24}$$

All factors are nonzero. Formula (9.1) completes both cases, since the semidimension $2mr$ is even. Only roots with algebraic multiplicity were used. □

Assume that $N_{m,c}(q)$ contains a fixed point free p-element. Write

$$d = \operatorname{ord}_p(q) = 2^s b_0, \quad b_0 \text{ odd}, \qquad \alpha_{p,q} = \begin{cases} \mathrm{i}^{(q-1)/2}, & s = 0, \\ \mathrm{i}^{(q-1)/2}\zeta_{2^{s+1}}^{\nu}, & s > 0, \end{cases} \tag{9.25}$$

where $\zeta_t = \exp(2\pi\mathrm{i}/t)$ and $\nu \equiv b_0^{-1} \pmod{2^{s+1}}$. Set $\lambda = \alpha_{p,q}^m$.

There are two constraints on the value of the character at the field generator. First, it must respect the relation (9.16), so that it defines an ordinary character. Second, on every field subgroup which occurs in an eligible centralizer, its restriction must equal the quadratic character already found there. The phase in the next lemma solves these constraints simultaneously. The descent calculation retains the discriminant because that discriminant appears in the second constraint.

Lemma 9.10. *This phase satisfies* (9.18)*. For every j in* (9.19) *arising from the centralizer of a fixed point free linear p-element, it also satisfies*

$$\lambda^j = \sigma_{p,q^j}(m/2). \tag{9.26}$$

Proof. The defining relation. First put $a = v_2(2c) \geq 1$, and $h_0 = \operatorname{ord}_p(Q)$. The possible unitary primary dimensions for nontrivial p-power roots are odd multiples of

$$f(h_0) = \begin{cases} 2h_0, & h_0 \text{ odd}, \\ h_0/2, & h_0 \equiv 2 \pmod 4, \\ h_0, & 4 \mid h_0. \end{cases} \tag{9.27}$$

Indeed, the Q^2-orbit of a root is self-dual for $z \mapsto z^{-Q}$ exactly when its Q-orbit has length twice an odd integer. Otherwise two such orbits must be paired. Orders p^k multiply these lengths by odd powers of p. Thus $f(h_0) \mid m$. There is a linear fixed point free element by Lemma 9.7. The resulting divisibilities give

$$s < a \Rightarrow 2 \mid m, \qquad s > a \Rightarrow v_2(m) \geq s - a + 1. \tag{9.28}$$

If $s = 0$, or $0 < s < a$, or $s > a$, these inequalities make both sides of (9.18) equal to one. If $s = a \geq 2$, the factor involving i in λ^{2c} is one and the other is $(-1)^m$. Since c is even, $(Q+1)/2$ is odd, as required. If $s = a = 1$, then c, ν are odd and the equality is

$$(-1)^{m((q-1)/2+\nu)} = (-1)^{m(q+1)/2} = (-1)^{m(Q+1)/2}.$$

This proves the defining relation.

The active field subgroups. For j as in (9.19), Lemma 9.8 makes m even and realizes a fixed point free symplectic p-element in dimension m over $\mathbb{F}_{q^j}$. Put

$$t = v_2(j), \qquad d_j = \operatorname{ord}_p(q^j) = d/\gcd(d, j).$$

Reciprocal Frobenius orbits give $d_j \mid m$ if d_j is even, and $2d_j \mid m$ if it is odd.

If $s = 0$, both sides of (9.26) equal $(-1)^{(m/2)j(q-1)/2}$. If $s = 1$, they equal one for even j; for odd j, both equal $(-1)^{(m/2)((q-1)/2+1)}$.

Suppose $s \geq 2$. The divisibilities just obtained give $2^s \mid mj$, so the i-factor disappears and

$$\lambda^j = (-1)^{mj/2^s}. \tag{9.29}$$

The first exponent $(m/2)(q^j - 1)/2$ in (9.7) is even: if j is even, use $q^j \equiv 1 \pmod 8$; if j is odd, use $4 \mid m$. When $t < s$, the remaining exponent m/d_j has the parity of $mj/2^s$, since their quotients by the common power of two are odd. When $t \geq s$, d_j is odd and has no remaining exponent, while $mj/2^s$ is even. This proves (9.26). □

The first part constructs an ordinary character. The second part determines its values on the active components; it cannot be replaced by the defining relation alone.

Theorem 9.11. *For every $c, m \geq 1$, there is an ordinary linear character Γ_p of $N = N_{m,c}(q)$, of 2-power order, such that*

$$\omega_V(us) = (-1)^m \Gamma_p(us) \quad (u \text{ a fixed point free } p\text{-element},\ s \in C_N(u)_{p'}). \tag{9.30}$$

If such elements exist, take

$$\Gamma_p|_{\mathrm{GU}_m(Q)} = \delta_Q \det_K, \qquad \Gamma_p(r) = \alpha_{p,q}^m. \tag{9.31}$$

Thus $\mathcal{L}_p(N, V)$ holds with $K = N$. No solvability assumption on N is required.

Proof. If there are no eligible elements, choose any 2-power extension from (9.18). Otherwise use (9.31). For a linear u, put $C = C_N(u)$ and $C_0 = C_{\mathrm{GU}_m(Q)}(u)$. Lemma 9.10 makes (9.31) an ordinary character. By Proposition 9.5,

$$\theta = \eta_u \Gamma_p^{-1}|_C$$

has 2-power order and is trivial on C_0. It factors through the cyclic field image of C. If that image has odd order, $\theta = 1$. Otherwise it suffices to evaluate θ on one lift of a generator of its Sylow 2-subgroup.

Use j, q_0, ℓ, u_0, H, h of Lemma 9.8. On the hermitian space with Gram matrix H, the map r^j is an isometry of the traced alternating form and centralizes u_0, because H, u_0 have entries in K_0. Consequently

$$s_0 = hr^j h^{-1} = Br^j, \qquad B = h(h^{-1})^{(q^j)}, \tag{9.32}$$

centralizes u. It induces the required field generator. Moreover, $(r^j)^\ell = -I$, so s_0 has order 2ℓ, and in particular is a p'-element.

Over K_0, identify this traced symplectic space with

$$(K,\ (x, y) \mapsto \mathrm{Tr}_{K/K_0}(\beta x \bar{y})) \otimes (K_0^m, H).$$

Here $u_0 r^j$ acts as $r^j \otimes u_0$. The first factor is symplectic of dimension ℓ, the second orthogonal of even dimension m, with determinant one. Their orders are a power of two and a power of p, respectively, and the second has no eigenvalue one. Their tensor is fixed point free. Lemma 9.9, with multipliers one and $D = 1$, gives

$$\omega_V(us_0) = 1. \tag{9.33}$$

Restriction of scalars is justified by (9.3). Since m is even and $\omega_V(u) = (-1)^m$, (9.11) yields $\eta_u(s_0) = 1$.

For the character value let $z = \det_K h$. From (9.20), $z^{Q+1} = \det H$, and from (9.32), $\det B = z^{1-q_0}$. Therefore

$$\begin{aligned} \Gamma_p(s_0) &= \delta_Q(z^{1-q_0})\lambda^j \\ &= (\det H)^{(1-q_0)/2}\lambda^j \\ &= \chi_{q_0}(\det H)\lambda^j = \sigma_{p,q_0}(m/2)^2 = 1. \end{aligned} \tag{9.34}$$

The last step uses (9.21) and (9.26). Hence $\theta(s_0) = 1$, so $\theta = 1$ on all of C. This proves (9.30) for every linear u.

For an arbitrary u, let l be its field order. By Lemma 9.7, $v = u^l$ is linear and fixed point free. For the original $s \in C_N(u)_{p'}$, (9.13) and the linear case give

$$\omega_V(us) = \omega_V(vs) = (-1)^m \Gamma_p(s).$$

Finally, $\Gamma_p(u) = 1$, since its order is a power of two and p is odd. This proves the stated identity for us itself. □

For example, in $N_{1,2}(3)$ at $p = 5$, (9.18) forces $\Gamma_p(r)^4 = -1$, so order eight is necessary. In $N_{1,4}(3)$ at $p = 41$, it forces order sixteen. An a priori bound of four on the compensating character would exclude these groups.

9.4. **Orthogonal tensor factors.** The separate tensor factors need not be ordinary representations of the operator group: their scalar ambiguities cancel only in the tensor product. For that reason the following constructions use quantities unchanged by inverse rescaling of the factors. Even orthogonal multiplicity gives a linear character directly. Odd orthogonal multiplicity gives an ordinary homomorphism to a smaller symplectic group. In each case the character or homomorphism is defined on the original operator group.

Lemma 9.12. *Let $T \in \mathrm{Sp}_{2r}(q)$ be fixed point free of odd order prime to the source characteristic. Let R be a similitude of multiplier ν, centralizing T, and assume $1 - \nu^{-1}(TR)^2$ invertible. Then*

$$\chi_q(\det(1 - \nu^{-1}(TR)^2)) = \chi_q(\nu)^r \chi_q(\det(1 - T^2)). \tag{9.35}$$

Proof. Put $B = TR$, $H_0 = T - T^{-1}$, $H = B - \nu B^{-1}$. The last two matrices commute, are skew-adjoint and are invertible. Thus $C = H_0^{-1}H$ is invertible, self-adjoint, and centralizes T. Over an algebraic closure, the eigenspaces of T are nondegenerately paired in pairs λ, λ^{-1}, with $\lambda \neq \pm 1$. Self-adjointness makes the two restrictions of C on a pair have the same characteristic polynomial. Hence its full characteristic polynomial is a monic square. Its unique monic square root is Frobenius invariant, so lies in $\mathbb{F}_q[X]$. In particular $\det C$ is a square in $\mathbb{F}_q^\times$.

Since $\det B = \nu^r$ and $\det T = 1$,

$$\det(1 - \nu^{-1}B^2) = \nu^{-r} \det H, \qquad \det(1 - T^2) = \det H_0.$$

Their ratio is $\nu^{-r} \det C$, proving (9.35). □

Now let A preserve a tensor decomposition $V = U \otimes W$, with U symplectic of dimension $2r$ and W orthogonal of dimension $2m$. Assume each $g \in A$ factors over $\mathbb{F}_q$ as $R_g \otimes S_g$, with inverse multipliers. Define

$$D(g) = \det(S_g)\mu_{S_g}^{-m}, \qquad \Lambda(g) = \chi_q(\mu_{R_g}), \qquad \sigma_{p,q}(r) = (-1)^{e_{p,q}(r)}. \tag{9.36}$$

Changing the factors by inverse scalars leaves both expressions unchanged. Multiplication of factors shows that D, Λ are ordinary quadratic characters.

Proposition 9.13. *For this tensor action, $\mathcal{L}_p(A,V)$ holds with $K = A$, $\varepsilon = 1$, and*

$$\Gamma_p = D^{e_{p,q}(r)}\Lambda^{mr}. \tag{9.37}$$

No solvability hypothesis is required.

Proof. For an eligible u of order p^a, its factors can be chosen as isometries T, S_u of p-power order. Indeed their p^a-powers are inverse scalars, so their multipliers are squares, since p^a is odd. Normalize to isometries; the remaining scalar p^a-powers are $\pm I$, removed by a simultaneous sign if necessary.

If $s \in C_A(u)_{p'}$, its factors commute with T, S_u. Tensor commutation initially gives commutation up to an inverse scalar. The form makes this scalar a sign; the odd order of T, S_u makes it one. Hence $us = (TR_s) \otimes (S_u S_s)$.

When $D(s) = 1$, Lemma 9.9 gives $\omega_V(us) = \Lambda(s)^{mr}$. Suppose $D(s) = -1$. Then S_u has a fixed vector. Otherwise its nontrivial eigenspaces pair by inversion, and any commuting orthogonal similitude, normalized over the algebraic closure, has determinant one on their sum, contradicting $D(s) = -1$. Eligibility therefore forces $C_U(T) = 0$.

Let $\nu = \mu_{R_s}$. Lemma 9.9 also gives the invertibility of $1 - \nu^{-1}(TR_s)^2$. Applying Lemmas 9.12 and 9.2 to its second case gives

$$\omega_V(us) = \Lambda(s)^{(m-1)r}\Lambda(s)^r\sigma_{p,q}(r) = D(s)^{e_{p,q}(r)}\Lambda(s)^{mr}.$$

Finally, D, Λ kill u, so this is (9.37) evaluated at us. □

Let $V = U \otimes W$, with $\dim U = 2r$ symplectic and $\dim W = 2m+1$ orthogonal. Again let A consist of tensors of similitudes of inverse multipliers over $\mathbb{F}_q$. The multiplier on an odd-dimensional orthogonal space is a square, since $(\det S)^2 = \mu_S^{2m+1}$. Thus both factors can be normalized to isometries, with only a simultaneous sign ambiguity.

Proposition 9.14. *The rule*

$$\rho(g) = (\det S_g)R_g \in \mathrm{Sp}(U,q) \tag{9.38}$$

defines an ordinary homomorphism. Every fixed point free p-element u has fixed point free image, and

$$\omega_V(x) = \omega_U(\rho(x)) \qquad (C_V(x) = 0). \tag{9.39}$$

Consequently localization for $\rho(A)$ pulls back to A.

Proof. A simultaneous change of sign changes $\det S_g$ and R_g by the same sign. It cancels in (9.38). The same cancellation in multiplying factorizations proves that ρ is a homomorphism, even though the two individual choices of factors need not be representations.

For isometries R, S, put $\epsilon = \det S$. Then

$$\mathrm{charpol}_S(X) = (X-\epsilon)X^m P(X+X^{-1}), \qquad \mathrm{charpol}_R(X) = X^r Q(X+X^{-1}),$$

where $P, Q \in \mathbb{F}_q[X]$ are monic. The inverse-pair calculation (9.23) gives

$$\det(1 - R \otimes S) = \det(1-\epsilon R)\,\mathrm{Res}(P,Q)^2. \tag{9.40}$$

If the tensor is fixed point free, all factors are nonzero. The initial Weil signs agree, since $r(2m+1) \equiv r \pmod 2$. Formula (9.1) proves (9.39).

For an eligible u, normalize its factors to p-elements as in Proposition 9.13. The odd-dimensional orthogonal factor has determinant one and an eigenvalue one. Hence its symplectic factor $T = \rho(u)$ is fixed point free. The last assertion is Lemma 9.1. □

This construction acts on the original group. It does not replace the lower character in Proposition 8.11 by a character of the image group. Only the geometric localizer and its linear character are pulled back.

9.5. **Reduction by dimension.** An action is *quasi-primitive* if its restriction to every normal subgroup is homogeneous, meaning a direct sum of copies of one irreducible module over the given ground field.

The reduction below has three branches. A nonhomogeneous restriction gives smaller Clifford components. A nonscalar cyclic normal subgroup gives a unitary field and hence a semilinear normalizer. In the remaining branch an extraspecial two-group gives a tensor decomposition. The parity of its orthogonal multiplicity determines which of the two tensor results applies. Only multiplicity one requires the extra analysis of the next three subsections.

Suppose $\mathcal{L}_p(A, W)$ holds with data $P, K, \Gamma, \varepsilon$, and $\dim W = 2a$. For $D \le S_t$, let $A \wr D$ act on the orthogonal sum of t isometric copies of W.

Proposition 9.15. *Property $\mathcal{L}_p(A \wr D, W^{\perp t})$ holds with*

$$K_t = K \wr D, \qquad \varepsilon_t = \varepsilon^t,$$
$$\Gamma_t((k_i), \pi) = \prod_i \Gamma(k_i)\ \mathrm{sgn}(\pi)^{(1-\varepsilon)/2 + a(q-1)/2}. \tag{9.41}$$

Proof. Choose $Q \in \mathrm{Syl}_p(D)$. Then $P^t \rtimes Q$ is a Sylow subgroup contained in K_t. The coset space is $(A/K)^t$. On a cycle of an eligible u, its cycle product is an element of P. It is fixed point free on W, since a fixed vector would propagate around the cycle to a fixed vector of u. It therefore fixes exactly one coset of A/K. The remaining coordinates are determined successively. Thus u fixes just the base coset globally, and its centralizer is contained in K_t.

For $x = us = ((k_i), \pi)$, let h_C be its product around a cycle C of π. We check that its p-part is eligible. Write $|C| = p^b d$, $p \nmid d$. The permutation of u has d cycles of length p^b on C. On one coordinate, the p-part of $h_C = x^{|C|}$ is $u^{|C|}$, namely the dth power of the corresponding cycle product of u. That product is fixed point free; raising a p-element to a p'-power preserves this property. This proves the assertion.

Each $h_C \in K$ now satisfies $\omega_W(h_C) = \varepsilon\Gamma(h_C)$: conjugate its p-part into P inside K and use (9.5). Block elimination on a cycle gives $\det(1 - x)|_C = \det(1 - h_C)$. Consequently (9.1) yields

$$\omega_{W^{\perp t}}(x) = \mathrm{sgn}(\pi)^{a(q-1)/2} \prod_C \omega_W(h_C). \tag{9.42}$$

If $c(\pi)$ counts cycles, then $\varepsilon^{c(\pi)} = \varepsilon^t \mathrm{sgn}(\pi)^{(1-\varepsilon)/2}$, and $\prod_C \Gamma(h_C) = \prod_i \Gamma(k_i)$. This proves (9.41) and the source identity simultaneously. □

Proposition 9.16. *To prove $\mathcal{L}_p(A, V)$ for every solvable faithful irreducible $A \le \mathrm{Sp}(V, q)$, it is enough to prove it when A has a normal extraspecial 2-subgroup of minus type acting faithfully and absolutely irreducibly on V.*

Proof. Argue by dimension, assuming the indicated cases have been proved.

Nonhomogeneous restriction. If V is not homogeneous over some normal subgroup, its Clifford components are paired only with their duals by the alternating form. The group permutes transitively either nondegenerate components, or hyperbolic pairs of totally isotropic dual components. In the first case the component stabilizer acts irreducibly on a smaller symplectic space. Apply induction, Proposition 9.15 and Lemma 9.1. In the second use Proposition 9.3 on the pair, then the same assembly and restriction. The embeddings are formed using actual isometries between components supplied by group elements.

Quasi-primitivity and a unitary field. Suppose now that A is quasi-primitive. A normal subgroup of source-characteristic prime-power order fixes a nonzero space and hence, by normality and irreducibility, acts trivially. Thus it is trivial. Every normal abelian subgroup acts homogeneously through a finite field and is cyclic. We use the Manz–Wolf structure theorem [21, Corollary 1.10] in the form stated by Baykalov [1, Lemma 3.2 and Remark 3.3]: the Fitting subgroup has the form ET; there is a cyclic normal $U \le T$ with $C_T(U) = U$; and the noncentral factors of E have normal extraspecial subgroups whose prime-order centers lie in $E \cap T$.

If U is not scalar, homogeneity gives a field $\mathbb{F}_q[U] = K$. The adjoint acts on it by a nontrivial involution: otherwise a generator would equal its inverse, and U would be scalar. Thus $K = \mathbb{F}_{q^{2c}}$, with involution $z \mapsto z^{q^c}$. Nondegeneracy of the trace pairing expresses the original alternating form

as the trace of β times a hermitian form on K^m. Since A normalizes this field, it embeds in (9.17). Apply Theorem 9.11 and Lemma 9.1.

An extraspecial normal subgroup. If U is scalar, then $T = C_T(U) = U \leq \{I, -I\}$. All the extraspecial factors therefore have prime two. At least one exists: otherwise the Fitting subgroup would be central, and its self-centralization in a solvable group would force A to be scalar, impossible here. Choose such a normal extraspecial subgroup E_0. Its central involution is $-I$, so the restriction of V is a sum of copies of its unique faithful absolutely irreducible representation W.

Write $V = W \otimes U_0$. The faithful representation W exists over $\mathbb{F}_q$; its invariant form is orthogonal for plus type and symplectic for minus type. These are the standard extraspecial representations and forms in Glasby–Howlett [10, Section 3]. The double centralizer gives factorizations of every element of A over $\mathbb{F}_q$. Preserving the form makes their factors similitudes of inverse multipliers.

For plus type, U_0 is symplectic and W is orthogonal of even dimension, so Proposition 9.13 applies. For minus type and even multiplicity, U_0 is orthogonal of even dimension and the same proposition applies with the factors interchanged. For minus type and odd multiplicity greater than one, Proposition 9.14 maps onto a smaller symplectic action containing E_0 faithfully and absolutely irreducibly. The image is solvable; induction and Lemma 9.1 apply. Multiplicity one is exactly the case assumed in the statement.

We record how the data return to the original action. Nonhomogeneous restriction uses spaces of strictly smaller dimension, followed by the wreath assembly of Proposition 9.15 and subgroup restriction in Lemma 9.1. The unitary-field branch uses Theorem 9.11 on an ambient group and then the same subgroup restriction; it makes no inductive call in the original dimension. The even tensor branches are the direct calculations of Proposition 9.13. The odd multiplicity branch uses the smaller symplectic image and the pullback assertion of Lemma 9.1. Each operation supplies an actual subgroup containing a Sylow subgroup, an ordinary linear character on it, and one sign for every fixed-point-free mixed section. Thus the output is precisely (9.4)–(9.5) on A, not a collection of unrelated identities on its centralizers. □

9.6. **Cyclic actions on the extraspecial quotient.** We have reduced to an extraspecial 2-group E_0 acting on the odd-characteristic space V. It is useful to distinguish this operator group from the extraspecial source group constructed in Section 8. The operator group acts on V; the source group has a complex representation whose character is ω_V. We first determine the cyclic spectra on V and then use those spectra in the determinant formula for ω_V. The occurrence of a 2-group here does not change the source characteristic, which remains odd throughout this section.

In this reduction, E denotes an extraspecial normal two-subgroup of the operator group A, as in Proposition 9.16. Two different actions occur here. The group E acts absolutely irreducibly on the finite symplectic space V of odd characteristic. Its quotient $E/Z(E)$ is a quadratic space over $\mathbb{F}_2$, on which an eligible operator also acts. Eligibility is fixed-point-freeness on V, not on that binary quotient. A fixed part of the quotient is therefore allowed. The cyclic spectrum calculation determines exactly how large that fixed part can be and how it contributes to a mixed source value.

The ordinary Weil character is the complex character associated with V. The cyclic spectra below belong to the faithful representation of the normal two-group on V; they decide which operators are eligible for the Weil-value formula. Thus the representation used to test eligibility and the Weil representation have distinct roles.

For the exceptional eligible order nine, the cube is useful for describing binary frames but is not fixed point free on V. The distinction between the two spaces is essential in that case.

We complete the case left in Proposition 9.16. Keep q odd and p odd, distinct from its characteristic, and assume

$$E \trianglelefteq A \leq \mathrm{Sp}(V, q), \qquad E \cong 2_-^{1+2d}, \qquad V|_E \text{ faithful and absolutely irreducible.} \tag{9.43}$$

Thus $\dim V = 2^d$, $d \geq 1$. Write $z = -I$ for the central involution of E. The quotient $\bar{E} = E/\langle z \rangle$ is a binary quadratic space: $e^2 = z^{Q_E(\bar{e})}$. Its polar form is denoted by b_E. The centralizer of E in the full symplectic group is $\{I, -I\}$. Conjugation consequently gives

$$A/E \hookrightarrow O_{2d}^-(2). \tag{9.44}$$

Indeed, the kernel of the action on $\bar{E}$ consists of inner automorphisms of E; after removing such an inner automorphism, an element centralizes E and lies in $\langle z\rangle$. For completeness, an automorphism trivial on the quotient is given by a binary linear functional, hence is inner by nondegeneracy of the polar form. Conversely, every quadratic isometry lifts: images of extraspecial generators preserve their squares and commutators, and therefore all the defining relations.

Lemma 9.17. *Let E_0 be extraspecial of order a power of two and $\varphi \in \mathrm{Aut}(E_0)$ have odd order n. Its faithful irreducible complex representation has a unique extension to $E_0 \rtimes \langle\varphi\rangle$ preserving its invariant bilinear form. The same assertion holds for its realization over a finite field of odd characteristic prime to n.*

Proof. The faithful irreducible representation is unique for its nontrivial central character. An intertwiner R for φ therefore exists. The invariant form is unique up to a scalar, so R is a similitude, of multiplier μ. Also $R^n = aI$ for a scalar a, giving $\mu^n = a^2$. Over a finite field, μ is thus a square because n is odd; over $\mathbb{C}$ this is automatic. Normalize R to an isometry. Its nth power is $\pm I$; changing its sign if needed makes that power I. This constructs the extension. Two such intertwiners differ by a scalar isometry, hence by $\pm I$. Their odd-order condition makes that sign positive, proving uniqueness. □

The faithful representations and their orthogonal or symplectic forms are those of Glasby–Howlett [10, Section 3]. Below, cyclic spectra can be computed over $\mathbb{C}$ and then used in characteristic different from two and p: the group $E_0 \rtimes C_{p^k}$ has order prime to that characteristic, so its ordinary lift preserves eigenvalue multiplicities. Uniqueness in Lemma 9.17 identifies the lift.

Let φ have order $n = p^k > 1$ on an extraspecial E_0 of order 2^{1+2d}. Its action on the binary quotient is faithful: the kernel of $\mathrm{Aut}(E_0) \to O(\bar{E}_0)$ is the inner automorphism group, a two-group.

Lemma 9.18. *The binary space $\bar{E}_0$ is the orthogonal sum of its nondegenerate fixed space and moving atoms of these types:*

atom	*dimension and type*	*order h of its action*
$Y \oplus Y^*$	$2r$, *plus*	$h \mid 2^r - 1$
self-dual irreducible	$2r$, *minus*	$h \mid 2^r + 1$.

(9.45)

Here $h > 1$, and Y in the first row is irreducible and not self-dual. The inverse images of the summands give a central product decomposition of E_0. Their faithful representations, including their normalized cyclic extensions, tensor to the representation of E_0.

Proof. The cyclic group has odd order, so its binary action is semisimple. The polar form pairs only dual primary modules. Its fixed space is nondegenerate and orthogonal to the moving space. A non-self-dual irreducible pairs with its dual; the two individual modules are totally singular, since an invariant linear functional on either is zero. Pairing bases gives the first row of (9.45).

On a self-dual nontrivial primary module the endomorphism field has an involution inducing inversion of the cyclic eigenvalue. The resulting hermitian multiplicity form diagonalizes, giving irreducible summands of dimension $2r$. On a moving module an invariant quadratic refinement is unique: two refinements differ by a fixed linear functional. For an irreducible self-dual summand one can write it on $\mathbb{F}_{2^{2r}}$ as

$$Q(x) = \mathrm{Tr}_{\mathbb{F}_{2^r}/\mathbb{F}_2}(ax^{1+2^r}), \qquad a \in \mathbb{F}_{2^r}^{\times}.$$

The norm has 2^r+1 elements in each nonzero fiber. Thus the number of zeros is $1+(2^r+1)(2^{r-1}-1) = 2^{2r-1} - 2^{r-1}$, which proves minus type. Inversion is Frobenius of degree r, so $h \mid 2^r + 1$. The first row instead has $h \mid 2^r - 1$.

The preimages of orthogonal summands commute and intersect in the common center. They are extraspecial, and their faithful representations tensor. On each moving factor the automorphism has order h, since its inner kernel is a two-group. On the fixed factor it acts trivially. Lemma 9.17 makes these tensor actions exactly the normalized extension, without a further odd linear twist. □

Write reg_{C_h} for the regular character. All spectra in this section refer to the form-preserving extension of Lemma 9.17.

Lemma 9.19. *For the two moving atoms in* (9.45)*, their cyclic characters are*

$$\begin{aligned} &\textit{plus atom:} && 1_{C_h} + \frac{2^r-1}{h}\,\mathrm{reg}_{C_h}, \\ &\textit{minus atom:} && \frac{2^r+1}{h}\,\mathrm{reg}_{C_h} - 1_{C_h}. \end{aligned} \tag{9.46}$$

In particular, the spectrum contains every hth root of unity, except when the atom is minus and $h = 2^r + 1$. In that case only the eigenvalue one is missing.

Proof. For a plus atom use its Schrödinger model with basis indexed by the binary space Y of dimension r. Lifts of a given odd cyclic quotient action are inner-conjugate by Schur–Zassenhaus, so Lemma 9.17 permits this model. The action on this basis is permutation of Y. Every nonzero orbit has length h: an irreducible cyclic eigenvalue has order h, so no proper power has a nonzero fixed vector. Zero contributes one fixed basis vector. This gives the first formula.

For a minus atom, put $L = 2^r + 1$. Multiplication by the norm-one torus C_L preserves its binary quadratic form and is fixed point free for every nonidentity element. The preimage of that torus in $\mathrm{Aut}(E_0)$ has a two-group kernel. Schur–Zassenhaus gives an odd complement. Its subgroup of order h realizes the given cyclic action up to inner conjugacy. Thus it is enough to determine the normalized character ϑ on C_L.

Its degree is $L - 1$. For $1 \neq t \in C_L$, conjugation permutes the homogeneous one-dimensional spaces in $\mathrm{End}(W)$ indexed by the binary quotient. Only the identity space is fixed. Hence $\vartheta(t)\vartheta(t^{-1}) = 1$, or $|\vartheta(t)|^2 = 1$. If m_λ is the multiplicity of $\lambda \in \mathrm{Irr}(C_L)$, then

$$\sum_\lambda m_\lambda = L - 1, \qquad \sum_\lambda m_\lambda^2 = \frac{(L-1)^2 + (L-1)}{L} = L - 1.$$

Each multiplicity is zero or one; exactly one is zero. Self-duality makes the missing character equal to its inverse. Since L is odd, it must be the trivial character. Thus $\vartheta = \mathrm{reg}_{C_L} - 1$. Restriction to C_h proves the second formula. □

The torus used here acts on the extraspecial binary quotient. It is not an assumed extension of the original group A; only its cyclic spectrum is used.

Theorem 9.20. *For φ of order $n = p^k > 1$ on E_0, the normalized faithful representation is fixed point free if and only if its moving binary space is one minus irreducible atom, of dimension $2c$, and*

$$n = 2^c + 1, \qquad E_0 = C_{E_0}(\varphi) * [E_0, \varphi], \qquad \vartheta|_{C_n} = 2^{d-c}(\mathrm{reg}_{C_n} - 1). \tag{9.47}$$

The only possible orders are a Fermat prime p, or $n = 9$ with $p = 3, c = 3$. In every other case the cyclic spectrum contains all nth roots of unity.

Proof. Since n is a prime power, some moving atom has action of order n. If its spectrum is full, its tensor with any other factor is still full. Otherwise Lemma 9.19 makes it a minus atom with $n = 2^c + 1$ and spectrum $\mu_n \setminus \{1\}$.

If another moving atom is present, the tensor of the other factors is nonscalar. A scalar operator could not implement a nontrivial automorphism of those extraspecial factors. It therefore has at least two distinct eigenvalues. Multiplying $\mu_n \setminus \{1\}$ by these two eigenvalues gives two sets with different missing members; their union is μ_n. Thus the full spectrum is again present. If there is no other moving atom, the fixed factor contributes multiplicity 2^{d-c}, proving (9.47). Conversely, that character has no trivial constituent.

If $p^k - 1 = 2^c$ and $k > 1$ is odd, the quotient $(p^k - 1)/(p - 1)$ is an odd divisor greater than one, which is impossible. If k is even, the powers of two $p^{k/2} - 1$, $p^{k/2} + 1$ differ by two and must be two and four. Hence $p = 3, k = 2, c = 3$. If $k = 1$, primality of $2^c + 1$ forces c to be a power of two. These are precisely the stated possibilities. The passage to the finite source field was explained after Lemma 9.17. □

In (9.43), an eligible u and its automorphism of E have the same order, since the symplectic centralizer of E is $\{I, -I\}$. The theorem therefore applies to the actual element u. It does not apply to an arbitrary homogeneous multiple of the faithful representation. In particular, its order-nine case is not obtained by replacing u by its cube.

9.7. **Centralizers and their ordinary characters.** The mixed centralizer preserves the fixed extraspecial factor. Its determinant gives the remaining source value. Large fixed factors have determinant one. For the small factors, the Sylow bound leaves the cases in which an ordinary character of the local subgroup is still required.

Return to (9.43), and let $u \in A$ be an eligible p-element. Put $n = |u| = 2^c + 1$ and $b = d - c$. Theorem 9.20 gives

$$F = C_E(u), \quad M = [E, u], \quad E = F * M, \quad V = U \otimes W, \quad \dim U = 2^b, \quad \dim W = n - 1. \tag{9.48}$$

Here M is minus. If $b > 0$, F is plus; if $b = 0$, read $F = Z(E)$, $\dim U = 1$. The form on U is symmetric, and that on W alternating. The element u acts as $I_U \otimes u_W$.

Lemma 9.21. *For this active factor,*

$$C_{\mathrm{Aut}(M)}(u|_M) = \langle u|_M \rangle. \tag{9.49}$$

Consequently every $s \in C_A(u)_{p'}$ acts exactly as

$$s = T \otimes I_W, \qquad T \in O(U, q). \tag{9.50}$$

The factors in (9.50) *are defined over the original field.*

Proof. Commuting with u preserves both F and M. On $M/Z(M)$, the element u is irreducible of dimension $2c$. Identify its endomorphism field with $\mathbb{F}_{2^{2c}}$. A commuting linear map is multiplication by a field element; preservation of the quadratic form is exactly norm one to $\mathbb{F}_{2^c}$. This centralizer is therefore cyclic of order $2^c + 1 = n$, and is already generated by the image of u.

The kernel of the centralizer map in (9.49) is the fixed part of $\mathrm{Inn}(M) \simeq M/Z(M)$. It is trivial because the action is fixed point free there. Thus that map is injective, and the powers of $u|_M$ exhaust its image. This proves (9.49).

The action induced by s on M has p'-order, so (9.49) makes it trivial. Since the faithful M-module W is absolutely irreducible, its commutant on $U \otimes W$ is $\mathrm{End}_{\mathbb{F}_q}(U) \otimes I_W$. This gives (9.50); preservation of the product form gives $T \in O(U, q)$. Similarly, centralization of F gives the asserted form of u, with the normalized action on W. □

This argument is specific to an irreducible extraspecial action. Arbitrary homogeneous tensor actions still require the multiplier characters of Proposition 9.13.

For n as above, put

$$f_n(X) = 1 + X + \cdots + X^{n-1}, \quad \delta_d = \chi_q((-1)^{2^{d-1}}), \quad \kappa_n = \chi_q(n). \tag{9.51}$$

For $T \in O(U, q)$, let $a_1(T)$ be the algebraic multiplicity of the eigenvalue one. It is not, in general, the dimension of $C_U(T)$.

Lemma 9.22. *For every eligible $u \in A$ and every $s \in C_A(u)_{p'}$, with T from* (9.50)*,*

$$\det(1 - us) = \det f_n(T), \qquad \omega_V(us) = \delta_d \kappa_n^{a_1(T)}. \tag{9.52}$$

These identities allow a nontrivial source-characteristic part in s.

Proof. The eigenvalues of u_W are exactly $\mu_n \setminus \{1\}$, each once. For each eigenvalue λ of T, multiplication over that set gives $\prod_{1 \neq \zeta \in \mu_n}(1 - \lambda\zeta) = f_n(\lambda)$. This includes $\lambda = 1$, where the value is n. It proves the determinant identity, using algebraic multiplicities even when T is not semisimple.

For $\lambda \neq \pm 1$, set $g_n(\lambda) = \lambda^{-(n-1)/2} f_n(\lambda)$. This is unchanged by inversion, and $f_n(\lambda) f_n(\lambda^{-1}) = g_n(\lambda)^2$. The product of g_n over inverse pairs, with multiplicity, is Frobenius invariant and lies in $\mathbb{F}_q$. Thus these pairs contribute a square to the determinant. The exceptional contributions are $f_n(1) = n$ and $f_n(-1) = 1$. All factors are nonzero, since the eigenvalues of T have p'-order. Formula (9.1) now proves (9.52). □

For later use the consequences are

$$\begin{aligned} &n = 9: && \omega_V(us) = 1, \\ &n = p,\ \kappa_p = 1: && \omega_V(us) = \delta_d, \\ &n = p,\ \kappa_p = -1,\ b \geq 1: && \omega_V(us) = \det T. \end{aligned} \tag{9.53}$$

Indeed, in even dimension $a_1(T)$ and the algebraic multiplicity of -1 have the same parity. Also $\delta_d = 1$ for $d \geq 2$. When $b = 0$ and $n = p$, the two values at $s = 1, z$ are respectively $\delta_d \kappa_p$ and δ_d.

Lemma 9.23. *If $F \cong 2_+^{1+2b}$ acts faithfully and absolutely irreducibly on the orthogonal space U over an odd field, and $b \geq 3$, then*

$$N_{O(U)}(F) \leq SO(U). \tag{9.54}$$

Proof. Use the tensor model for the plus extraspecial group, with basis indexed by $\mathbb{F}_2^b$. The published generator description needed here is Nebe–Rains–Sloane [25, Theorem 2.2 and Corollary 2.3]. Its orthogonal normalizer is generated over the reals by quadratic sign diagonals, affine permutations of the index set, and a normalized Hadamard matrix in one tensor factor. We calculate their determinants.

A constant, linear, or quadratic binary monomial has weight 2^b, 2^{b-1}, or 2^{b-2}. All are even for $b \geq 3$; sums of their support indicators still have even weight. Hence every quadratic sign diagonal has determinant one. A translation has 2^{b-1} transpositions, and a linear transvection has 2^{b-2}. They generate the affine permutations, all of positive sign in this range. Finally, for $H = 2^{-1/2} \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$,

$$\det(H \otimes I_{2^{b-1}}) = (-1)^{2^{b-1}} = 1.$$

To pass to an odd finite characteristic, work first over its algebraic closure, where a square root of two exists. These same matrices are isometries; their conjugations on the extraspecial generators are given by the same binary transformations. They therefore generate the full automorphism image there as well. Two isometric intertwiners inducing the same automorphism differ by $\pm I$, also of determinant one. This covers every normalizer element over the closure, and hence over the original field. No square root of two in that original field was assumed. □

In (9.50), T normalizes F. Thus the last line of (9.53) is constant one whenever $b \geq 3$. The possible nonconstant cases have $b = 0, 1, 2$, with a separate simultaneous treatment when $p = 3, d = 3$.

Proposition 9.24. *Suppose $p = 2^c + 1$ is a Fermat prime, $d = c + b$ with $0 \leq b \leq 2$, and $(p, d) \neq (3, 3)$. If (9.43) has an eligible element, every Sylow p-subgroup P of A has order p. The subgroup $K = N_A(P)$ contains P, and every eligible element of P fixes a unique coset in A/K.*

Proof. By (9.44), the p-part of $|A|$ divides that of

$$|O_{2d}^-(2)| = 2^{d(d-1)+1}(2^d + 1) \prod_{i=1}^{d-1} (2^{2i} - 1). \tag{9.55}$$

Here $\mathrm{ord}_p(2) = 2c$. For $p \geq 5$, we have $c \geq 2$ and $c \leq d \leq c + 2 \leq 2c$. If $d = c$, the only divisible factor is $2^c + 1 = p$. If $d > c$, it is $2^{2c} - 1 = p(p - 2)$, with index $i = c$. In both cases its valuation is one. The order formula also gives valuation one for $p = 3, d = 1, 2$. The excluded case $p = 3, d = 3$ has three-part 3^4. An eligible element supplies the nontrivial p-part, so $|P| = p$.

Every nonidentity element of P generates P. It lies in only one Sylow subgroup, namely P, and its subnormalizer is $N_A(P)$. Lemma 5.2 gives the required unique fixed coset. □

This bound will be used only when the source has not already become constant. For $p \geq 5$, all eligible elements have order p by Theorem 9.20. For $p = 3, d \geq 4$, the elements of order three have $b = d - 1 \geq 3$, while the order-nine elements already have source value one by (9.53). Thus those larger dimensions need no small-factor localizer.

The order formula (9.55) is the full binary orthogonal group order, not its subgroup of index two. Only its odd-prime valuation is used in the argument.

Keep the hypotheses of Proposition 9.24 and suppose $\kappa_p = -1$. Fix $P \in \mathrm{Syl}_p(A)$ and $K = N_A(P)$. All nonidentity elements of P have the same fixed extraspecial factor $F = C_E(P)$.

Lemma 9.25. *There is an ordinary linear character Γ of K, of two-power order, satisfying*

$$\omega_V(us) = \varepsilon \Gamma(us) \quad (1 \neq u \in P,\ s \in C_A(u)_{p'}), \qquad \varepsilon = \begin{cases} -\delta_d, & b = 0, \\ 1, & b = 1, 2. \end{cases} \tag{9.56}$$

Proof. If $b \geq 1$, then F is normal in K. Work over an algebraic closure of the source field. For $g \in K$, choose an isometric intertwiner T_g for its automorphism of the faithful orthogonal F-module U. It is unique up to $\pm I$. Since $\dim U = 2^b$ is even, the value

$$\Gamma(g) = \det T_g \in \{1, -1\} \tag{9.57}$$

is independent of this ambiguity. Products of intertwiners differ from the intertwiner of the product by $\pm I$, so Γ is an ordinary quadratic character of the original group K. No rational choice of square roots for all the intertwiners is required. On the mixed centralizer, (9.50) supplies an actual isometry over $\mathbb{F}_q$, so (9.57) is exactly $\det T$ in (9.53). Also $\Gamma(u) = 1$. This proves (9.56).

If $b = 0$, Lemma 9.21 gives $C_A(P) = Z(E)P$. Moreover $K \cap E = C_E(P) = Z(E)$: an element of E normalizing P centralizes it, since $E \cap P = 1$. On the binary space P is irreducible and generates its scalar field. Its normalizer is therefore semilinear; its quotient by the norm-one torus P embeds in the cyclic Galois group of order $2c$. Using (9.44) yields

$$1 \longrightarrow Z(E) \longrightarrow K/P \longrightarrow D \longrightarrow 1, \qquad D \leq C_{2c}. \tag{9.58}$$

This is a central extension of a cyclic two-group by C_2. Thus K/P is an abelian two-group. Choose a linear character with $\Gamma(z) = -1$ and inflate it to K. It has two-power order. The p'-elements of $C_A(u) = Z(E)P$ are precisely $1, z$. The two values after (9.53) are $-\delta_d, \delta_d$, proving (9.56) for both. □

One choice of Γ works for the entire Sylow family. In the case $b = 0$, it need not be quadratic.

9.8. **The six-dimensional case and completion.** At three, one six-dimensional orthogonal action remains. Its forty anisotropic frames give a subgroup with a unique fixed frame for both orders three and nine, and its frame sign corrects both orders. This completes the extraspecial case and hence the dimension reduction.

The remaining nonconstant case is $p = 3, d = 3, \kappa_3 = -1$. Let Ω be the set of unordered orthogonal decompositions of the binary minus six-space $\bar{E}$ into three anisotropic planes. Call these decompositions *frames*.

Lemma 9.26. *There are* 120 *anisotropic planes and* 40 *frames. Each anisotropic plane lies in exactly one frame.*

Proof. A minus six-space has 36 nonsingular vectors. For one such vector v, its polar hyperplane $v^\perp$ has 32 elements. Translation by v interchanges the two quadratic values there, so 16 of them are nonsingular. Consequently there are 20 nonsingular w with $b_E(v, w) = 1$. Each pair (v, w) spans an anisotropic plane, and each such plane has six ordered bases. There are therefore $36 \cdot 20/6 = 120$ of these planes.

The perpendicular space to one is plus of dimension four. In a hyperbolic coordinate model $Q(x) = x_1x_2 + x_3x_4$, its six nonsingular vectors form two disjoint triangles under the relation $b_E(x, y) = 1$. Explicitly the two triples are

$$\{(1,1,0,0),(1,0,1,1),(0,1,1,1)\}, \qquad \{(1,1,1,0),(1,1,0,1),(0,0,1,1)\}.$$

Each triple is the set of nonzero vectors of an anisotropic plane. These are the only such planes; they are perpendicular and give the unique decomposition of that plus four-space into two anisotropic planes.

Thus each plane of the minus six-space completes in exactly one way to a frame. Each frame has three planes, proving $|\Omega| = 120/3 = 40$. □

All counting here is in the binary quotient. An anisotropic plane lifts to a quaternion subgroup of E; a frame lifts to a central product of three such subgroups. The action of A on frames factors through (9.44). It is not necessary that its image be the full orthogonal group.

Proposition 9.27. *In* (9.43) *with* $p = 3, d = 3$*, every eligible element of order three or nine fixes exactly one member of* Ω*.*

Proof. An eligible element of order three has binary moving space an anisotropic plane and fixed space a plus four-space, by Theorem 9.20. It cannot cycle the three planes of a fixed frame: such a cycle has fixed dimension two, since its third power is the identity. Thus it preserves each plane, and acts nontrivially on exactly one. That plane is its moving space; its perpendicular completes the frame uniquely by Lemma 9.26.

Now let $|u| = 9$. Its binary action is irreducible of dimension six. Its cube j is fixed point free on that binary space and defines its structure as a three-dimensional $\mathbb{F}_4$-space. Choose $\mathbb{F}_4^\times = \langle \zeta \rangle$ with j acting as ζ. The formula

$$H(x, y) = b_E(x, jy) + \zeta b_E(x, y) \tag{9.59}$$

is nondegenerate hermitian, linear in the first variable, and $H(x, x) = Q_E(x)$. These facts follow from $j^2 + j + 1 = 0$, invariance of b_E, Q_E, and $b_E(x, jx) = Q_E(x)$. The element u is unitary.

A j-fixed frame cannot have its planes cycled by j, since that would give fixed dimension two. It therefore consists of three perpendicular $\mathbb{F}_4$-lines of norm one. In an orthonormal basis the norm-one lines are the three coordinate axes and the nine lines represented by $(1, \zeta^r, \zeta^s)$, $r, s \in \mathbb{F}_3$. An axis is perpendicular only to the other axes among these lines. Two of the other nine are perpendicular exactly when their exponent pairs differ by $(1, 2)$ or $(2, 1)$. Their three frames are labeled by $r + s$. Thus j fixes four frames in all.

The group $\langle u \rangle$ fixes at least one of these four. Relative to such a frame, its monomial permutation is a three-cycle; a diagonal matrix here would have order at most three. A diagonal change of basis puts it in the form

$$(x_1, x_2, x_3) \longmapsto (\zeta^a x_3, x_1, x_2), \qquad a \in \{1, 2\}.$$

It fixes the coordinate frame and sends the labels of the other three to $r + s + a$. Hence exactly one of the four is fixed by u. Every u-fixed frame is j-fixed, proving uniqueness in Ω. □

The cube is used only to count binary frames. In the source module V, that cube is not eligible.

Fix $P \in \mathrm{Syl}_3(A)$ in the exceptional case. Since $3 \nmid 40$, its action on Ω fixes a frame $\mathcal{F}$. Put

$$K = \mathrm{Stab}_A(\mathcal{F}), \qquad \Gamma(g) = \mathrm{sgn}(\text{its permutation of } \mathcal{F}) \quad (g \in K). \tag{9.60}$$

Proposition 9.28. *These data satisfy $\mathcal{L}_3(A, V)$, with $\varepsilon = 1$, for both eligible orders simultaneously.*

Proof. If there are eligible elements in P, each has the unique fixed frame of Proposition 9.27. Since P fixes $\mathcal{F}$, it must be that frame for every one of them. The orbit map $A/K \to \Omega$ is injective and equivariant, so each eligible $u \in P$ has just one fixed coset. In particular $C_A(u) \le K$.

Suppose first $|u| = 3$, and let $s \in C_A(u)_{3'}$. The element u fixes each of the three planes. The two fixed binary planes give $F = Q_8 * Q_8$, with faithful orthogonal module $U = W_1 \otimes W_2$ of dimension four; each W_i is symplectic of dimension two. The element s preserves the moving plane and may interchange the two fixed ones. If it preserves them, its operator T in (9.50) factors as $T_1 \otimes T_2$. The factors are similitudes of inverse multipliers μ_1, μ_2, so

$$\det T = (\det T_1)^2 (\det T_2)^2 = (\mu_1 \mu_2)^2 = 1.$$

If it interchanges them, an additional tensor flip has determinant -1 on the four-space. Thus $\det T = \Gamma(s)$ in either case. Formula (9.53) gives $\omega_V(us) = \Gamma(s) = \Gamma(us)$. No normalization of both small factors to isometries over the original field was used.

If $|u| = 9$, its permutation of the planes is a three-cycle. The permutation of s commutes with it and has $3'$-order, so it is trivial. Consequently $\Gamma(us) = 1$, which is also the source value in (9.53). This proves the mixed identity for both orders with the same sign and character. If the eligible set is empty, only the coset condition and identities are required on an empty family. □

The group K contains a full Sylow of the original A. It is not assumed to be the full monomial normalizer of $Q_8 * Q_8 * Q_8$.

Theorem 9.29. *Under* (9.43)*, for every $P \in \mathrm{Syl}_p(A)$ there exist $P \le K \le A$, an ordinary linear character Γ of K of two-power order, and one sign ε, such that every eligible $u \in P$ has a unique fixed coset in A/K, and*

$$\omega_V(us) = \varepsilon \Gamma(us) \qquad (s \in C_A(u)_{p'}). \tag{9.61}$$

The same data work for the whole eligible family. Solvability of A is not required.

Proof. If the eligible set is empty, take $K = A$, $\Gamma = 1$, $\varepsilon = 1$. Otherwise Theorem 9.20 makes p a Fermat prime and allows only orders p, and possibly nine when $p = 3$.

Constant source values. If $\kappa_p = 1$, formula (9.53) is constant δ_d on the order-p sections. The order-nine sections, if present, have $d \geq 3$, so their value one is also δ_d. Thus $K = A$, $\Gamma = 1$, $\varepsilon = \delta_d$ works for the entire family.

Small fixed factors. Suppose $\kappa_p = -1$. For $p \geq 5$, put $c = \log_2(p-1)$ and $b = d - c$. If $b \geq 3$, Lemma 9.23 makes every mixed value one, and we again take $K = A, \Gamma = 1$. If $b \leq 2$, Propositions 9.24 and Lemma 9.25 give a single $K = N_A(P)$ and Γ, with the indicated common sign. The unique coset condition is part of Proposition 9.24.

The prime three. For $p = 3, d = 1, 2$, the same two small-factor results apply. For $p = 3, d = 3$, use Proposition 9.28, which includes both eligible orders. Finally, if $p = 3, d \geq 4$, an eligible element of order three has fixed width $b = d - 1 \geq 3$, so Lemma 9.23 gives value one; an eligible element of order nine has value one directly by (9.53). Take $K = A, \Gamma = 1, \varepsilon = 1$. These cases exhaust the possibilities. □

Every chosen K contains the full Sylow P. The fixed coset assertion yields both $C_A(u) \leq K$ and $\mathrm{Sub}_A(u) \leq K$. No induction on irreducible characters or change of a lower character occurs in this theorem.

Corollary 9.30. *Let q be odd, let p be odd and distinct from its characteristic, and let $A \leq \mathrm{Sp}(V, q)$ be solvable, faithful and irreducible. Then $\mathcal{L}_p(A, V)$ holds for the canonical ordinary Weil source.*

Proof. Proposition 9.16 reduces the assertion, by dimension, to (9.43). Theorem 9.29 proves that case, with precisely the unique fixed coset condition and two-power character required by (9.4)–(9.5). This discharges the assumption in that reduction and proves the assertion. □

For an odd source characteristic in Proposition 8.11, identify the actual source action with the canonical one by Lemma 8.8, using the parameter normalization of Corollary 8.7. Corollary 9.30 supplies (8.3). The localizer and its character are then used in the original smaller group, over the original invariant lower character. Lemma 8.9 performs the compensation if the explicitly smaller relative instance is available. Its row signs remain common on the whole specified family; its preservation claim is for the p-parts of degrees.

10. Sources in characteristic two

We now prove the characteristic-two source part of Theorem 8.1; the prime p remains odd. The source representation is still over $\mathbb{C}$. Characteristic two describes the finite quotient of the extraspecial group and its square form.

Here we construct an ordinary unitary source first and extend it to the semilinear group. A norm power reduces an eligible semilinear operator to a linear one. Self-duality, where available, makes the local ratios quadratic; a common field phase identifies those ratios simultaneously. Hyperbolic sources and tensor permutations then give the dimension reduction. Finally, the elementary cocycle calculation of Lemma 7.5 identifies the constructed action with the given action on the source.

10.1. **An ordinary unitary source.** We now prove the binary source assertion 10.1. Here p is odd. Let E be extraspecial of order 2^{1+2M}, write $Z(E) = \langle z \rangle$, and put $V = E/Z(E)$. Its quadratic and polar forms are determined by

$$e^2 = z^{Q_E(\bar{e})}, \qquad [e, f] = z^{B_E(\bar{e}, \bar{f})}. \tag{10.1}$$

The faithful irreducible complex representation ρ_E has degree 2^M. Its invariant bilinear form is symmetric for plus type and alternating for minus type.

Our goal is the following statement for every solvable faithful irreducible $A \leq O(V, Q_E)$. There is a genuine lift of this action to E and an ordinary implementer σ, together with $\epsilon \in \{1, -1\}$ and $\Gamma_p \in \mathrm{Lin}(A)$ of two-power order, such that

$$\psi(us) = \epsilon \Gamma_p(us), \qquad \psi = \mathrm{Tr}\,\sigma, \quad C_V(u) = 0, \quad |u| = p^a > 1, \quad s \in C_A(u)_{p'}. \tag{10.2}$$

The same choices work for the entire mixed family. We will also identify the constructed action with any given genuine lift. The localizer in (8.3) is $K_A = A$.

For an action $\alpha : A \to \mathrm{Aut}(E)$, the operators satisfy $\sigma(ab) = \sigma(a)\sigma(b)$ and $\sigma(a)\rho_E(e)\sigma(a)^{-1} = \rho_E(\alpha(a)e)$. Thus ρ_E extends to $E \rtimes A$. A projective implementer does not suffice. Neither a splitting of the full orthogonal automorphism sequence nor a characteristic two analogue of the odd symplectic Weil representation is assumed.

The proof first constructs ordinary unitary semilinear and hyperbolic sources, including their phases. It then assembles orthogonal components by tensor permutation and uses the center of the Fitting subgroup in the quasi-primitive case. The final identification of lifts is a separate step.

Theorem 10.1. *Let $A \le O(V, Q_E)$ be solvable, faithful and irreducible. For odd p, assertion* (10.2) *holds. It holds for any given genuine lift of the A-action to E, after choosing its ordinary implementer. The localization data can be chosen with $K_A = A$, and the binary tensor construction preserves this assertion.*

Let $q = 2^f$, $K = \mathbb{F}_{q^{2c}}$, $Q = q^c$, and let h be a nondegenerate hermitian form on K^m, for $\bar{a} = a^Q$. Put $U = \mathrm{GU}_m(Q)$; this notation denotes the isometry group. Set

$$\mathcal{H} = \{(v, a) : a + \bar{a} = h(v, v)\}, \qquad (v, a)(w, b) = (v + w, a + b + h(v, w)). \tag{10.3}$$

Its center is $\{(0, a) : a \in \mathbb{F}_Q\}$. Let $\lambda(a) = (-1)^{\mathrm{Tr}_{\mathbb{F}_Q/\mathbb{F}_2}\, a}$.

Lemma 10.2. *The quotient $E_h = \mathcal{H}/\ker\lambda$ is extraspecial of order 2^{1+2M}, where $M = m \log_2 Q$, with forms*

$$B(v, w) = \mathrm{Tr}_{K/\mathbb{F}_2}\, h(v, w), \qquad Q_0(v) = \mathrm{Tr}_{\mathbb{F}_Q/\mathbb{F}_2}\, h(v, v). \tag{10.4}$$

It has type $(-1)^m$. Its faithful irreducible source extends ordinarily to $E_h \rtimes U$, with unitary trace

$$\tau(g) = (-1)^m (-Q)^{\dim_K \ker(g - I)}. \tag{10.5}$$

Proof. Commutators in (10.3) are $h(v, w) + h(w, v)$, and squares are $h(v, v)$. Applying λ gives (10.4). The polar form is nondegenerate by the trace pairing. Thus the center and derived subgroup of the quotient both have order two. One hermitian line has quadratic sum

$$1 + (Q + 1) \sum_{a \in \mathbb{F}_Q^\times} (-1)^{\mathrm{Tr}_{\mathbb{F}_Q/\mathbb{F}_2}\, a} = -Q.$$

Orthogonal multiplication of these sums gives the stated type. The source degree is Q^m.

The ordinary extension and formula (10.5) are the published input of Imai–Tsushima [15, Lemma 2.2], for the group defined in their Section 2.1. Its prescribed central character is λ, so the extension factors through $\ker\lambda$. The statement includes characteristic two and $(m, Q) = (2, 2)$. □

In (10.4) the two trace fields are different: $\mathrm{Tr}_{K/\mathbb{F}_2}\, h(v, v) = 0$ would give the wrong quadratic form. The quotient, its type, and subsequent normalizations are deductions made here, rather than further assertions attributed to the published trace formula.

Lemma 10.3. *The map $\pi : \mathrm{Aut}(E) \to O(V, Q_E)$ is onto with kernel $\mathrm{Inn}(E) = V$. If T is an odd cyclic subgroup of $O(V, Q_E)$ with $C_V(T) = 0$, it has a lift $\widehat{T}$, and*

$$N_{\mathrm{Aut}(E)}(\widehat{T}) \xrightarrow{\ \cong\ } N_O(T). \tag{10.6}$$

The corresponding centralizers also map isomorphically.

Proof. The kernel consists of linear central corrections on V, which are inner by nondegeneracy of B_E. Quadratic isometries lift by sending an extraspecial generating set to elements with the same squares and commutators. Schur–Zassenhaus gives complements to V in $\pi^{-1}(T)$, all conjugate by V. A lift of an element of $N_O(T)$ therefore becomes a normalizer of $\widehat{T}$ after an inner correction. The kernel of the restricted map is $C_V(T) = 0$. An element centralizing T centralizes its lift because the projection on that lift is injective. This also proves the last assertion. □

Apply this to the scalar torus $T = \mu_{Q+1}$ in (10.4). After fixing its lift, (10.6) supplies a genuine action of

$$N = U \rtimes \langle r \rangle, \qquad r = \mathrm{Frob}_q, \qquad r^{2c} = 1. \tag{10.7}$$

The unitary action from Lemma 10.2 agrees with this action: its scalar torus can be aligned first, and it then lies in the same normalizer in (10.6).

There is also a useful consequence of the nonzero real values in (10.5). The unitary extension is self-dual. Indeed, its dual extends the same source and differs by a linear character of U; comparison on U forces that character to be trivial. Its invariant form has the type of E_h.

If R implements the field automorphism on E_h, then $R\sigma(g)R^{-1}$ and $\sigma(g^{(q)})$ differ by a scalar. Formula (10.5) gives equal, nonzero traces for the two operators, so that scalar is one. Hence field covariance is exact. This argument does not require a description of the abelianization of U, including in the small unitary case.

Proposition 10.4. *The unitary extension in Lemma 10.2 extends ordinarily to $E_h \rtimes N$. If m is even, it can be chosen self-dual, with*

$$\psi(r^j) = q^{jm/2} \qquad (j \mid 2c). \tag{10.8}$$

If m is odd, it can be chosen with real trace on every element whose field image has odd order.

Proof. Write $m = 2h_0$ first, and use the hermitian matrix $J = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. A model for the extraspecial cocycle is

$$\beta((x,y),(x',y')) = \mathrm{Tr}_{K/\mathbb{F}_2}(\bar{x}^T y'). \tag{10.9}$$

On the basis e_t, $t \in K^{h_0}$, one may take $\rho(x,y)e_t = (-1)^{\mathrm{Tr}_{K/\mathbb{F}_2}(\overline{t+x}^T y)} e_{t+x}$. The scalar $a \in \mu_{Q+1}$ and Frobenius are implemented by $P_a e_t = e_{at}$ and $Re_t = e_{t^q}$. These are exact automorphisms of the cocycle model. For $a \neq 1$, both P_a and the aligned unitary operator have trace one, so their scalar ratio is one. Lemma 10.3 now aligns R with the field action in (10.7).

Exact field covariance was proved in the preceding argument. Together with $R^{2c} = I$, it gives an ordinary extension. The source has its standard symmetric form, preserved by the self-dual unitary extension and the permutation R. Counting the fixed basis elements of R^j gives (10.8).

For odd m, the unitary extension preserves an alternating form. Choose an isometric intertwiner R_0 for r. Schur's lemma gives $R_0^{2c} = \pm I$. Write $2c = 2^a c_0$, with c_0 odd. If the sign is negative, multiply R_0 by a primitive 2^{a+1}-st root; otherwise leave it unchanged. The resulting R has $R^{2c} = I$ and retains exact covariance, so again gives an ordinary extension. Its dual differs by a two-power linear character inflated from the field group. This character is trivial on elements with odd-order field image, where the trace is therefore real. □

For odd m, no self-duality on the whole semilinear group is asserted. That stronger statement is unnecessary. The normalization is fixed before any eligible element is chosen.

10.2. **Norms and centralizer characters.** The semilinear norm must preserve eligibility, not merely preserve a determinant. Once this is known, odd powers and the self-duality of the source show that the ratio on each linear centralizer is a quadratic character. A further compatibility argument is needed to obtain one character on all those centralizers.

Lemma 10.5. *Let x be a θ-semilinear invertible map on a K-space, with $\theta \in \mathrm{Gal}(K/\mathbb{F}_q)$ of order ℓ. Put $K_0 = K^\theta$, $B = x^\ell$, and $D = \det_K(1-B)$. Then $D \in K_0$, and*

$$\begin{aligned} \det_{\mathbb{F}_q}(1-x) &= N_{K_0/\mathbb{F}_q}(D), \\ \det_{\mathbb{F}_q}(1-x^\ell) &= N_{K_0/\mathbb{F}_q}(D)^\ell. \end{aligned} \tag{10.10}$$

In particular x is fixed point free if and only if x^ℓ is fixed point free.

Proof. This is Lemma 9.7, with field automorphism θ and order ℓ. That determinant lemma and its proof are valid in characteristic two: they use cyclic permutation of the field embeddings and block elimination, with no division by the field-automorphism order. The two resulting identities are (10.10). Their determinants vanish simultaneously, which gives the fixed-point-free assertion. □

We will apply the last assertion also on the absolute binary space; extending or restricting the ground field does not change whether the kernel of $1-x$ is zero.

Two elementary observations will be used repeatedly. If u is a fixed point free p-element and $s \in C(u)_{p'}$, then us is fixed point free: a vector fixed by us is fixed by $(us)^{|s|} = u^{|s|}$, which generates $\langle u \rangle$. If the field image of u has order ℓ_u, then u^{ℓ_u} is linear and remains eligible by (10.10).

This is a specific norm power. A general passage from u to an element of order p need not preserve eligibility, and no such passage occurs in the proof.

Lemma 10.6. *Let ψ be the trace of an ordinary source implementer. If x is fixed point free on $E/Z(E)$, then $|\psi(x)| = 1$. Suppose that $k > 0$ is odd, x^k is also fixed point free, and $\psi(x^d)$ is real for every $d \mid k$. Then*

$$\psi(x^k) = \psi(x). \tag{10.11}$$

Proof. The source endomorphism algebra has a basis indexed by $E/Z(E)$, from homogeneous extraspecial units. Conjugation by x permutes these basis lines, possibly with signs. For a fixed point free action only the identity line contributes, with eigenvalue one. Therefore $\psi(x)\overline{\psi(x)} = 1$. A real such trace is a sign.

Every power x^d with $d \mid k$ is fixed point free, since a vector fixed by it would be fixed by x^k. For an odd prime $a \mid k$, reduction of eigenvalue sums modulo a prime ideal above a gives $\psi(y^a) \equiv \psi(y)^a$. If the two traces are signs, their difference is zero or ± 2; the latter cannot belong to that ideal, whose intersection with $\mathbb{Z}$ is $a\mathbb{Z}$. The traces are equal. Apply this successively along a factorization of k. □

The reality hypothesis is essential: a value of absolute value one need not be a sign. Proposition 10.4 supplies reality everywhere in even unitary multiplicity, and on the required odd field images in odd multiplicity.

A second useful congruence involves commuting u, s, with u a p-element. After simultaneous diagonalization over $\mathbb{C}$, every eigenvalue of u reduces to one modulo a prime above p. Hence

$$\psi(us) \equiv \psi(s) \pmod{\mathfrak{p}}. \tag{10.12}$$

When both sides under consideration are rational integers, this is a congruence modulo p. We will use it to determine a sign from an explicitly known source trace. It is not a statement about commuting homology with a Brauer quotient.

Lemma 10.7. *Suppose that a source extension to $E \rtimes A$ is self-dual. For a fixed point free p-element $u \in A$, there is a quadratic linear character η_u of $C_A(u)$ such that*

$$\psi(us) = \psi(u)\eta_u(s) \qquad (s \in C_A(u)_{p'}). \tag{10.13}$$

Proof. Decompose the absolute binary space into primary summands for the odd-order semisimple map u. A pair of distinct reciprocal polynomials gives a hyperbolic pair and a general linear centralizer. A self-reciprocal polynomial gives a unitary centralizer. There is no $X - 1$ factor. On a hyperbolic half the quadratic form is linear and u-invariant, and is therefore zero. On the self-dual summands the trace-hermitian description gives the original quadratic form: an alternative refinement would differ by an invariant linear form, also zero.

For a general linear factor use the permutation source on a lagrangian. Its mixed trace at us is one, because the only fixed basis element is zero. For each unitary factor use Lemma 10.2. Its mixed trace is $(-1)^b$, where b is the hermitian multiplicity in that factor. Here us is fixed point free by the observation after Lemma 10.5. Tensoring gives an ordinary self-dual source for the full orthogonal centralizer, with a constant mixed value ϵ_u.

These are actual extraspecial actions. Align their lift of $\langle u \rangle$ with the given lift. Lemma 10.3, applied to this fixed point free odd cyclic group, identifies the centralizer actions exactly. The ratio of the two ordinary implementers is a linear character of $C_A(u)$. Both are self-dual, so its square is one. It is trivial on u, of odd order. Comparison at u gives $\epsilon_u = \psi(u)$, and comparison at us gives (10.13). □

This proves local linearity on the full centralizer, but does not by itself give one character for all eligible u. The common field phase in the following arguments supplies that additional compatibility. Parity of a homological degree would only control a sign and would not prove this lemma.

10.3. **One phase on the semilinear group.** The phase calculation has a useful way of being read. The ordinary unitary trace determines the values on linear centralizers. The quotient of a semilinear centralizer by its linear part is cyclic, so a two-power character on that quotient is determined on its Sylow two-subgroup. The element obtained by descent gives one generator there. Its known source trace determines the required sign modulo the odd prime p. The following calculations show that one phase, chosen before u, has that value on every centralizer that occurs.

Lemma 10.8. *Let $u \in U$ be an eligible p-element. Suppose the field image of $C_N(u)$ has Sylow two-subgroup of order $\ell > 1$. Set $j = 2c/\ell$ and $q_0 = q^j$. Then m is even, and for the hermitian matrix J used in Section 10.1 there are*

$$u_0 \in \mathrm{Sp}_m(q_0), \quad k \in U, \quad u = ku_0k^{-1}, \quad s_0 = kr^jk^{-1} \in C_N(u). \tag{10.14}$$

In particular $|s_0| = \ell$ and $\psi(s_0) = q_0^{m/2}$ for the extension of Proposition 10.4.

Proof. The characteristic polynomial of u is invariant under the indicated field subgroup, hence has coefficients in $\mathbb{F}_{q_0}$. Since $j \mid c$, the hermitian involution fixes that field. Unitarity now makes the polynomial reciprocal, with no root one.

Realize its distinct reciprocal pairs on hyperbolic summands over $\mathbb{F}_{q_0}$. Realize a self-reciprocal factor by a trace-hermitian pairing on its extension field, with its prescribed multiplicity. This constructs a semisimple isometry with that polynomial and a nondegenerate symmetric bilinear form. In characteristic two the square root of the diagonal of this form is a linear invariant functional. It is zero because $u_0 - I$ is invertible. The form is alternating, so m is even and it can be identified with J.

The resulting u_0, viewed over K, is unitary. Semisimple unitary elements with the same characteristic polynomial are conjugate in U: on each self-dual primary factor this follows from uniqueness of a finite hermitian form in a fixed dimension, and on a reciprocal pair from the perfect dual pairing. Choose such a k. The entries of u_0 are fixed by r^j, proving (10.14). Conjugation preserves the order and trace of r^j, and (10.8) gives the last assertion. □

For odd m, this proves that $C_N(u)$ has odd-order field image for every eligible linear u. The conclusion is about the entire image, not merely the cyclic subgroup generated by a chosen commuting element.

Assume m is even and that eligible elements exist. Write

$$d = \mathrm{ord}_p(q) = 2^s b, \quad b \text{ odd}, \qquad \lambda = \begin{cases} 1, & s = 0, \\ \zeta_{2^{s+1}}^{m\nu}, & s > 0, \end{cases} \quad \nu b \equiv 1 \pmod{2^{s+1}}. \tag{10.15}$$

The convention for $s = 0$ needs no ν.

Lemma 10.9. *There is a two-power ordinary linear character of N given by*

$$\Gamma_p|_U = 1, \qquad \Gamma_p(r) = \lambda. \tag{10.16}$$

For every $j \mid c$ occurring in Lemma 10.8, put $d_j = \mathrm{ord}_p(q^j)$. Then

$$\lambda^j = \begin{cases} 1, & d_j \text{ odd}, \\ (-1)^{m/d_j}, & d_j \text{ even}. \end{cases} \tag{10.17}$$

Proof. First an eligible semilinear element has an eligible linear norm power by Lemma 10.5. Thus the linear dimension divisibilities apply. If $h = \mathrm{ord}_p(Q)$, the minimal unitary primary dimension is $2h, h/2, h$, according as h is odd, is two modulo four, or is zero modulo four. This follows from orbits of roots under $a \mapsto a^{-Q}$. For roots of order p^a only an odd p-power factor is added to this dimension. Consequently, with $a_0 = v_2(2c)$, $s > a_0$ implies $v_2(m) \geq s - a_0 + 1$. If $s \leq a_0$, the assumed evenness of m suffices. In either case $v_2(2cm) \geq s + 1$ for $s > 0$. Hence $\lambda^{2c} = 1$, proving that (10.16) is ordinary.

For an active j, Lemma 10.8 gives a fixed point free symplectic map in dimension m over $\mathbb{F}_{q^j}$. Its primary dimensions show $2d_j \mid m$ if d_j is odd and $d_j \mid m$ otherwise. Now $d_j = d/\gcd(d, j)$. If $v_2(j) < s$, these divisibilities give $2^s \mid mj$, and $\lambda^j = (-1)^{mj/2^s} = (-1)^{m/d_j}$. The odd factors, including ν, do not change parity. If $v_2(j) \geq s$, then d_j is odd and the evenness of m gives $\lambda^j = 1$. The case $s = 0$ is immediate. □

The formula contains no odd-characteristic Gaussian factor. The order of Γ_p need not be two, even though its values on the mixed sections below are signs.

Proposition 10.10. *For the even-multiplicity extension of Proposition 10.4 and the character in* (10.16)*,*

$$\psi(us) = \Gamma_p(us) \tag{10.18}$$

for every eligible linear $u \in U$ and $s \in C_N(u)_{p'}$.

Proof. Formula (10.5) gives $\psi(u) = 1$ and $\psi(us) = 1$ whenever $s \in C_U(u)_{p'}$. Lemma 10.7 therefore gives a quadratic character η_u of $C_N(u)$, trivial on $C_U(u)$. To justify this last statement on the whole linear centralizer, its odd-order elements are killed by quadraticity, and its even-order elements have the same character value as their two-parts, which are p'.

If the field image has a nontrivial Sylow two-subgroup, use s_0 from (10.14). Lemmas 10.6–10.8 give a sign $\psi(us_0)$, with the integer congruence

$$\psi(us_0) \equiv q_0^{m/2} \pmod{p}. \tag{10.19}$$

Let $d_j = \mathrm{ord}_p(q_0)$. If d_j is odd, its inverse-paired root orbits imply $2d_j \mid m$, so the right side is one modulo p. If d_j is even, $d_j \mid m$ and $q_0^{m/2} \equiv (-1)^{m/d_j}$. These assertions allow root orders p^a: their orbit lengths differ from d_j by odd factors. Since p is odd, a sign is determined by its residue modulo p.

By (10.17), the result is precisely λ^j. Thus $\eta_u(s_0) = \Gamma_p(s_0)$. The two-power character $\eta_u\Gamma_p^{-1}$ is trivial on $C_U(u)$, and so factors through the cyclic field image. It is trivial on its Sylow two-subgroup by this calculation, and on its odd part by its order. If that Sylow subgroup was trivial, the latter argument applies at once. Hence $\eta_u = \Gamma_p$ on the centralizer. Finally $\Gamma_p(u) = 1$, proving (10.18). □

The field phase was defined by p, q, m before choosing u. The argument verifies its restriction on every active field component; it does not choose separate phases and then assume that they glue.

Theorem 10.11. *For N in* (10.7)*, an ordinary source extension and a two-power linear character can be chosen so that*

$$\psi(us) = (-1)^m \Gamma_p(us) \tag{10.20}$$

for every eligible p-element $u \in N$ and $s \in C_N(u)_{p'}$. For even m, use (10.16)*; for odd m, use $\Gamma_p = 1$. If there are no eligible elements, take $\Gamma_p = 1$.*

Proof. For an eligible u, let ℓ_u be the order of its field image. It is odd, and $v = u^{\ell_u} \in U$ remains eligible by Lemma 10.5. Also $C_N(u) \le C_N(v)$.

Suppose first that m is odd and use Proposition 10.4. By Lemma 10.8 the field image of $C_N(v)$ has odd order. Thus for $x = us$ the field order ℓ_x is odd. The norm power x^{ℓ_x} is linear and fixed point free. Every intermediate divisor power is fixed point free and has odd-order field image. Its trace is real by Proposition 10.4 and is a sign by Lemma 10.6. That lemma therefore gives $\psi(x) = \psi(x^{\ell_x}) = -1$, using (10.5) at the last step. This proves (10.20) for odd m, without requiring global self-duality.

For even m the extension is self-dual. Choose a positive odd integer k such that $k \equiv \ell_u \pmod{|u|}$ and $k \equiv 1 \pmod{|s|}$. The Chinese remainder theorem gives one, imposing oddness separately if needed. Then $(us)^k = vs$, which is fixed point free because v is eligible and $s \in C_N(v)_{p'}$. Lemma 10.6 and Proposition 10.10 give $\psi(us) = \psi(vs) = \Gamma_p(vs)$. The two-power character kills the odd-order elements u, v, so $\Gamma_p(vs) = \Gamma_p(us)$. □

The norm power can lower the order of u; it never silently discards the original semilinear element from the conclusion. Formula (10.20) treats all its commuting p'-parts. The sign and character are common to all eligible sections in this model.

10.4. **Decomposition and tensor actions.** We next assemble the unitary construction. Hyperbolic pairs use Fourier operators, permuted tensor factors use cycle traces, and Clifford decomposition respects the square form. Each operation retains an ordinary action and one character on the whole family.

Let $Y = \mathbb{F}_q^n$, and put $Q_H(y,\phi) = \mathrm{Tr}_{q/2}\,\phi(y)$ on the absolute binary space of $Y \oplus Y^*$. Write $h_B = \mathrm{diag}(B, B^{-T})$ and let $N_h = \mathrm{GL}_n(q) \rtimes \langle J \rangle$, with $J^2 = 1$ interchanging the two halves.

Proposition 10.12. *The hyperbolic source has an ordinary self-dual extension to N_h and a quadratic character Γ_p such that $\psi(us) = \Gamma_p(us)$ on every eligible mixed section. One can take $\Gamma_p|_{\mathrm{GL}} = 1$, and, for even n with a nonempty eligible family,*

$$\Gamma_p(J) = \begin{cases} 1, & \mathrm{ord}_p(q) \text{ odd}, \\ (-1)^{n/\,\mathrm{ord}_p(q)}, & \mathrm{ord}_p(q) \text{ even}. \end{cases} \tag{10.21}$$

For odd n, or an empty eligible family, take $\Gamma_p(J) = 1$.

Proof. On the basis e_y, use permutation operators for $\mathrm{GL}_n(q)$ and the real Fourier matrix

$$\mathcal{F}_{y,t} = q^{-n/2}(-1)^{\mathrm{Tr}_{q/2}(y^T t)} \tag{10.22}$$

for J. Then $\mathcal{F}^2 = I$ and the contragredient covariance is exact. These operators normalize the extraspecial source model and preserve its symmetric form; thus the extension is ordinary self-dual.

An odd-order u lies in $\mathrm{GL}_n(q)$, say $u = h_T$. Every linear mixed trace is one. Lemma 10.7 shows that the remaining ratio is quadratic and trivial on the linear centralizer. An outer centralizer exists only if T is similar to T^{-T}. Since T is semisimple and has no root one, its primary pairing then provides an invertible alternating matrix B with $TBT^T = B$. In particular n is even. The element $s_0 = h_B J$ is an involution centralizing u, and (10.22) gives

$$\psi(s_0) = q^{-n/2} \sum_{y \in Y} (-1)^{\mathrm{Tr}_{q/2}(y^T B y)} = q^{n/2}. \tag{10.23}$$

The congruence (10.12), followed by the root-orbit argument of Section 10.3, determines $\psi(us_0)$ as the sign in (10.21). It determines the entire outer coset of the local quadratic character. If there is no outer centralizer, the formula is already true on the linear one. □

Proposition 10.13. *Suppose an ordinary source for a binary quadratic space W, with operator group A_0, satisfies $\psi_0(ys) = \epsilon\Gamma_0(ys)$ on all eligible mixed sections. Its tensor permutation extension to $A_0 \wr D$, on t orthogonal copies, has data*

$$\epsilon_t = \epsilon^t, \qquad \Gamma_t((g_i), \pi) = \prod_i \Gamma_0(g_i)\ \mathrm{sgn}(\pi)^{(1-\epsilon)/2}. \tag{10.24}$$

These restrict to every subgroup. If Γ_0 has two-power order, so does Γ_t.

Proof. Take the central product of the extraspecial groups on the orthogonal summands. The tensor source is faithful irreducible, and tensor permutation gives an ordinary action normalizing it. The displayed product and sign are ordinary linear characters of the wreath product.

Let $x = us$, where u is eligible of p-power order and $s \in C(u)_{p'}$. Consider a permutation cycle of x of length $p^a e$, $p \nmid e$, with chosen summand W_i. The p-part of the permutation is that of u, with cycles of length p^a inside this cycle. Its cycle product $u^{p^a}|_{W_i}$ is fixed point free; a fixed vector would extend around the u-cycle to a fixed vector for u. The p-part of the full x-cycle product is

$$\left(x^{p^a e}|_{W_i}\right)_p = \left(u^{p^a}|_{W_i}\right)^e. \tag{10.25}$$

It is still eligible because e is prime to p. This justifies applying the base mixed formula to the whole cycle product, including its commuting p'-part.

The trace of a tensor permutation operator is the product of the cycle traces. There is one ϵ for each cycle, while the product of all Γ_0-values is $\prod_i \Gamma_0(g_i)$. If there are b cycles, $\epsilon^b = \epsilon^t\,\mathrm{sgn}(\pi)^{(1-\epsilon)/2}$. This proves (10.24). Restriction preserves both the ordinary action and the whole mixed identity. □

There is no odd Weil determinant correction in (10.24). The permutation operators here act directly on the tensor source. No correspondence on irreducible characters of an overgroup is being restricted.

Lemma 10.14. *Let $A \leq O(V, Q_E)$ be faithful irreducible and solvable. If its restriction to a normal subgroup is not homogeneous, then V is an orthogonal sum, transitively permuted by A, of either nondegenerate smaller components or hyperbolic pairs of dual components. The resulting action embeds in an isometry wreath product of the appropriate component stabilizer, or of a hyperbolic normalizer.*

Proof. Restriction of an irreducible module to a normal subgroup D is semisimple here even if $|D|$ is even: its nonzero D-socle is A-invariant, hence is the whole module. Clifford theory permutes its isotypic components transitively. The polar form pairs only dual types. Thus a self-dual component is nondegenerate, while a non-self-dual component pairs nondegenerately with its dual and is orthogonal to all other components.

In the latter case each half is totally isotropic for the polar form. Its quadratic restriction is therefore a binary linear functional, invariant under D. The component has nontrivial simple type: a trivial type anywhere, by transitivity, would make D act trivially everywhere, contradicting nonhomogeneity. It has no trivial quotient, so the functional is zero. The pair is quadratically hyperbolic.

The component stabilizer acts irreducibly on a nondegenerate component. Otherwise the sum of translates of a proper invariant subspace would be a proper A-submodule of V. Pass to its faithful image. In the paired case the pair stabilizer acts in the full hyperbolic normalizer. Choose isometries from one block to each other block using elements of A. Relative to these identifications, the action is an injective subgroup of the stated wreath product. The embeddings retain the quadratic forms. □

The induction is now transparent in this case. On a nondegenerate component apply the binary assertion in smaller dimension. On a hyperbolic pair apply Proposition 10.12, with $q = 2$. Proposition 10.13 supplies an ordinary source with the common data (10.2) on the entire wreath product and hence on A.

This constructs a genuine extraspecial action for the given quadratic quotient. Identifying it with a previously specified extraspecial action is still required; that is done in Section 10.5, after the quasi-primitive case.

10.5. **Localization for the original action.** For a quasi-primitive action, the center of the Fitting subgroup generates the required unitary field. The dimension induction then constructs an ordinary source. The final step identifies its lift with the prescribed lift, so that all the preceding trace formulas apply to the original action.

Proposition 10.15. *If $A \leq O(V, Q_E)$ is solvable, faithful, irreducible and quasi-primitive over $\mathbb{F}_2$, it embeds in a group* (10.7) *with $q = 2$, with quadratic form* (10.4)*. Consequently it has the ordinary source and common data in* (10.2)*.*

Proof. We have $O_2(A) = 1$: a normal two-subgroup has a nonzero fixed space, which is A-invariant, so faithfulness and irreducibility make it trivial. Thus $F(A)$ is odd. Every normal abelian subgroup is cyclic, since its homogeneous faithful action generates a finite field.

There is a shorter reduction here than in odd source characteristic. The group A is nontrivial, since a trivial group cannot act irreducibly on the nonzero even-dimensional binary space. Solvability gives $F(A) \neq 1$. Take simply

$$U_0 = Z(F(A)) \neq 1.$$

It is normal, abelian and odd, hence cyclic by the preceding paragraph. No nonidentity element of U_0 is a scalar over $\mathbb{F}_2$.

Quasi-primitivity gives a field $K = \mathbb{F}_2[U_0]$. The adjoint from B_E acts on it by inversion on U_0. This involution is nontrivial: otherwise every element of the odd group U_0 would equal its inverse. Consequently $K = \mathbb{F}_{2^{2c}}$, with adjoint $a \mapsto a^{2^c}$.

For precision, define $h(v, w)$ by $B_E(av, w) = \mathrm{Tr}_{K/\mathbb{F}_2}(ah(v, w))$ for every $a \in K$. The trace pairing makes this unique. The adjoint identity makes it sesquilinear, and symmetry of B_E makes it hermitian. It is nondegenerate and $B_E = \mathrm{Tr}_{K/\mathbb{F}_2} h$. Its quadratic refinement in (10.4) is U_0-invariant. The difference from Q_E is a U_0-invariant linear functional. It is zero, because the nontrivial homogeneous action of U_0 has no fixed space, also on the dual.

Normalizing U_0, the group A acts semilinearly on K^m. In a standard hermitian basis, preservation of B_E, and hence of its trace-pairing reconstruction, puts it in $\mathrm{GU}_m(2^c) \rtimes \mathrm{Gal}(K/\mathbb{F}_2)$. Apply Theorem 10.11 and restrict. □

This binary reduction uses only the nontrivial center of the Fitting subgroup and quasi-primitivity. The stronger Manz–Wolf description used in Proposition 9.16 is not needed here. The ordinary lifts and mixed traces are still supplied by the earlier unitary construction.

Proof of Theorem 10.1. We construct a model and then compare its action with the prescribed action.

Induct on $\dim_{\mathbb{F}_2} V$. The non-quasi-primitive case follows from Lemma 10.14, the smaller faithful component actions, Proposition 10.12 and Proposition 10.13. The quasi-primitive case is Proposition 10.15. Each step constructs an extraspecial model with the specified quadratic quotient, a genuine action, an ordinary implementer and the common two-power data.

Identify that model with the given E, using an isometry of quotients and the extraspecial presentations. It remains to compare the two genuine actions. The group is nontrivial, since its nonzero even-dimensional module is irreducible. Apply Lemma 7.5 to the faithful irreducible binary module. Its elementary averaging argument gives

$$H^1(A, V) = 0. \tag{10.26}$$

By Lemma 10.3, two genuine lifts inducing the same quadratic action differ by a cocycle in $\mathrm{Inn}(E) = V$, as in the proof of Lemma 8.8. Equation (10.26) makes the cocycle a coboundary. Hence a single inner automorphism conjugates the two lifts. Conjugating the constructed ordinary implementer by the corresponding source matrix identifies the actions and preserves every trace. This proves (10.2) for the original lift. □

If σ_0 was an earlier ordinary implementer for that action, then $\sigma = \alpha\sigma_0$ for an ordinary linear character α of A. Its order need not be a power of two. The two-power assertion concerns the chosen Γ_p in (10.2). Corollary 8.7 adjusts the projective parameter by α^{-1}, retaining its factor and its lower character.

Equation (10.26) identifies genuine lifts already constructed. It does not prove that the full orthogonal automorphism sequence splits, and it does not turn a projective source into an ordinary one. The preceding models supplied the ordinary sources before this identification.

Theorem 10.1 supplies (8.3) for the binary case in the algebraic setup of Section 8. The quotient action there is faithful irreducible, and Proposition 8.4 gives the genuine extraspecial subgroup and its square form. The theorem applies to that action, rather than to an unrelated quadratic model.

Here $K_A = A$, so its inverse image in the original B is $K = B$. In Lemma 8.9 take the identity relative map. Inflate Γ_p through B/L. On all the enlarged eligible mixed sections, the permutation is simply

$$F(\eta) = \Gamma_p \otimes \eta, \qquad \psi\eta = \epsilon F(\eta). \tag{10.27}$$

It preserves the fiber over ξ, all degrees, and zero profiles. The full correspondence 8.6 followed by F therefore gives the primitive matching of Proposition 8.11. The source degree 2^M is prime to the odd prime p, so the resulting matching preserves degree p-parts; it need not preserve full upper and lower degrees.

Proof of Theorem 8.1. Proposition 8.4 constructs an invariant extraspecial subgroup E inside $\mathcal{S}^\times$. It spans $\mathcal{S}$, and the action on its quotient is the original faithful irreducible action of A on V. Thus an ordinary implementer of this action on E implements the action on the whole source algebra.

Suppose first that ℓ is odd. Identify E with its exponent- ℓ Heisenberg model. The commutator form makes V a symplectic space, and the canonical ordinary Weil action supplies an implementer for the model action. Corollary 9.30 supplies the common localization data (8.3) for its character. Lemma 8.8 conjugates this genuine model action to the given action, and conjugation of the implementer preserves every trace.

If $\ell = 2$, use the square form of the actual extraspecial group constructed in Proposition 8.4. Theorem 10.1 provides an ordinary implementer for the given genuine action, with the common data (8.3) and $K_A = A$.

In either case the action and the source character are defined on the original group A. If a previous ordinary implementer was used to label the parameters, Corollary 8.7 changes those labels by the inverse linear twist. It retains the original lower character and its projective parameter factor. This proves the theorem without a character-theoretic inductive call. □

11. Completion of the solvable proof

The source localization theorem is now proved for both source characteristics. We return to the relative character problem with the inductive hypothesis available for all smaller groups. First we prove the fully ramified assertion left in Section 7, and then complete the relative correspondence. The block preliminaries identify its visible pairs; the final step supplies the height-zero characters whose selected profiles vanish.

11.1. **The fully ramified case.**

Proof of Proposition 7.9. Apply Theorem 8.1 to the fully ramified primitive section with $X = G$. Choose a Sylow subgroup P_B of G inside B, and use its image in $A = B/L$. The theorem supplies an ordinary implementer with the common localization data (8.3). The complete correspondence and its lower parameters are those of Theorem 8.6 and Corollary 8.7.

Let K be the inverse image in B of the resulting localizer. Proposition 8.11 applies to the family enlarged to all fixed-point-free elements in P_B. When $K < B$, its relative instance $\mathcal{R}(B, K; L, \xi, Y_*)$ is available by the group-order induction, since $B < G$. When $K = B$, use the identity. The proposition verifies the lower and Sylow hypotheses, transports the linear twist by one permutation of parameters, and restricts the result to the original family, removing zero profiles on both sides.

The source degree is $n = \ell^m$, with $\ell \neq p$. The degree formula (8.9) and the compensation therefore preserve degree p-parts. One sign works on the entire original mixed family, as required. □

11.2. **Proof of the relative assertion.**

Proof of Theorem 7.1. We use induction on $|G|$ simultaneously for all the relative parameters. The cases $Y = \varnothing$ and $H = G$ were settled in Section 7.1. Otherwise choose B and L as in Lemma 7.2.

Partition by G-orbits in $\mathrm{Irr}(L \mid \lambda_0)$. Every character with nonzero selected values lies over a potentially contributing orbit. Lemma 7.3 identifies its unique potentially contributing local B-orbit. If the inertia is proper, Proposition 7.4 constructs the correspondence on that Clifford family. If the constituent is invariant, Lemma 7.6 gives the dichotomy. Propositions 7.8 and 7.9 treat its two branches. Each construction preserves degree p-parts and gives a common sign on X_Y. All zero induced profiles are removed simultaneously.

The Clifford families are disjoint on both sides. Their union is therefore an exhaustive matching from G to B, over λ_0. To pass from B to H, the original data still satisfy $\mathrm{Sub}_B(u) \leq H$: this follows by restricting the unique-coset condition on G/H to B/H. Also $C_B(u) = C_G(u)$, since the latter lies in H. Apply the smaller relative theorem on B, with Z_0, λ_0, Y unchanged. The mixed sets agree. We may therefore compose the two correspondences, multiplying their signs, while retaining exactly the characters with nonzero selected values. This proves (7.3). □

The group order decreases in every character call: either to $T < G$, or to $B < G$, both for compensation and for the final composition. No call is made on a larger central cover. The matching applies to all active ordinary characters, not just those of height zero.

The relative theorem is now proved. Applying the block comparison in Section 5.5 identifies the visible blocks of the resulting mixed matching. No block condition was imposed on the intermediate parameter twists.

11.3. **Completion of the solvable case.**

The nonzero mixed profiles now lie in the correct blocks and have the correct heights. Their integral spans give the positive-height part of the assertion. The only characters not yet counted are height-zero characters whose selected profile is zero. We supply that count separately before completing the proof.

Return to solvable G and odd p. Take the full eligible family in $H = \mathrm{Sub}_G(Q)$, and use $Z_0 = 1$ in Theorem 7.1. If $H = G$, use the identity instead.

Corollary 11.1. *For every visible pair (b, c_b), the active mixed profiles admit a signed bijection preserving every ordinary height. In particular,*

$$\mathrm{ev}_X\, C_+(G, b) = \mathrm{ev}_X\, C_+(H, c_b) \tag{11.1}$$

as integral lattices.

Proof. Theorem 7.1 gives the exhaustive active matching for all characters. Lemma 5.6 restricts it exhaustively to the prescribed block pair and preserves its heights. The profiles of its positive-height rows are therefore the same, up to signs. Their integral spans are equal. The remaining positive-height rows have zero mixed profile and contribute nothing to either lattice. □

By Proposition 5.7, (11.1) is precisely preservation of the positive-height images by the canonical value map in Conjecture B(a). This proves its lattice assertion with the stated integral convention. Rational span or saturation of C_+ is not substituted for that image.

The same active matching already pairs all active height-zero rows. It has not yet counted height-zero rows whose entire profile is zero. Those rows are absent from both selected sets in Theorem 7.1, so their number does not follow from that theorem.

The next step uses a published numerical theorem only for that missing count. Its bijection is not used to replace the mixed matching already constructed.

Proposition 11.2. *Conjecture B(a) holds for solvable G and odd p. Its height-zero congruences can be chosen with zero positive-height correction.*

Proof. For a visible pair choose its defect D. Lemma 5.6 gives the common normalizer $N_G(D) = N_H(D)$. By Brauer's first main theorem, b and c_b have the same correspondent there. Apply the published p-solvable Alperin–McKay theorem to both groups. It gives

$$k_0(b) = k_0(c_b). \tag{11.2}$$

This is the Alperin–McKay theorem for p-solvable groups of Okuyama–Wajima [26, Section 3].

Remove the height-zero characters with nonzero selected values already paired in Corollary 11.1. The remaining two sets, consisting exactly of zero mixed profiles, have equal cardinality. Pair them arbitrarily, with positive sign. Their degree p-parts both equal $|P : D|$, by height zero. Combine this with the active matching to obtain a bijection $f_b : \mathrm{Irr}_0(G, b) \to \mathrm{Irr}_0(H, c_b)$, with one sign per pair, for which

$$\mathrm{ev}_X\, \chi = \epsilon_\chi\, \mathrm{ev}_X\, f_b(\chi). \tag{11.3}$$

Equation (5.10) now gives the ambient congruence

$$\chi_H - \epsilon_\chi f_b(\chi) \in \widehat{J}_H. \tag{11.4}$$

By (5.14), its block projection belongs to $\widehat{J}_{H,c_b}$ as well. Thus the required congruence holds with zero member of $C_+(H, c_b)$. Together with (11.1) this proves Conjecture B(a). □

The denominator remains $C_+ + \mathrm{sat}\, J$. Equation (11.4) does not assert membership in unsaturated J. Nor does numerical Alperin–McKay supply any value compatibility: all nonzero values came from Theorem 7.1, and both rows in each added pair vanish on X.

Proof of Theorem D. Part (a) is Proposition 11.2. Since G is solvable and p is odd, the prescribed global radical is $R_p(G) = G$, and its local intersection is H. Every Clifford family is a singleton, so every Clifford level is zero. Hence

$$F_1^R(K, d; Y) = 0, \qquad A_0^R(K, d; Y) = \{\chi \in \mathrm{Irr}(K, d) : \mathrm{ev}_Y\, \chi \neq 0\}. \tag{11.5}$$

Restrict the exact full mixed matching of Corollary 11.1 to any requested pure eligible family Y. A nonzero pure profile is a nonzero mixed profile. Exact signed equality pairs the zero pure profiles with one another, so deleting these pairs gives an exhaustive bijection on the sets in (11.5). It preserves degree p-parts and is exact on Y, proving (b), including the multiplicities of equal profiles. For empty Y the sets are empty.

Lemma 6.6 now proves (c): at each eligible u, the nonzero values are paired up to sign with the same ordinary height, so their weighted generators agree at every level. This completes the proof of all three parts. □

12. Groups with a normal complement

The normal-complement case has a different proof. Canonical extensions supply the mixed values, and the Glauberman correspondence supplies the lower characters. The point requiring care is that the same relative correspondent and the same sign must work on the entire eligible family.

Theorem 12.1. *Let p be odd, and suppose G has a normal p-complement. For every Q, every $H = \mathrm{Sub}_G(Q)$, and every visible block pair (b, c), there is a bijection of all irreducible characters in these two blocks, preserving degree p-parts and heights, with one exact signed equality on all eligible mixed sections. All three parts of Conjecture B hold.*

12.1. **Canonical extensions and relative correspondents.** We now prove 12.1 separately. Throughout this section,

$$p > 2, \qquad G = N \rtimes P, \quad p \nmid |N|, \quad P \in \mathrm{Syl}_p(G), \quad P \le H \le G, \quad C = N \cap H. \tag{12.1}$$

Thus $H = C \rtimes P$, and C is P-invariant. Neither N nor G is assumed solvable. All characters and extensions in this section are ordinary complex characters. For $1 \ne U \le P$, call U admissible if $C_N(U) \le C$. An element is eligible when its cyclic subgroup is admissible.

This agrees with the earlier definition. Indeed the Sylow subgroups containing U are P^n, with $n \in C_N(U)$: if $U \le P^n$, then $U^{n^{-1}} \le P$, and projection onto $G/N = P$ forces $u^{n^{-1}} = u$ for every $u \in U$. Also $N_N(P) = C_N(P)$. The normalizers of these Sylow subgroups generate

$$\mathrm{Sub}_G(U) = \langle P, C_N(U) \rangle. \tag{12.2}$$

For the reverse containment, $C_N(U) \le \mathrm{Sub}_G(U)$ also follows from the transporter criterion in Lemma 5.2. Consequently $\mathrm{Sub}_G(U) \le H$ is equivalent to admissibility. In particular $C_G(U) \le H$.

We first compare canonical extensions on a complete mixed section. We then glue the relative Glauberman maps and their signs. Modular idempotents identify the visible blocks; Clifford induction uses the same cosets on both sides. This gives a bijection of entire visible blocks, including characters with zero profile. No numerical completion is needed.

We use canonical extensions and the cyclic mixed formula from Glauberman [11, Theorems 2–5], together with the multiplicity congruence, naturality and transitivity for a normal operator subgroup.

The compatible relative maps also follow from Souza [30, Theorem C]. We give the particular gluing argument needed here.

Lemma 12.2. *Let u have order $p^a > 1$, acting on a p'-group N, and let $\theta \in \mathrm{Irr}(N)$ be u-invariant. Its canonical extension $\widehat{\theta}$ to $N \rtimes \langle u \rangle$ satisfies*

$$\widehat{\theta}(us) = e(\theta, u)\, g_{N,\langle u \rangle}(\theta)(s) \quad (s \in C_N(u)), \qquad e(\theta, u) \in \{1, -1\}. \tag{12.3}$$

The extension is normalized by determinant one on $\langle u \rangle$. Restricting the canonical extension to any operator subgroup preserves this normalization.

Proof. We include the integral character calculation behind the classical cyclic formula. Put $B = C_N(u)$, and decompose the restriction to $B \times \langle u \rangle$ as

$$\widehat{\theta}|_{B \times \langle u \rangle} = \sum_{\lambda \in \mathrm{Irr}(\langle u \rangle)} \gamma_\lambda \times \lambda. \tag{12.4}$$

Each γ_λ is a character of B, possibly zero. A Galois automorphism fixing all $|N|$-th roots preserves the canonical extension, by its uniqueness. It follows that γ_λ depends only on the order of λ. The sums of the primitive p^j-th roots at a generator are $1, -1, 0$ for $j = 0, 1, \ge 2$, respectively. Hence

$$F(s) := \widehat{\theta}(us) = \gamma_1(s) - \gamma_{\lambda_p}(s) \in \mathbb{Z}\,\mathrm{Irr}(B), \qquad \theta_B \equiv F \pmod{p\mathbb{Z}\,\mathrm{Irr}(B)}. \tag{12.5}$$

Here λ_p is any character of order p.

Every element of Nu is N-conjugate to su with $s \in B$: its p-part generates a complement and can be conjugated to u by Schur–Zassenhaus. Two such elements are N-conjugate exactly when

their p'-parts are B-conjugate, and $C_N(su) = C_B(s)$. Matrix coefficient orthogonality for a unitary representation affording θ gives

$$1 = |N|^{-1} \sum_{n \in N} |\widehat{\theta}(nu)|^2 = \sum_{[s] \subseteq B} \frac{|F(s)|^2}{|C_B(s)|} = \langle F, F \rangle_B. \tag{12.6}$$

An integral virtual character of norm one is a signed irreducible. The congruence in (12.5) identifies that irreducible with the Glauberman correspondent. This proves (12.3). Restriction preserves both the extension and its determinant normalization, so uniqueness proves the last assertion. □

The local calculation also works for $p = 2$. The use of $p > 2$ below will distinguish a common sign from its negative.

For each admissible $U \leq P$, the centralizer $B_U = C_N(U) = C_C(U)$ is common to N and C. Define

$$b_U = g_{C,U}^{-1} g_{N,U} : \mathrm{Irr}(N)^U \longrightarrow \mathrm{Irr}(C)^U. \tag{12.7}$$

Lemma 12.3. *The maps b_U agree on overlaps and give a P-equivariant bijection on the unions of their domains and ranges. If $\varphi = b_U(\theta)$, then $P_\theta = P_\varphi$. Whenever $V \leq P_\theta$ is admissible, $b_V(\theta) = \varphi$.*

Proof. First suppose $U \trianglelefteq A \leq P$, with U admissible and θ fixed by A. Naturality of $g_{N,U}$ makes its image A/U-invariant; naturality and injectivity of $g_{C,U}$ then make $b_U(\theta)$ fixed by A. Transitivity gives

$$g_{N,A}(\theta) = g_{B_U,A/U}(g_{N,U}(\theta)) = g_{C,A}(b_U(\theta)). \tag{12.8}$$

Here $C_N(A) \leq B_U \leq C$, so A is admissible as well. Injectivity proves $b_A(\theta) = b_U(\theta)$. The identical argument applies to the inverse maps.

Every subgroup of a finite p-group is subnormal. Applying the preceding step along a subnormal chain proves the same assertion for $U \leq A$, without normality. If θ is fixed by two admissible subgroups U, V, apply this assertion in $A = \langle U, V \rangle$. Thus $b_U(\theta) = b_A(\theta) = b_V(\theta)$. The inverse assertion proves that the glued map is bijective. Naturality under conjugation by P proves equivariance.

For completeness, let $\varphi = b_U(\theta)$. The preceding argument with $A = P_\theta$ shows that φ is fixed by P_θ. The inverse argument with $A = P_\varphi$ gives the reverse inclusion. This proves equality of stabilizers. The assertion for V now follows from the overlap assertion. □

In particular, equality of the correspondents at one admissible subgroup gives equality at every admissible cyclic subgroup in the common inertia group. It does not require the smaller centralizers to be P-invariant. This is the compatibility that will be needed in each summand of an induction formula.

12.2. **A common sign and the visible blocks.** The relative correspondence must agree for different eligible cyclic subgroups. Once that agreement is proved, the degree formula fixes one sign for the entire family. We then identify the blocks and induce the corresponding parameters over the same cosets.

Lemma 12.4. *Suppose $A = P_\theta$ is admissible, and set $\varphi = b_A(\theta)$. Let $\widehat{\theta}$ and $\widehat{\varphi}$ be the canonical extensions to NA and CA. There is a unique sign ϵ_θ satisfying*

$$\theta(1) \equiv \epsilon_\theta \varphi(1) \pmod{p}. \tag{12.9}$$

For every eligible $u \in A$ and every $s \in C_N(u)$,

$$\widehat{\theta}(us) = \epsilon_\theta \widehat{\varphi}(us). \tag{12.10}$$

Proof. The Glauberman multiplicity congruence at A gives $\theta_{C_N(A)} \equiv e_N \nu$ and $\varphi_{C_N(A)} \equiv e_C \nu$ modulo p, for signs e_N, e_C and the same irreducible ν. Their degrees are prime to p. Evaluation at one gives (12.9), with $\epsilon_\theta = e_N / e_C$. Since $p > 2$, this sign is unique.

Fix an eligible $u \in A$. Lemma 12.3 gives a common cyclic correspondent ν_u of θ and φ. Lemma 12.2 gives

$$\widehat{\theta}(us) = e(\theta, u)\nu_u(s), \qquad \widehat{\varphi}(us) = e(\varphi, u)\nu_u(s). \tag{12.11}$$

To compare these signs, reduce the eigenvalues of u modulo a prime above p in a cyclotomic integer ring. All roots of p-power order reduce to one. Thus $\widehat{\theta}(u) \equiv \theta(1)$ and $\widehat{\varphi}(u) \equiv \varphi(1)$. The traces at u are integers by (12.11), so these are ordinary congruences modulo p. At $s = 1$, (12.11) and (12.9) imply

$$e(\theta, u)\nu_u(1) \equiv \epsilon_\theta e(\varphi, u)\nu_u(1) \pmod{p}. \tag{12.12}$$

The degree $\nu_u(1)$ is prime to p; cancellation and $p > 2$ give equality of the two signs. Substitution in (12.11) proves (12.10) for all s, including those at which $\nu_u(s) = 0$. No character value was divided out. □

The sign depends on the lower orbit, rather than on u, s, or the later parameter in $\mathrm{Irr}(A)$. It may be negative, as the example on Section 12.3 shows.

We recall the normal-complement block description. A p-block of NP is determined by a P-orbit on $\mathrm{Irr}(N)$; over θ it has defect group $A = P_\theta$. All its ordinary characters are the Clifford inductions of the characters above θ in NA. One way to see the block assertion is to reduce the orbit idempotent in a splitting modular system. Its inertia corner is Morita equivalent to kA: the canonical extension removes the factor set. The algebra of a p-group in characteristic p has one block. The defect group is the inertia Sylow subgroup A.

Lemma 12.5. *Let $H = \mathrm{Sub}_G(Q)$ and let b be a visible block with defect group $Q \le A \le P$. Choose θ in its orbit so that $P_\theta = A$. Then A is admissible, and the block c over $b_A(\theta)$ is exactly the visible Brauer correspondent of b.*

Proof. Put $\varphi = b_A(\theta)$. Since $C_N(A) \le C_N(Q) \le C$, the map is defined; Lemma 12.3 gives $P_\varphi = A$. We verify the block identification, rather than infer it from the equal defect orders.

Write $B = C_N(A)$, and work in a splitting modular system with residue field k of characteristic p. For an A-invariant $\xi \in \mathrm{Irr}(N)$ with Glauberman correspondent $\nu \in \mathrm{Irr}(B)$, the coefficient formula for primitive idempotents gives

$$\mathrm{Br}_A(\overline{e}_\xi) = \overline{e}_\nu. \tag{12.13}$$

Indeed $\xi_B \equiv e\nu \pmod{p}$, $\xi(1) \equiv e\nu(1) \pmod{p}$, and $|N| \equiv |B| \pmod{p}$. Substituting these in $|N|^{-1}\xi(1)\sum_{n\in N}\xi(n^{-1})n$ proves (12.13) coefficient by coefficient. All denominators are units at p. Nonfixed A-orbits of characters contribute zero to the Brauer image of an orbit sum.

The A-fixed members of the P-orbit of θ and of that of φ correspond equivariantly by Lemma 12.3. Formula (12.13), applied also in C, therefore gives identical Brauer images of b and c in $kC_G(A) = kC_H(A)$. Here the centralizers are equal because A is admissible. Also $N_G(A) \le H$: if $A^g = A$, then $Q^g \le H$, so 5.2 gives $g \in H$. Brauer's first main theorem now identifies c as the unique block of defect A inducing to b. This is the specified visible correspondent. □

Retain the visible pair (b, c), and the lower characters θ, φ with common inertia A. For $\alpha \in \mathrm{Irr}(A)$, put

$$\chi_\alpha = \mathrm{Ind}_{NA}^G(\widehat{\theta}\,\mathrm{Inf}\,\alpha), \qquad \psi_\alpha = \mathrm{Ind}_{CA}^H(\widehat{\varphi}\,\mathrm{Inf}\,\alpha). \tag{12.14}$$

Clifford theory and Gallagher's theorem give bijections from $\mathrm{Irr}(A)$ onto $\mathrm{Irr}(b)$ and $\mathrm{Irr}(c)$. In particular every irreducible occurs, including those which will have identically zero eligible profile.

Proposition 12.6. *The map $\chi_\alpha \mapsto \psi_\alpha$ is a bijection of the entire visible blocks. It preserves degree p-parts and ordinary heights, with*

$$\chi_\alpha(1)_p = \psi_\alpha(1)_p = |P : A|\alpha(1), \qquad h_b(\chi_\alpha) = h_c(\psi_\alpha) = v_p(\alpha(1)). \tag{12.15}$$

Proof. The p'-degrees $\theta(1), \varphi(1)$ do not affect degree p-parts; an irreducible degree of the p-group A is itself a power of p. Moreover $[G : NA] = [H : CA] = |P : A|$. This gives the first formula. Both blocks have defect group A, by Lemma 12.5, and both ambient groups have Sylow order $|P|$. The definition of height now gives the second formula. □

Full ordinary degrees need not agree: they contain the possibly different factors $\theta(1)$ and $\varphi(1)$. Nor does the argument require A to be abelian. Its nonlinear characters account for the positive heights in (12.15).

Changing θ inside its P-orbit conjugates A, φ, the canonical extensions and the parameters simultaneously. Naturality of (12.7) and uniqueness of canonical extensions preserve the resulting

paired block characters. The degree congruence (12.9) also preserves ϵ_θ. Thus the construction does not require unrelated choices for different sections.

Proposition 12.7. *For the correspondence in* (12.14),

$$\chi_\alpha(us) = \epsilon_\theta \psi_\alpha(us) \quad \textit{whenever } u \in H \textit{ is eligible and } s \in C_G(u)_{p'}. \tag{12.16}$$

The same sign works for the whole family, for every $\alpha \in \mathrm{Irr}(A)$.

Proof. First take $u \in P$. The image of the p'-element s in $G/N = P$ is trivial, so

$$s \in C_N(u) \le C. \tag{12.17}$$

This inclusion is the feature specific to a normal p-complement. Cosets of NA in G and of CA in H have the same representatives in P. Use representatives t for the right cosets $A\backslash P$; put ${}^t x = txt^{-1}$. For $x = us$, membership of ${}^t x$ in either inertia group is equivalent to ${}^t u \in A$. For the local assertion this uses ${}^t s \in C$, which holds because C is P-invariant. The two induction formulas are therefore

$$\begin{aligned} \chi_\alpha(us) &= \sum_{At:\, {}^t u \in A} \widehat{\theta}({}^t u\, {}^t s)\alpha({}^t u), \\ \psi_\alpha(us) &= \sum_{At:\, {}^t u \in A} \widehat{\varphi}({}^t u\, {}^t s)\alpha({}^t u). \end{aligned} \tag{12.18}$$

Every ${}^t u$ in this sum is eligible, and ${}^t s \in C_N({}^t u)$. Lemma 12.4 applies to each summand with the same ϵ_θ; summing proves (12.16).

For arbitrary eligible $u \in H$, conjugate it into P by an element of H. Eligibility and the mixed section are transported at the same time. Both characters are class functions on H, giving (12.16) in general. □

No conclusion is inferred from pure values alone. The element s occurs throughout (12.17)–(12.18), and (12.10) is an identity on its entire centralizer. Cancellation among induction summands is harmless: the same signed sum occurs on both sides. In particular zero mixed profiles correspond in both directions.

12.3. **Proof of the normal-complement theorem.**

Proof of Theorem 12.1. Choose a Sylow subgroup P containing the specified defect group $D \ge Q$. Then $P \le H$, so (12.1) applies. The normal-complement block description allows a representative θ with $P_\theta = D$. Lemma 12.5 identifies the required local block. Propositions 12.6–12.7 prove the asserted full bijection and its value identities. If $H = G$, use the identity correspondence instead.

To verify the value component, take the full union of eligible p-sections X_H from Lemma 5.3. Equation (12.16) identifies the two integral images of the positive-height lattices exactly; no saturation of that positive-height image is taken. The height-zero rows, including zero rows, are already matched by (12.14). Theorem 5.4 identifies the kernel of evaluation with $\widehat{J}_H$; Proposition 5.7 identifies the canonical block projection with the identity on these value lattices. Thus each paired restriction differs by an element of $\widehat{J}_H$, and its block projection by an element of $\widehat{J}_{H,c}$. The positive-height correction can be taken to be zero. The results of Section 5 hold for arbitrary finite groups, independently of the solvable theorem 7.1.

For part (b), a normal p'-subgroup with p-group quotient makes G p-solvable, even when N is not solvable. Hence the specified global radical is $R_p(G) = G$, and $N_H = H$. All Clifford levels are zero: $F_1^R = 0$, and A_0^R consists of precisely the nonzero pure profiles. Restrict (12.16) to any prescribed eligible pure family and remove the paired zero profiles. This gives an exhaustive signed bijection with the same degree p-parts. Equal vectors belonging to distinct characters retain their multiplicities. This proves part (b) of Conjecture B. The exact bijection preserves ordinary heights, so Lemma 6.6 also gives part (c) at every weighted level. □

This is 12.1, independent of the source geometry in Sections 8–10 and of the solvable induction 7.1. It neither proves the result for all p-solvable groups nor supplies a statement for $p = 2$.

13. The unitary family

Theorem 13.1. *Let $q = p^f \geq 3$, with p odd, and $S = \mathrm{PSU}_3(q)$. For every $1 \neq Q \leq P \in \mathrm{Syl}_p(S)$, every visible block pair (b, c), every eligible u, and every $r \geq 0$,*

$$\mathcal{M}_r(S, b; u) = \mathcal{M}_r(\mathrm{Sub}_S(Q), c; u) = p^r\mathbb{Z}.$$

In particular, B(c) holds for these simple groups without a restriction on $\gcd(3, q+1)$.

The subgroup and the visible blocks. Put $d = \gcd(3, q+1)$. The unitary group has a rank-one action on its q^3+1 isotropic lines. Its standard Borel subgroup is the normalizer of P, and distinct Borel subgroups have a p'-intersection. This follows also from the two-cell Bruhat decomposition $S = B \cup BwB$, with $B \cap B^w$ a torus. Thus Sylow p-subgroups are TI, and

$$H = \mathrm{Sub}_S(Q) = N_S(P) = P \rtimes C_{(q^2-1)/d} \qquad (1 \neq Q \leq P).$$

Every nonidentity p-element fixes exactly one isotropic line, so precisely these p-elements are eligible.

The subgroup P is normal in H and $C_H(P) = Z(P)$. Every block of H consequently has defect P. Its Brauer image at P is a nonzero central idempotent of $kZ(P)$, where k is algebraically closed of characteristic p. This algebra has no nontrivial idempotent decomposition, so H has just one block. Brauer's first and third main theorems identify the only visible pair as the principal pair, with defect P.

Integral values, including the split unipotent classes. It is useful to verify integrality without inferring it from the unsplit case. In $\mathrm{SU}_3(q)$ use the antidiagonal Hermitian form and write

$$U(a,t) = \begin{pmatrix} 1 & a & -a^{q+1}/2 + t \\ 0 & 1 & -a^q \\ 0 & 0 & 1 \end{pmatrix}, \qquad a \in \mathbb{F}_{q^2}, \quad t + t^q = 0.$$

These matrices form a Sylow subgroup. Its centre consists of $U(0,t)$. For $a \neq 0$, conjugation inside this Sylow subgroup changes t by every trace-zero element: the relevant map in the conjugating parameter c is, up to sign,

$$c \longmapsto ca^q - ac^q,$$

which is a nonzero $\mathbb{F}_q$-linear map onto the one-dimensional trace-zero space.

The diagonal element $\mathrm{diag}(s, s^{q-1}, s^{-q})$ acts by

$$a \longmapsto s^{q-2}a, \qquad t \longmapsto s^{-q-1}t.$$

The first multiplier has image of index d in $\mathbb{F}_{q^2}^\times$. If $d = 1$ it is surjective. If $d = 3$, then $p \equiv -1 \pmod 3$ and f is odd, so $3 \nmid p - 1$. Every element of $\mathbb{F}_p^\times$ is therefore a cube, and is again in that image. For each $j \in \mathbb{F}_p^\times$, a torus conjugation matches the first parameter of $U(a,t)^j = U(ja, jt)$; a Sylow conjugation then matches the second. If $a = 0$, the surjectivity of the norm multiplier s^{-q-1} gives the same conclusion directly.

Hence every p-element is conjugate in the Borel subgroup to each of its prime-to-p powers. Its order is p, since p is odd. All ordinary character values at these elements are rational algebraic integers, hence integers, both locally and globally. This argument survives the central quotient to $\mathrm{PSU}_3(q)$.

Positive heights and the weighted modules. The generic table of Simpson–Frame [29, Table 2] uses the notation $\mathrm{PSU}(3, q^2)$ for our group. Its character degrees show that, apart from Steinberg, the only degrees divisible by p are

$$q(q-1) \quad \text{once}, \qquad q(q^2-q+1) \quad (q+1)/d - 1 \text{ times}.$$

Their unipotent profiles are respectively $(-q, 0, \ldots, 0)$ and $(q, 0, \ldots, 0)$: the first coordinate is the nontrivial central Sylow type, and the other d coordinates are the regular types. Their height in the full-defect principal block, whenever present there, is f. Steinberg has defect zero and is not in that block. We do not use the erroneous printed nonzero Steinberg entry at the central unipotent type in the old table; its value there is zero.

For the local computation, P has q^2 linear characters and $q-1$ nonlinear characters of degree q. The torus modulo the scalar centre acts transitively on the nontrivial central characters. A

nonlinear character has cyclic inertia quotient of order $(q+1)/d$, to which it extends. Induction gives exactly $(q+1)/d$ local irreducibles of degree $q(q-1)$, all of height f. Their values are $-q$ at the central nonidentity type and zero outside $Z(P)$. All other local degrees are prime to p: the linear inflations and the d induced rows of degree $(q^2-1)/d$. This also follows from the Borel character description in Hoyer–Nebe [14, Proposition 4.1].

Thus on both sides every character in the visible block satisfies

$$\chi(u) \in p^{h(\chi)}\mathbb{Z}.$$

Every weighted generator lies in $p^r\mathbb{Z}$, and the trivial character supplies p^r. This proves Theorem 13.1.

14. The symplectic rank-two family

Theorem 14.1. *Let $q = p^f \geq 3$, with p odd, and $S = \mathrm{PSp}_4(q)$. For every nontrivial $Q \leq P \in \mathrm{Syl}_p(S)$, every visible pair (b, c), every eligible p-element u, and every integer $r \geq 0$,*

$$\mathcal{M}_r(S, b; u) = \mathcal{M}_r(H, c; u).$$

In particular B(c) holds for this entire family of simple groups. If $H < S$, the pair is principal with full Sylow defect. At a regular unipotent element the common module is $p^r\mathcal{M}_0(S, b; u)$. At an eligible nonregular element it is $p^r\mathbb{Z}$.

Two dependencies must be kept separate.

The unweighted comparison. Proposition 5.7, identifies the integral blockwise eligible value profiles by block projection of restriction and induction. In particular, for a visible pair and any eligible u,

$$\mathcal{M}_0(S, b; u) = \mathcal{M}_0(H, c; u). \tag{14.1}$$

This is an unweighted result; no part of B(c) is being assumed in (14.1).

Published input. Malle [20, Propositions 3.7(2), 4.6 and Theorem 3.2] gives these facts for $\widetilde{S} = \mathrm{Sp}_4(q)$, q odd:

(i) Proper cyclic subnormalizers occur only for regular unipotents, or the anisotropic subregular type with centralizer order $2q^3(q+1)$; their subnormalizers are respectively a Borel and a Siegel parabolic.
(ii) At regular unipotents, nonvanishing irreducibles of $\widetilde{S}$ and its Borel have p'-degree.
(iii) At the specified subregular type, nonzero values of p-divisible-degree characters on either side are $\pm q$ or $\pm 2q$.
(iv) The proved correspondence on nonvanishing characters preserves degree p-parts.

Only these proved facts, not a universal conjecture from that paper, are used. Geometric names avoid any dependence on parabolic numbering. We use Malle's published verification of these assertions.

14.1. **Subnormalizers, blocks, and the central quotient.** Write $B = N_S(P)$. Since $Q \leq P$, we have $B \leq H$. The overgroups of a Borel in the split rank-two BN-pair are

$$B, \qquad H_{\mathrm{Si}}, \qquad H_{\mathrm{Kl}}, \qquad S, \tag{14.2}$$

where H_{Si} stabilizes a Lagrangian plane and H_{Kl} an isotropic line. No assertion that an arbitrary guessed parabolic is $\mathrm{Sub}_S(Q)$ is needed.

For each proper group K in (14.2), its normal p-subgroup $U = O_p(K)$ satisfies

$$C_K(U) = \begin{cases} U, & K = H_{\mathrm{Si}}, \\ Z(U), & K = H_{\mathrm{Kl}} \text{ or } B. \end{cases} \tag{14.3}$$

These follow directly from the natural matrix actions: the Siegel radical is the additive group of symmetric 2×2 matrices, the Klingen radical is Heisenberg, and the torus action is faithful after quotienting by $\{\pm I\}$.

Every block defect of K contains its normal subgroup U, and hence its block idempotent has nonzero Brauer image at U. By (14.3), $kC_K(U)$ is a p-group algebra and has no nontrivial central idempotent decomposition. Orthogonal nonzero Brauer images are impossible. Thus K has one

block, the principal block, with defect P. Brauer's first and third main theorems, since $N_S(P) \le K$, identify its visible global partner as the principal full-defect block. In particular, in all proper cases

$$h(\chi) = v_p(\chi(1)). \tag{14.4}$$

If $H = S$, equality in the theorem is tautological for every visible pair.

Here is the useful consequence of eligibility, with no assumption that Q is cyclic.

For an eligible u, the inclusion $\mathrm{Sub}_S(\langle u \rangle) \le H$ is the definition. Equivalently, u fixes just the base coset of S/H, by Lemma 5.2.

The central map $\pi : \widetilde{S} \to S$ has kernel of order two. The full preimage of a p-subgroup is the direct product of this kernel with its unique Sylow p-subgroup. Normalizers, the defining Sylow families, generated subnormalizers and the coset actions therefore correspond under π. A p-element has a unique p-element lift. Character inflation preserves degree and value. Consequently the published inputs apply to the required elements in S, even though no central-character-preserving choice of the published bijection is assumed.

By eligibility and input (i), a proper H only requires two cases: regular elements, and the anisotropic subregular type. In the latter case $H = H_{\mathrm{Si}}$, since the cyclic subnormalizer is already maximal. A Siegel radical element $U(C)$ represents that type exactly when the nonsingular binary form C is anisotropic.

14.2. **The Siegel calculation.** It is simplest to calculate before the central quotient. Write

$$\widetilde{H}_{\mathrm{Si}} = U \rtimes \mathrm{GL}_2(q), \qquad U = \mathrm{Sym}_2(\mathbb{F}_q)^+.$$

The trace pairing identifies $\mathrm{Irr}(U)$ with symmetric forms. The orbits under congruence have rank zero, rank one (two square types), and rank two (split and nonsplit). A linear character of U extends to its semidirect inertia group by taking the complement trivially. Clifford theory then gives degrees as orbit size times a character degree of the stabilizer. Passing to the central quotient simply retains characters trivial on $-I$.

Lemma 14.2. *Every irreducible character of H_{Si} has degree p-part either 1 or q. Every character with degree divisible by p vanishes at every regular unipotent element of H_{Si}.*

Proof. The relevant orbit data are

rank	orbit size	stabilizer before the quotient
0	1	$\mathrm{GL}_2(q)$
1	$(q^2-1)/2$	$C_2 \times (\mathbb{F}_q^+ \rtimes \mathbb{F}_q^\times)$
2	$q(q+1)/2$ or $q(q-1)/2$	$O_2^+(q)$ or $O_2^-(q)$.

The rank-one stabilizer modulo the scalar center is $\mathbb{F}_q^+ \rtimes \mathbb{F}_q^\times$; all its character degrees are prime to p. Rank-two stabilizers have order $2(q-1)$ or $2(q+1)$, prime to p, so those induced degrees have p-part exactly q. For rank zero, the elementary degree list for $\mathrm{GL}_2(q)$ is contained in $\{1, q, q-1, q+1\}$; its only p-divisible irreducibles are determinant twists of Steinberg.

If u is regular on the four-dimensional space, its image in the Siegel Levi is a nontrivial unipotent of $\mathrm{GL}_2(q)$. Otherwise $u \in U$ would have square-zero nilpotent part. Such a Levi image cannot lie in any conjugate of a rank-two stabilizer, so all rank-two induced values are zero. Rank-one rows have p'-degree. The rank-zero rows of p-divisible degree vanish because the rank-one Steinberg character vanishes at a nontrivial unipotent. These observations prove both assertions. □

For clarity, the degree lists for $\mathrm{GL}_2(q)$ and $\mathrm{SL}_2(q)$ used here and below are the elementary rank-one lists. For $\mathrm{SL}_2(q)$, the possible degrees lie in $\{1, q, q-1, q+1, (q-1)/2, (q+1)/2\}$; the sole p-divisible type is Steinberg. No generic odd-q character table of a rank-two parabolic has been computed in this argument.

The anisotropic subregular element. Take $u = U(C)$ with C anisotropic and nonsingular. Every scalar multiple jC, $j \in \mathbb{F}_q^\times$, is congruent to C: identify the form with a scalar multiple of the norm form on $\mathbb{F}_{q^2}$ and use surjectivity of the norm. Since u has order p, its prime-to-p powers are therefore conjugate inside the Siegel parabolic. All ordinary character values at u are integers, both locally and globally.

Lemma 14.2 and published input (iv), applied first in $\mathrm{Sp}_4(q)$, show that a nonzero p-divisible-degree value globally can only have degree p-part q. Restricting to center-trivial rows is sufficient; there is no need to restrict a particular bijection. With (14.4) and input (iii), on either side every row satisfies

$$\chi(u) \in p^{h(\chi)}\mathbb{Z}; \qquad \chi(u) \neq 0 \Longrightarrow h(\chi) \in \{0, f\}. \tag{14.5}$$

For $r \geq 0$, (14.5) places every generator of (1.4) in $p^r\mathbb{Z}$. The trivial character supplies p^r. Hence

$$\mathcal{M}_r(S, b; u) = \mathcal{M}_r(H_{\mathrm{Si}}, c; u) = p^r\mathbb{Z}. \tag{14.6}$$

This genuinely handles surviving positive height: inflate the degree-q Steinberg character of the Siegel Levi. It descends through $-I$, belongs to the unique local block, has height f, and takes the nonzero value q at u. In particular the family is not a height-zero-only verification, and the relevant height is unbounded as f increases.

14.3. **A Heisenberg vanishing lemma for Klingen parabolics.** Use a basis with block sizes $(1, 2, 1)$ and put

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad J_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & J_2 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The Klingen radical consists of

$$Q(v, z) = \begin{pmatrix} 1 & v^t J_2 & z \\ 0 & I_2 & v \\ 0 & 0 & 1 \end{pmatrix}, \qquad v \in \mathbb{F}_q^2, \quad z \in \mathbb{F}_q,$$

with multiplication

$$Q(v, z)Q(w, t) = Q(v + w, z + t + v^t J_2 w). \tag{14.7}$$

It has center $Z(U) = \{Q(0, z)\}$, and the full Levi before quotienting is $\mathbb{F}_q^\times \times \mathrm{SL}_2(q)$.

Lemma 14.3. *Every irreducible character of H_{Kl} with degree divisible by p vanishes at a regular unipotent element.*

Proof. Lift the element to its unique p-element in $\mathrm{Sp}_4(q)$. Its Levi part is $(1, A)$, where $A \in \mathrm{SL}_2(q)$ is nontrivial unipotent. Thus

$$u = Q(v, z)\,\mathrm{diag}(1, A, 1).$$

Put $L = \ker(A - I_2)$. Direct multiplication shows that $(u - I_4)^3 \neq 0$ if and only if $v \notin L$. For example, for $A = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, $t \neq 0$, the top-right entry of $(u - I_4)^3$ is $-tv_2^2$. Hence regularity is exactly $v \notin L$.

For every $w \in L$, multiplication in (14.7) gives

$$u^{Q(w,0)} = Q(0, 2v^t J_2 w)u. \tag{14.8}$$

Since p is odd and $v \notin L$, the scalar $2v^t J_2 w$ runs over all of $\mathbb{F}_q$ as w runs over L.

Suppose an irreducible lies over a nontrivial central character λ of $Z(U)$. In its Clifford inertia group, each $Q(0, t)$ acts as the scalar $\lambda(Q(0, t))$. That inertia group contains U. If a regular u lies in it, (14.8) and invariance under conjugation force

$$\eta(u) = \lambda(Q(0, t))\eta(u) \qquad (t \in \mathbb{F}_q),$$

and hence $\eta(u) = 0$. If u is outside it, use the induction formula: every conjugate contributing to that formula is again regular, so every term vanishes by the same argument. No choice of extension or normalization of a Weil representation enters this step.

It remains to handle characters trivial on $Z(U)$. The quotient $U/Z(U)$ is an elementary abelian group of order q^2. The Levi is transitive on its nontrivial linear characters, with orbit size $q^2 - 1$. Their inertia quotient after dividing by $\{\pm I\}$ is

$$\mathbb{F}_q^+ \rtimes C_{(q-1)/2}.$$

Its character degrees are prime to p, because the additive normal subgroup is abelian and its complement is cyclic of p'-order. The linear radical character extends to its semidirect inertia group; consequently all corresponding induced degrees are prime to p as well.

The remaining characters inflate from $(\mathbb{F}_q^\times \times \mathrm{SL}_2(q))/\langle(-1,-I_2)\rangle$. Their p-divisible-degree rows are Steinberg twists in the $\mathrm{SL}_2(q)$ factor, which vanish at A. This exhausts the irreducibles and proves the lemma. □

The coefficient 2 in (14.8) is an essential odd-characteristic feature. The argument supplies no theorem at $p = 2$.

14.4. **Completion of the family proof.** Assume $H < S$ and u is regular. Input (ii), Lemma 14.2, and Lemma 14.3 cover respectively S, B, the Siegel parabolic, and the Klingen parabolic. Thus every nonzero value in the visible blocks has height zero. It follows directly from (1.4) that

$$\mathcal{M}_r(S,b;u) = p^r\mathcal{M}_0(S,b;u), \qquad \mathcal{M}_r(H,c;u) = p^r\mathcal{M}_0(H,c;u).$$

Apply the unweighted identity (14.1). The subregular case was proved in (14.6), and (14.2) exhausts H. This completes Theorem 14.1.

A necessary caution. One must not replace the regular module by $p^r\mathbb{Z}$ without further proof. Regular elements can have nonrational character values. The unweighted bridge, rather than a rationality assertion, is precisely what is used in this case.

We do not assert here a signed correspondence on whole eligible profiles for this family.

15. Subnormalizers in symplectic dimension six

Lemma 15.1. *Let $u \in P \in \mathrm{Syl}_p(G)$ and $L = \mathrm{Sub}_G(\langle u\rangle)$. Then u fixes exactly one coset in G/L. Moreover, if $N_G(P) \le H \le G$ and u is eligible in G/H, then $L \le H$.*

Proof. Every Sylow subgroup containing u lies in L. If $u^x \in P$, then $P^{x^{-1}} \le L$ and Sylow conjugacy in L gives an $h \in L$ with $x^{-1}h \in N_G(P) \le L$. Thus $x \in L$. If $u^{g^{-1}} \in L$, conjugate it inside L into P and apply this transporter assertion to obtain $g \in L$. This is precisely uniqueness of the fixed coset.

For the second assertion, $|G : H|$ is prime to p. Every Sylow T containing u has a fixed coset, which must be the unique u-fixed coset H. Hence $T \le H$. Sylow conjugacy in H yields $N_G(T) \le H$; generate over all T. □

15.1. **The visible blocks.**

Lemma 15.2. *Let $S = \mathrm{PSp}_6(q)$, with q odd. If $H = \mathrm{Sub}_S(Q) < S$, then H has one block, the principal block of full Sylow defect. Its only visible partner in S is the principal block. In this pair ordinary height is degree valuation.*

Proof. Since H contains a Borel, it is one of $P_1, P_2, P_3, P_{12}, P_{13}, P_{23}, B$, where subscripts specify the dimensions in an isotropic flag. In the maximal cases the radicals have orders q^5, q^7, q^6, respectively. Their centralizers in the projective groups are

$$C_{P_1}(O_p(P_1)) = Z(O_p(P_1)), \quad C_{P_2}(O_p(P_2)) = Z(O_p(P_2)), \quad C_{P_3}(O_p(P_3)) = O_p(P_3).$$

For Siegel, the kernel of the symmetric-matrix action is the scalar sign. For the middle parabolic, centralizing its symmetric-matrix centre first forces the GL_2 coordinate to be $\pm I$; the radical quotient forces the same sign in the SL_2 coordinate. The Heisenberg centre and natural four-dimensional quotient give the Klingen assertion. The common scalar sign disappears in S.

A smaller parabolic contains a maximal parabolic's radical as a normal subgroup of its own p-core. Its p-core therefore also has a p-group centralizer. Every block defect contains the normal p-core, and every block idempotent has nonzero Brauer image there. The algebra of a p-group centralizer has no nontrivial idempotent decomposition, so two orthogonal nonzero block images are impossible. Thus H has only its principal block, of full defect. Since $N_S(P) \le H$, Brauer's first and third main theorems identify the visible global block as principal. Both defect indices have p-part one. □

If $H \le M$ both contain a full Sylow subgroup, eligibility in G/H implies eligibility in G/M: the fibre above a fixed M-coset has p'-size and therefore has a fixed point for the p-element. Distinct fixed M-cosets would give distinct fixed H-cosets.

15.2. **Uniform geometry in dimension six.** Let $q = p^f$ be odd and let V be symplectic of dimension six. The central quotient $\mathrm{Sp}(V) \to \mathrm{PSp}(V)$ preserves Sylow families, normalizers, generated subnormalizers and the relevant coset actions. Indeed, the inverse image of a p-subgroup is a direct product with the center of order two and has a unique Sylow p-subgroup. We may work with unipotent lifts in $\mathrm{Sp}(V)$.

For a unipotent u, use the Cayley transform

$$X = (u - I)(u + I)^{-1}, \qquad u = (I + X)(I - X)^{-1}. \tag{15.1}$$

Then X is nilpotent and skew-adjoint. It has the same Jordan block sizes and invariant subspaces as u. This avoids using a logarithm, which would be invalid uniformly in small characteristic.

The possible nonidentity Jordan partitions are

$$(6),\ (4,2),\ (4,1^2),\ (3^2),\ (2^3),\ (2^2,1^2),\ (2,1^4). \tag{15.2}$$

We use the standard symplectic signed-Jordan decomposition, with odd blocks paired and square-class parameters on even block sizes; see Gonshaw–Liebeck–O'Brien [12, Proposition 2.3]. In particular there are four finite-group classes of type $(4,2)$ and two regular classes. The calculations that follow decide which invariant isotropic spaces are unique; that conclusion is not being inferred merely from the partition list.

Write P_i for a stabilizer of an isotropic i-space, $i = 1, 2, 3$. Thus P_1 is Klingen and P_3 is Siegel. These are the three maximal overgroups of a chosen Borel. Every overgroup of that Borel is standard parabolic, including the whole group.

Lemma 15.3. *A nonidentity unipotent has exactly one invariant isotropic line, or exactly one invariant isotropic two-space, if and only if it is regular. It has exactly one invariant Lagrangian if and only if it is regular or has type* $(4,2)$ *with the anisotropic relative form defined below.*

Proof. Invariant lines lie in $\ker X$. Every line is isotropic, so uniqueness holds exactly when there is one Jordan block. In that case every invariant subspace of each dimension is the corresponding kernel in the single chain; the spaces of dimensions $1, 2, 3$ are isotropic.

For two-spaces in the nonregular cases, the following constructions give at least two. For $(4,2)$ and (3^2), the kernel is an isotropic two-space, and the bottom two vectors in one longer chain give a different invariant isotropic two-space. For $(4,1^2)$, combine the bottom vector of the four-chain with two different lines in the fixed symplectic plane. In each remaining case the kernel contains an isotropic three-space, and its distinct two-subspaces suffice.

For Lagrangians, partitions with a fixed nondegenerate summand give several by varying a Lagrangian in that summand while retaining one in its orthogonal complement. This covers $(4,1^2), (2^2,1^2), (2,1^4)$. For (3^2) the two paired three-chains themselves give distinct invariant Lagrangians.

For (2^3), $K = \ker X = \operatorname{im} X$ is Lagrangian. The form induced by $\langle v, Xv\rangle$ on V/K is a nondegenerate ternary quadratic form. Such a form is isotropic over $\mathbb{F}_q$. Choose $v \notin K$ with $\langle v, Xv\rangle = 0$. The plane $\langle v, Xv\rangle$ is isotropic and invariant. A vector in $(K \cap v^\perp) \setminus \langle Xv\rangle$ extends it to an invariant Lagrangian different from K.

It remains to analyze $(4,2)$. Choose orthogonal chains $a_1, \dots, a_4$ and b_1, b_2, with $Xa_i = a_{i-1}$, $Xb_2 = b_1$, and nonzero pairings

$$\langle a_1, a_4\rangle = a, \quad \langle a_2, a_3\rangle = -a, \quad \langle b_1, b_2\rangle = b.$$

The relative binary form is

$$F(s,t) = as^2 - bt^2. \tag{15.3}$$

Its anisotropy is independent of chain rescaling. There is always the invariant Lagrangian $W_0 = \langle a_1, a_2, b_1\rangle$.

An invariant Lagrangian W cannot contain a four-chain. If $X^2W = 0$, then $W \le \ker X^2 = \langle a_1, a_2, b_1, b_2\rangle$. Its isotropy forces it to contain the radical $\langle a_1, a_2\rangle$ of that four-space, and invariance forces its remaining line to be $\langle b_1\rangle$. Hence $W = W_0$.

Otherwise W is a cyclic three-chain generated by $v = \alpha a_3 + \beta b_2 + \gamma a_2 + \delta a_1 + \epsilon b_1$, with $\alpha \neq 0$. Its isotropy requires

$$\langle v, Xv\rangle = a\alpha^2 - b\beta^2 = 0.$$

The other chain pairings are already zero. Thus such a W exists exactly when a/b is a square. If a/b is nonsquare, W_0 is unique. If a/b is square, the two choices of β/α already give Lagrangians distinct from W_0. This completes the classification. □

Theorem 15.4. *For $S = \mathrm{PSp}_6(q)$, q odd, the cyclic subnormalizer of a nonidentity unipotent is proper exactly in these cases:*
(i) type (6)*: the subnormalizer is its unique Borel;*
(ii) type $(4,2)$ *with a/b nonsquare in* (15.3)*: the subnormalizer is its unique Siegel parabolic.*
There are two finite-group classes of each eligible type. In all other nonidentity unipotent classes the cyclic subnormalizer is S. For arbitrary $1 \neq Q \leq P$, an eligible element in a proper $H = \mathrm{Sub}_S(Q)$ must therefore be regular, or of type (ii); in case (ii), H is Siegel.

Proof. The cyclic subnormalizer contains a Borel. If it is proper, it lies in a maximal parabolic, and Lemma 15.1 implies eligibility there. Lemma 15.3 leaves only the stated types. A regular element has a unique invariant full isotropic flag, and hence lies in a unique Sylow subgroup, giving its Borel.

An element of type (ii) is eligible in its Siegel parabolic, so its cyclic subnormalizer is contained there. A smaller proper parabolic containing the chosen Borel would also lie in P_1 or P_2. Eligibility would descend to that maximal parabolic, contradicting Lemma 15.3. Thus the cyclic subnormalizer is Siegel. Among the four signed $(4,2)$ classes, exactly two have a/b nonsquare. The final assertion follows from the second part of Lemma 15.1 and maximality of the Siegel subgroup. □

16. The Siegel value lattice

16.1. **The local theorem.** Let $q = p^f$ be odd and let H be the Siegel parabolic of $\mathrm{PSp}_6(q)$. In the standard symplectic model,

$$H = \mathrm{Sym}_3(\mathbb{F}_q)^+ \rtimes (\mathrm{GL}_3(q)/\{\pm I\}), \qquad |H|_p = q^9. \tag{16.1}$$

Its abelian radical U satisfies $C_H(U) = U$: the kernel of the action of $\mathrm{GL}_3(q)$ on symmetric matrices is $\{\pm I\}$. Thus H has a single block, principal of full defect. Indeed the normal p-core is contained in every block defect group, so every block idempotent has nonzero Brauer image there; orthogonal nonzero images in the algebra of a p-group centralizer are impossible. Consequently the ordinary block height is

$$h(\chi) = v_p(\chi(1)). \tag{16.2}$$

No Clifford level is substituted for this height.

Fix a nontrivial additive character ψ of $\mathbb{F}_q$, let η be the quadratic multiplicative character of $\mathbb{F}_q^\times$ (extended by $\eta(0) = 0$), and set

$$G = \sum_{x \in \mathbb{F}_q} \psi(x^2), \qquad G^2 = \eta(-1)q, \qquad \omega = \frac{G-1}{2}, \qquad A_q = \mathbb{Z}[\omega]. \tag{16.3}$$

The two choices of the Gauss root exchange ω with $-1-\omega$ and give the same lattices below. For a nonregular eligible $u \in H$, write

$$L_0 = \sum_{h(\chi)=0} \mathbb{Z}\chi(u), \quad T_j = \sum_{h(\chi)\geq j} \mathbb{Z}\chi(u), \quad \mathcal{M}_r = \sum_{\chi \in \mathrm{Irr}(H)} p^{\max(0, r-h(\chi))}\mathbb{Z}\chi(u). \tag{16.4}$$

Theorem 16.1. *At either of the two anisotropic* $(4,2)$ *classes in H, a nonzero irreducible value has ordinary height zero or f, and both heights occur. Moreover,*

$$L_0 = T_0 = \mathbb{Z} + q\mathbb{Z}\omega, \qquad T_j = \begin{cases} qA_q, & 1 \leq j \leq f, \\ 0, & j > f, \end{cases} \tag{16.5}$$

and, for every integer $r \geq 0$,

$$\boxed{\mathcal{M}_r = p^r\mathbb{Z} + p^{\max(r,f)}\mathbb{Z}\omega.} \tag{16.6}$$

In particular $\mathcal{M}_r = p^r A_q$ once $r \geq f$.

Theorem 15.4 proves that every nonregular eligible element for a proper subnormalizer in $\mathrm{PSp}_6(q)$ has this type and forces the actual local group to be Siegel. Its signed-Jordan coordinates give the relative form $as^2 - bt^2$, anisotropic precisely when a/b is nonsquare. A unique invariant Lagrangian certifies eligibility. Thus Theorem 16.1 calculates the full local target in the only nonregular branch left by that geometry, for arbitrary Q.

16.2. **An explicit representative and Clifford reduction.** Write

$$U(B) = \begin{pmatrix} I_3 & B \\ 0 & I_3 \end{pmatrix}, \qquad L(C) = \mathrm{diag}(C, C^{-t}).$$

The Cayley representatives from the signed chains above take the form

$$u = U(B)L(C), \quad C = I_3 + 2E_{12}, \quad B = \begin{pmatrix} \gamma & \gamma & 0 \\ \gamma & \gamma & 0 \\ 0 & 0 & \delta \end{pmatrix}, \qquad \gamma = -2/a, \quad \delta = 2/b. \tag{16.7}$$

The two anisotropic classes have $(a, b) = (1, \nu), (\nu, 1)$ for a nonsquare ν. Equivalently $\eta(-\gamma\delta) = -1$. All calculations therefore use an actual representative of the specified eligible class, not merely a global table position.

Parametrize $\mathrm{Irr}(U)$ by symmetric matrices S, with $\lambda_S(U(B)) = \psi(\mathrm{Tr}(SB))$. The dual action is $S \mapsto T^t S T$. There are seven orbits: zero and two of each nonzero rank. A linear radical character extends trivially to its stabilizer. Let K_S be that stabilizer in $\mathrm{GL}_3(q)$ and retain $\rho \in \mathrm{Irr}(K_S)$ with $\rho(-I) = \rho(1)$. Abelian Clifford theory gives every irreducible of H, with

$$\chi(1) = [\mathrm{GL}_3(q) : K_S]\rho(1), \qquad \chi(u) = \sum_{\substack{S' \in S^{\mathrm{GL}_3(q)} \\ CC^t S' C = S'}} \psi(\mathrm{Tr}(S'B))\rho(TCT^{-1}), \quad T^t S T = S'. \tag{16.8}$$

Solving $C^t S' C = S'$ gives exactly

$$S' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & y \\ 0 & y & z \end{pmatrix}, \qquad \mathrm{Tr}(S'B) = \gamma x + \delta z. \tag{16.9}$$

In particular no rank-three orbit contributes a nonzero value. We treat the other ranks separately.

Rank zero. These characters inflate from $\mathrm{GL}_3(q)/\{\pm I\}$. Values at the transvection C are integers, since its prime-to-p powers are conjugate inside $\mathrm{GL}_3(q)$. The standard GL_3 table has only these p-divisible degree types:

degree	p-part	value at C
$q(q+1)$	q	q
$q(q^2+q+1)$	q	q
q^3	q^3	0.

(16.10)

These are the Steinberg–Green data recorded in the primary generic table [34]; the classical origins are [31, 13]. Determinant twists do not change the displayed value. The untwisted character of degree $q(q+1)$ descends through $-I$, so a value q actually occurs at ordinary height f.

16.3. **Rank one: generators and the height-zero lattice.** A rank-one fixed form is $\sigma v v^t$ on the last two coordinates, where $v = (s, t) \neq 0$ and σ is a representative square class. The parameter v is taken modulo $\{\pm 1\}$. Its phase is $\psi(\sigma(\gamma s^2 + \delta t^2))$. Define

$$A = -\frac{q+1}{2}, \qquad W_\sigma = \frac{-1 + \eta(\sigma\gamma)G}{2}. \tag{16.11}$$

The sum of all phases is A, because the binary form is anisotropic and its complete additive sum is $-q$. The sum on the axis $t = 0$, excluding zero and dividing by two, is W_σ.

The rank-one orbit index is $(q^3 - 1)/2$, prime to p, and its projective inertia quotient is $\mathrm{AGL}_2(q)$. For $t \neq 0$ the transported C induces a nontrivial transvection on the radical plane of S'; its affine translation can be removed by conjugation. Indeed its full three-dimensional Jordan type is $(2, 1)$, so the translation lies in the image of that transvection minus identity. For $t = 0$ it is a nonzero pure affine translation.

For an inertia character ρ, let $D = \rho(1)$, R its value at the linear transvection, and V its value at a nonzero translation. Formula (16.8) becomes

$$\chi(u) = RA + (V - R)W_\sigma. \tag{16.12}$$

The complete list of possible triples is

inertia family	D	R	V
inflated linear	1	1	1
inflated Steinberg	q	0	q
inflated principal series	$q+1$	1	$q+1$
inflated cuspidal	$q-1$	-1	$q-1$
noninflated, linear little row	q^2-1	$q-1$	-1
noninflated, nonlinear little row	$(q^2-1)(q-1)$	$-(q-1)$	$-(q-1)$.

(16.13)

The first four rows use the ordinary GL_2 table [34]. For the last two, use Clifford theory for the normal affine plane. Its nonzero dual orbit has size q^2-1 and vector stabilizer $\mathrm{AGL}_1(q)$. That stabilizer has $q-1$ linear rows and one row of degree $q-1$, taking values 1 and -1 at a nonzero translation. At the linear transvection, $q-1$ dual vectors are fixed; at a nonzero affine translation their full Fourier sum is -1. These give the displayed R, V without requiring a generic parabolic table.

The only positive degree p-part in (16.13) is q. It contributes exactly qW_σ at ordinary height f. Every other row has $V - R$ divisible by q, so its value lies in $\mathbb{Z} + q\mathbb{Z}\omega$. A principal-series inertia row exists for every odd $q \geq 3$ and has value $A + qW_\sigma$. Together with the trivial character, this supplies $q\omega$ in the height-zero lattice. The two square classes exchange $W_\sigma = \omega$ and $-1 - \omega$.

Thus rank one both supplies the nonintegral direction needed in (16.5) and gives its positive-height counterpart. It remains to show that rank two does not enlarge either proposed lattice.

16.4. **Rank two: exact Fourier transforms.** Put $A_0 = \left(\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}\right)$ and let $\epsilon = \eta(-\det A_0) \in \{+1, -1\}$ distinguish the split and anisotropic binary types. The full rank-two inertia group has the form

$$N \rtimes (O_2^\epsilon(q) \times \mathbb{F}_q^\times), \qquad N \cong \mathbb{F}_q^2. \tag{16.14}$$

After taking the common central quotient, every inertia character still has p'-degree: N is abelian and its complement has p'-order. The global dual orbit index is

$$[\mathrm{GL}_3(q) : K_S] = \frac{q(q+\epsilon)(q^3-1)}{2}, \tag{16.15}$$

with p-part exactly q. Hence every rank-two induced row has ordinary height f.

For a fixed A_0, the element C is in N and pairs with a dual vector $w = (w_1, w_2)$ as $\psi(2w_1)$. The nonzero dual orbits in N^* are classified by $\eta(w^t A_0 w) = +1, -1$, and also by the nonzero isotropic vectors in the split case. Restriction of an inertia character to N is the corresponding orbit sum, multiplied by the integer degree of its little row. Define

$$F_\epsilon = \sum_{\eta(-\det A_0)=\epsilon} \psi(\gamma x + \delta z),$$

$$D_{\epsilon,k} = \sum_{\eta(-\det A_0)=\epsilon} \psi(\gamma x + \delta z) \sum_{\substack{w \neq 0 \\ \eta(w^t A_0 w)=k}} \psi(2w_1), \qquad k = 0, +1, -1.$$

Thus all rank-two values are integer multiples of the following transforms.

Lemma 16.2. *For $\eta(-\gamma\delta) = -1$,*

$$F_\epsilon = \frac{q}{2}(1 - \epsilon), \qquad D_{\epsilon,0} = 0, \tag{16.16}$$

and

$$D_{\epsilon,+} + D_{\epsilon,-} = -F_\epsilon, \qquad D_{\epsilon,+} - D_{\epsilon,-} = \frac{q\eta(\gamma)}{2}(1 - \epsilon\eta(-1)q)G. \tag{16.17}$$

Every transform in (16.16)–(16.17) *belongs to qA_q.*

Proof. The full Fourier sum over symmetric binary matrices is zero. The rank-zero contribution is one, and the two rank-one contributions are each $-(q+1)/2$. Therefore $F_+ + F_- = q$. Summing the determinant sign instead gives $F_+ - F_- = -q$: for $z \neq 0$ substitute $d = xz - y^2$ and use two one-dimensional Gauss sums; the remaining sum $\sum_{z\neq 0} \psi(\delta z)$ is -1. The terms with $z = 0$ sum to zero because $\gamma \neq 0$. This proves the first formula in (16.16).

For a nonsingular A_0, completion of the square gives

$$\sum_{w\in\mathbb{F}_q^2} \eta(w^t A_0 w)\psi(2w_1) = q\eta(z). \tag{16.18}$$

One direct derivation expands $\eta(t)$ as $G^{-1}\sum_{r\neq 0}\eta(r)\psi(rt)$, then completes the square in w. The resulting one-dimensional Gauss sum uses $(A_0^{-1})_{11} = z/\det A_0$ and $G^2 = \eta(-1)q$, giving (16.18), including zero when $z = 0$.

It follows that the difference in (16.17) is qE_ϵ, where

$$E_\epsilon = \sum_{\eta(-\det A_0)=\epsilon} \eta(z)\psi(\gamma x + \delta z) = \frac{\eta(\gamma)G}{2}(1 - \epsilon\eta(-1)q). \tag{16.19}$$

For the last identity, $z = 0$ contributes zero. For $z \neq 0$, write $x = (d + y^2)/z$, with $\eta(d) = \epsilon\eta(-1)$. The y-sum is $\eta(\gamma/z)G$; the restricted d-sum is $(-1 + \epsilon\eta(-1)\eta(\gamma/z)G)/2$. Summing over z and using $\eta(\gamma\delta) = -\eta(-1)$ gives (16.19).

In the anisotropic case there are no nonzero isotropic vectors. In the split case their Fourier sum at $(2, 0)$ is $q\mathbf{1}_{z=0} - 2$. Its total against the outer phase is zero: the $z = 0$ sum vanishes and $F_+ = 0$. Thus $D_{\epsilon,0} = 0$. Finally the Fourier sum over all $w \neq 0$ is -1, proving the sum formula in (16.17).

For $\epsilon = +1$, the two nonzero-norm transforms are

$$\pm q\eta(\gamma)\frac{1 - \eta(-1)q}{4}G \in q\mathbb{Z}[G],$$

since $(1 - \eta(-1)q)/4$ is an integer. For $\epsilon = -1$ they are

$$q\frac{-1 \pm mG}{2}, \qquad m = \eta(\gamma)\frac{1 + \eta(-1)q}{2},$$

and m is odd. Both therefore lie in $q\mathbb{Z}[\omega]$. □

Rank two can only contribute height f and values in qA_q. Rank zero and rank one already supply q and $q\omega$ at height f, while all higher-height values vanish. They also supply the full proposed height-zero lattice, with every remaining value lying inside it. This proves (16.5). The weighted sum now is

$$p^r(\mathbb{Z} + q\mathbb{Z}\omega) + p^{\max(0,r-f)}qA_q = p^r\mathbb{Z} + p^{\max(r,f)}\mathbb{Z}\omega,$$

proving Theorem 16.1.

17. Regular unipotent elements

Theorem 17.1. *Let $q = p^f$ be odd. Let H be any proper parabolic of $\mathrm{Sp}_6(q)$ or its image in $\mathrm{PSp}_6(q)$, and let $u \in H$ be regular unipotent in the ambient six-dimensional group. If $\chi \in \mathrm{Irr}(H)$ and $p \mid \chi(1)$, then $\chi(u) = 0$.*

We use the following lower-rank degree-vanishing inputs. In $\mathrm{GL}_2(q)$ and $\mathrm{SL}_2(q)$ only Steinberg types have p-divisible degree; they vanish at nontrivial unipotents. Their Borels have only p'-degrees. In $\mathrm{GL}_3(q)$, p-divisible-degree characters vanish at a regular unipotent by Green–Lehrer–Lusztig, as recalled in [20, proof of Proposition 4.1]. In $\mathrm{Sp}_4(q)$ and its Borel, the same vanishing is stated in [20, proof of Proposition 4.6]. The connected-center hypothesis of Proposition 4.1 is used for GL_3, *not* for Sp_6.

The rank-two local lemmas above give the same vanishing for both maximal parabolics of $\mathrm{Sp}_4(q)$. Briefly, in its Siegel parabolic a regular Levi image is a nontrivial GL_2 unipotent: rank-two symmetric-form inertia groups have p'-order and cannot contain it; rank-one Clifford rows have p'-degree; inflated Steinberg rows vanish. In its Klingen parabolic the Heisenberg central-shift argument used below eliminates nontrivial central characters, and the other noninflated rows have

p'-degree. These proofs apply before quotienting by the center, so all central-character sectors needed below are covered.

A regular six-chain has a unique invariant subspace of each dimension. Its induced actions on the natural Levi quotients of the invariant isotropic flag are single chains. This justifies the regular or nontrivial unipotent Levi images used in all three arguments below.

17.1. **Siegel: P_3.** Use the symmetric-form Clifford parametrization of Section 16, with a nontrivial additive character ψ of $\mathbb{F}_q$. Normalize the regular Levi image as

$$C = (I+Y)(I-Y)^{-1}, \qquad Y = E_{12} + E_{23}.$$

For $u = U(B)L(C)$, a direct multiplication gives the top-right 3×3 block of $(u-I)^5$ as $16B_{33}E_{11}$. Thus regularity means $B_{33} \neq 0$. The fixed symmetric forms are exactly

$$S(a,b) = \begin{pmatrix} 0 & 0 & a \\ 0 & -a & 0 \\ a & 0 & b \end{pmatrix}. \tag{17.1}$$

There are no rank-two forms in (17.1). The rank-zero rows inflate from $\mathrm{GL}_3(q)$ and their positive-degree values vanish by the input above.

For rank one, the full form stabilizer modulo $\{\pm I\}$ is $\mathrm{AGL}_2(q)$; before this quotient it is the direct product with the central sign, which does not affect degrees or vanishing. The orbit index $(q^3-1)/2$ is prime to p. In $\mathrm{AGL}_2(q)$ all p-divisible-degree rows are inflated Steinberg twists of $\mathrm{GL}_2(q)$: the other Clifford orbit has size q^2-1 and its vector stabilizer has an abelian normal q-subgroup and a p'-torus. The induced action of C on the radical plane of a fixed rank-one form is nontrivial unipotent. Thus these positive-degree values vanish term by term in the Clifford value formula.

For rank three, put $D_c = I + cY^2$. It commutes with C and

$$D_c^t S(a,b) D_c = S(a, b+2ca).$$

For fixed $a \neq 0$, varying c runs through every $b \in \mathbb{F}_q$ within one dual orbit. Choose transporters $T_b = T_0 D_c$; then $T_b C T_b^{-1}$ is constant. The phase in the Clifford value formula is

$$\psi\big(a(2B_{13} - B_{22}) + bB_{33}\big).$$

Its sum over b is zero. Hence all rank-three induced values vanish, irrespective of degree. This proves Theorem 17.1 for P_3.

17.2. **Klingen and the line–Lagrangian flag: P_1, P_{13}.** Use block sizes $(1,4,1)$ and a nondegenerate alternating matrix J on the middle four-space. The common radical is

$$Q(v,z) = \begin{pmatrix} 1 & v^t J & z \\ 0 & I_4 & v \\ 0 & 0 & 1 \end{pmatrix}, \qquad u = Q(v,z)\,\mathrm{diag}(1,A,1). \tag{17.2}$$

The remaining middle group M is $\mathrm{Sp}_4(q)$ for P_1, and a Siegel parabolic of $\mathrm{Sp}_4(q)$ for P_{13}. Its Levi factor here is $(\mathbb{F}_q^\times \times M)/\{\pm I\}$ in the projective case. For regular u, A is regular on the four-space and $v \notin \mathrm{im}(A-I)$. Put $\ell = \ker(A-I)$; then $\mathrm{im}(A-I) = \ell^\perp$.

For $w \in \ell$ direct multiplication gives

$$u^{Q(w,0)} = Q(0, 2v^t J w)u. \tag{17.3}$$

The scalar runs over all of $\mathbb{F}_q$. In a Clifford inertia group over a nontrivial central radical character, that central character is scalar and the inertia group contains the radical. Conjugacy in (17.3) forces every value at regular u to vanish. For an induced character the same applies to every conjugate contributing to its induction formula. This argument requires no extension or normalization of a Weil character.

The remaining characters factor through the abelian radical quotient of dimension four. Identify its dual with the natural module by J. Its nonzero fixed vectors lie on ℓ. Their orbit size is q^4-1 for $M = \mathrm{Sp}_4(q)$ and q^2-1 for M Siegel; in the latter case ℓ is inside the fixed Lagrangian plane. These indices are prime to p. The inertia quotient is the stabilizer of the line ℓ in M, up to the

central quotient: respectively a rank-two Klingen parabolic or a rank-two Borel. The scalar factor is determined by the action on that line. Transported A remains regular on the four-space. The lower-rank vanishing inputs therefore eliminate every positive-degree little-group row. The inflated rows vanish by the corresponding input for M. This proves the two cases.

17.3. **The middle family: P_2, P_{12}, P_{23}, B.** Use blocks $(2,2,2)$ and $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The common radical is

$$Q(V,Z) = \begin{pmatrix} I_2 & V^tJ & Z + \frac{1}{2}V^tJV \\ 0 & I_2 & V \\ 0 & 0 & I_2 \end{pmatrix}, \qquad Z = Z^t. \tag{17.4}$$

The remaining Levi group is $L_A \times L_B$, modulo the common sign in the projective case, with L_A either $\mathrm{GL}_2(q)$ or its upper Borel, and L_B either $\mathrm{SL}_2(q)$ or its upper Borel. The four choices are precisely the four groups in this subsection. The action on V is $V \mapsto BVA^t$ and on Z is $Z \mapsto AZA^t$.

For a regular element normalize its Levi coordinates as

$$A = I + tE_{12}, \quad B = I + sE_{12}, \quad t, s \neq 0, \qquad u = Q(V,Z)\,\mathrm{diag}(A, B, A^{-t}).$$

The top-right block of $(u-I)^5$ is $t^2 s V_{22}^2 E_{11}$. Thus $V_{22} \neq 0$.

A central radical character fixed by u has a symmetric dual matrix satisfying $A^tSA = S$, hence $S = cE_{22}$. If it is nontrivial then $c \neq 0$. For any $d \in \mathbb{F}_q$ put

$$W = \begin{pmatrix} 0 & d \\ -td/s & 0 \end{pmatrix}.$$

It satisfies $BWA^t = W$ and direct multiplication yields

$$u^{Q(W,0)} = Q(0, \Delta Z)u, \quad \Delta Z = V^tJW - W^tJV, \quad \Delta Z_{22} = -2V_{22}d. \tag{17.5}$$

As d varies the character's scalar phase is nontrivial. The same Clifford central-shift argument as in (17.3) forces all these values to be zero.

After factoring out the center the radical is the abelian space $\mathrm{Mat}_2(q)$. Pair its dual by $\psi(\mathrm{Tr}(T^tV))$; the dual action is $T \mapsto B^tTA$. The fixed matrices are

$$T(y,z) = \begin{pmatrix} 0 & y \\ -sy/t & z \end{pmatrix}. \tag{17.6}$$

For rank two, $y \neq 0$. The element $D = I + dE_{12} \in L_A$ centralizes A and sends z to $z - (s/t)yd$. Hence, within every relevant orbit, each fixed y occurs with every z. Choosing the corresponding transporters by right multiplication with D leaves the conjugated Levi element constant. Summing the phases

$$\psi(yV_{12} - (s/t)yV_{21} + zV_{22})$$

over z gives zero. Thus every rank-two induced value vanishes.

For rank one all nonzero fixed matrices zE_{22} are in one orbit. Their stabilizer before the common central quotient consists of

$$A = \begin{pmatrix} a & c \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \lambda^{-1} & b \\ 0 & \lambda \end{pmatrix}, \qquad a, \lambda \neq 0. \tag{17.7}$$

It lies in both Borels, so the same stabilizer works for all four Levi choices. It has an abelian normal subgroup of order q^2 and a p'-torus complement. All its character degrees are prime to p. Its orbit index is $(q-1)(q+1)^2$, q^2-1, or $q-1$, according as two, one, or zero full rank-one factors are present. These indices are also prime to p, so no rank-one induced row has positive p-degree.

Finally, rank-zero rows inflate from $L_A \times L_B$. A p-divisible tensor factor is Steinberg in a full factor, and vanishes at the nontrivial unipotent A or B; Borel factors have only p'-degree. This proves the last four cases and completes Theorem 17.1.

Proposition 17.2. *Every irreducible character of $\mathrm{Sp}_6(q)$ of positive degree p-part vanishes at every regular unipotent element, for every odd q. Consequently B(c) holds in every regular eligible branch of S.*

Proof. There are two ordinary Gelfand–Graev characters Γ_1, Γ_2. Their Alvis–Curtis duals are supported on the two regular rational unipotent classes and are independent, by Digne–Lehrer–Michel [4, Theorem 2.12], as restated in [2, Theorem 14.7]. They therefore span the two regular class indicator functions.

Every irreducible constituent of $D\Gamma_i$ has p'-degree. One way to check the degree assertion without a connected-center assumption is to induce to $\mathrm{CSp}_6(q)$. Both Γ_i induce to its ordinary Gelfand–Graev character. Its dual has only semisimple constituents, of p'-degree. Duality commutes with restriction through the regular embedding; every constituent under discussion lies under one of these semisimple characters. The restriction has one or two constituents of equal degree, so their degree p-parts are still one, since p is odd.

A positive-degree-p-part irreducible is orthogonal to both indicator functions and hence vanishes at each regular class separately. Theorem 17.1 gives the same local vanishing. Both weighted modules are therefore p^r times their unweighted modules, and the unconditional unweighted projection theorem proves equality. □

18. The symplectic rank-three family

18.1. **The conformal table.**

Proposition 18.1. *Let $\theta \in \mathrm{Irr}(\mathrm{CSp}_6(q))$, with $p \mid \theta(1)$. At a unipotent element u of type $(4,2)$, a nonzero value implies*

$$\theta(1)_p = q, \qquad \theta(u) \in \{q, -q, 2q, -2q\}.$$

At a regular unipotent, every such character vanishes.

Proof. Use the complete generic table `CSp6.1.jl` from [34]. Its first ten columns have central semisimple part; setting the class parameter $i_1 = 0$ makes that part the identity. In these columns, every cyclotomic exponent vanishes. Summing its polynomial coefficients therefore gives an exact polynomial in q, independently of the character parameters. Column 1 is the degree, columns 8 and 9 are the two conformal classes of type $(4,2)$, and column 10 is regular.

For completeness, the following list gives every type of degree divisible by p with a nonzero entry in columns 8 or 9. Type numbers are those of the generic table; they label families, not individual characters.

Column 8	Column 9	Character types
q	0	2, 3, 36, 38
0	q	4, 5, 37, 39
q	q	14, 24, 26, 48, 54, 55, 59, 74, 86
$-q$	$-q$	19, 51, 62, 63, 66, 78, 84
$2q$	0	25, 58, 76, 88
0	$-2q$	32, 82
0	$-q$	42, 44
$-q$	0	43, 45
$-2q$	0	67, 80
0	$2q$	90

All these 37 types have degree q-exponent one. Every other type of positive degree q-exponent has zero in both columns. Column 10 is zero for every positive exponent. Across all 100 types the degree exponents are $0, 1, 2, 3, 4, 6, 9$. After removing that power of q, the constant coefficient is ± 1 or $\pm 1/2$, a unit at every odd prime. Hence the exponent records the degree p-part.

The source package contains the pinned generic table, its SHA-256 digest, the 100-row extraction and an exact polynomial verifier in `certificates/conformal/`. The verifier checks all first-ten-column entries, including the degrees. Empty parameter ranges at small q simply remove types; the polynomial identities remain valid. □

Restriction from $\mathrm{CSp}_6(q)$ to $\mathrm{Sp}_6(q)$ is multiplicity free, since the quotient is cyclic. The conjugation action factors through $\mathrm{CSp}_6(q)/(Z(\mathrm{CSp}_6(q))\,\mathrm{Sp}_6(q))$, of order two, so a restriction is

irreducible or the sum of two diagonal conjugates. Degree p-parts are unchanged, as p is odd. Proposition 18.1 therefore controls the irreducible restrictions. For a split restriction it controls only $\chi(u) + \chi^{\delta}(u)$: zero can result from cancellation. In particular it does not prove the corresponding constituent-wise higher-height vanishing or q-divisibility.

The same degree calculation gives one further global restriction: all degree p-parts in $\mathrm{Sp}_6(q)$, and hence in its simple quotient, belong to $\{1, q, q^2, q^3, q^4, q^6, q^9\}$. In a visible full-defect principal block the possible ordinary heights are consequently contained in

$$\{0, f, 2f, 3f, 4f, 6f\}.$$

The case of p-part q^9 has character defect zero and lies outside that block. Thus possible nonzero values above height f at the eligible classes must be ruled out for split restrictions of heights $2f, 3f, 4f, 6f$.

18.2. **Characters supplied by Harish–Chandra induction.** Throughout this section, η_q is the quadratic character of $\mathbb{F}_q^{\times}$ and $\epsilon = \eta_q(-1)$. Let ψ run over two additive characters in different square classes. In the Schrödinger model let $\rho^{\epsilon}_{m,\psi}$ be the parity ϵ half of the Weil representation of $\mathrm{Sp}_{2m}(q)$. Its degree is $(q^m + \epsilon)/2$. The central element $-I$ acts on it as ϵ^{m+1}. Let P_1 be the Klingen parabolic and let η be the quadratic character of its $\mathrm{GL}_1(q)$ Levi factor. Define

$$R_{\psi} = R_{P_1}^{\mathrm{Sp}_6(q)}(\eta \boxtimes \rho^{\epsilon}_{2,\psi}). \tag{18.1}$$

Both the inducing representation and R_{ψ} have trivial central sign, since $\eta(-1) = \epsilon$ and $\rho^{\epsilon}_{2,\psi}(-I) = \epsilon$. They therefore define characters on the appropriate projective groups.

Lemma 18.2. *There are distinct irreducibles $\alpha_{\psi}, \beta_{\psi}$ such that*

$$R_{\psi} = \rho^{\epsilon}_{3,\psi} + \alpha_{\psi} + \beta_{\psi}. \tag{18.2}$$

Their respective degrees are

$$d_0 = \frac{q^3 + \epsilon}{2}, \quad d_1 = \frac{q(q+\epsilon)(q^4 + q^2 + 1)}{2}, \quad d_2 = \frac{q^2(q^2+1)(q^3+\epsilon)}{2}. \tag{18.3}$$

All three belong to the principal block, with ordinary heights $0, f, 2f$.

Proof. Use the regular embedding into the conformal group. The relevant quadratic series has connected dual centralizer of type D_3^{ϵ}; in the Klingen Levi the connected centralizer has type D_2^{ϵ}. Under $D_3 = A_3$, unipotent induction of the trivial character from this D_2 Levi gives the partitions $(4), (31), (22)$, each once. In the split case this is the rule $\mathrm{Ind}_{\mathfrak{S}_2 \times \mathfrak{S}_2}^{\mathfrak{S}_4} 1 = [4] + [31] + [22]$; its Frobenius-stable unitary version gives the same three labels. Compatibility of Harish–Chandra induction with Jordan decomposition is used here only for *classical groups with connected center*, as in Fong–Srinivasan [8] and Geck–Malle, Theorem 4.7.2 [9]. No general disconnected-center compatibility conjecture is used.

The three conformal degrees are $2d_0, 2d_1, 2d_2$. They are families 13–15 if $\epsilon = 1$, and 18–20 if $\epsilon = -1$, in the generic conformal table [34]. The order-two component action fixes each of the five unipotent labels of D_3^{ϵ}: their degrees are distinct. The associated conformal characters therefore each restrict into a diagonal pair. Restriction of the induced conformal character is $R_{\psi} + R_{\psi'}$, where ψ' is in the other square class. Each member of each of the three pairs has total coefficient one in that sum. Diagonal conjugation exchanges $R_{\psi}, R_{\psi'}$, so each contains exactly one member of each pair. The Weil Jacquet module identifies the degree-d_0 member as $\rho^{\epsilon}_{3,\psi}$: radical invariants are the parity-ϵ Weil space in dimension four, with the GL_1 factor η.

The degree identity

$$\frac{q^2 + \epsilon}{2} \frac{q^6 - 1}{q - 1} = d_0 + d_1 + d_2$$

also checks exhaustiveness. The factors after removing $1, q, q^2$ in (18.3) are p-units. None of these degrees has p-part q^9, so none is the Steinberg character. The stated principal-block heights follow. □

Lemma 18.3. *At either anisotropic $(4,2)$ element, the two characters $\Phi_\psi = R_\psi - \rho^\epsilon_{3,\psi} = \alpha_\psi + \beta_\psi$ have values*

$$F_+ = \frac{q(\epsilon + g_q)}{2}, \qquad F_- = \frac{q(\epsilon - g_q)}{2} \tag{18.4}$$

in some order.

Proof. Take signed Cayley chains $Xa_i = a_{i-1}$, $Xb_2 = b_1$ with $\langle a_1, a_4\rangle = a$, $\langle a_2, a_3\rangle = -a$, $\langle b_1, b_2\rangle = b$, and a/b nonsquare. Then $u = 1 + 2X + 2X^2 + 2X^3$. Its fixed lines are $\ell_t = \langle a_1 + tb_1\rangle$ for $t \in \mathbb{F}_q$, and $\ell_\infty = \langle b_1\rangle$.

For finite t, use the quotient basis $w = b_1, x = a_2, y = a_3, z = tba_4 - ab_2$ of $\ell_t^\perp/\ell_t$. Writing $N = u - 1$ on this quotient gives

$$Nw = 0, \quad Nx = -2tw, \quad Ny = 2x - 2tw, \quad Nz = 2tby + 2tbx - 2(t^2b + a)w. \tag{18.5}$$

For $t = 0$ this is type $(2,2)$, with Weil trace $-q$: its two-dimensional quadratic phase has diagonal coefficients $-2/a, 2/b$ and $\eta_q(4/(ab)) = -1$. For $t \neq 0$ it is regular of type (4) and

$$\langle N^3 z, z\rangle = 8ab^2t^2. \tag{18.6}$$

At ℓ_∞, the regular quotient is the a-chain and the corresponding quantity is $8a$. These coefficients differ by squares. The finite-field Weil trace formula [35] therefore gives the *same* trace σg_q on all q regular quotients. Replacing ψ by a nonsquare multiple changes σ.

The total Weil trace on the six-dimensional element is $-q$. This also follows directly in the Siegel coordinates above: the fixed-vector phase sum is the anisotropic binary Gauss sum. On a unipotent element the parity-ϵ half has trace (total Weil trace $+\,\epsilon$)/2. The GL_1 component at each fixed line is 1. The induction formula now gives

$$R_\psi(u) = \frac{-q + \epsilon}{2} + q\frac{\sigma g_q + \epsilon}{2}, \qquad \rho^\epsilon_{3,\psi}(u) = \frac{-q + \epsilon}{2}.$$

Subtracting proves (18.4). □

18.3. **The simple group.**

Theorem 18.4. *For every odd prime power $q = p^f$ and the data above, $\mathcal{M}_r(S, b; u) = \mathcal{M}_r(H, c; u)$ for every integer $r \geq 0$. Consequently B(c), whose requested range is $r \geq 2$, holds for this entire infinite simple family at its defining prime.*

We first recall the reductions proved above. If $H = S$, the assertion is immediate. Otherwise H is a parabolic and eligible u has type (6), or type $(4,2)$ with anisotropic relative binary form. In the latter case the *actual* H is the Siegel parabolic. These facts apply to arbitrary Q, not merely to cyclic subgroups. The local block is principal of full defect $9f$, and its actual global partner is principal. The global heights under consideration therefore equal $v_p(\chi(1))$. The other defining-characteristic global block is the defect-zero Steinberg block. Section 17 treats the regular branch using dual ordinary Gelfand–Graev characters; every positive-height local or global character vanishes there.

It remains to consider an anisotropic $(4,2)$ element. Choose

$$\epsilon = \eta_q(-1), \qquad g_q^2 = \epsilon q, \qquad \omega_q = (g_q - 1)/2, \qquad A_q = \mathbb{Z}[\omega_q]. \tag{18.7}$$

Either choice of Gauss-root sign gives the same order. If f is even, $A_q = \mathbb{Z}$. Theorem 16.1 gives

$$\mathcal{M}_0(H, c; u) = \mathbb{Z} + q\mathbb{Z}\omega_q, \qquad \mathcal{M}_r(H, c; u) = p^r\mathbb{Z} + p^{\max(r,f)}\mathbb{Z}\omega_q. \tag{18.8}$$

Nonzero local values occur only at heights $0, f$; the positive-height values generate qA_q. The unconditional unweighted projection theorem of Proposition 5.7 gives

$$\chi(u) \in \mathbb{Z} + q\mathbb{Z}\omega_q \subseteq A_q \quad \text{for every relevant global } \chi. \tag{18.9}$$

18.4. **The two subregular sign functions.** Work temporarily in $G = \mathrm{Sp}_6(q)$ and inflate simple-group characters. Let δ be a nonsquare diagonal automorphism. A Cayley representative of type $(4,2)$ has skew-adjoint chains $a_1, \dots, a_4$ and b_1, b_2,

$$Xa_i = a_{i-1}, \quad Xb_2 = b_1, \quad \langle a_1, a_4\rangle = a, \quad \langle a_2, a_3\rangle = -a, \quad \langle b_1, b_2\rangle = b,$$

with $a, b \neq 0$, and $u = (1+X)(1-X)^{-1}$. There are four rational classes, indexed by $(\eta(a), \eta(b))$. The automorphism δ reverses both signs. Anisotropy means $\eta(a/b) = -1$.

Use the following real sign functions, extended by zero away from their indicated geometric classes:

function	Y_A	Y_C	Y_T	Y_R	Y_D
class	42	411	21111	6	222
value	$\epsilon\eta(a)$	$\epsilon\eta(a)$	$\eta(a)$	$\eta(a)$	$\epsilon\eta(abc)$

(18.10)

For 222, the three length-two chain coefficients are a, b, c. For 411, the remaining two-dimensional summand is fixed, and for 21111 the remaining four-dimensional summand is fixed. On 42 also put $Y_B = \eta(b)$, up to an immaterial overall sign. The two functions Y_A, Y_B span its diagonally anti-invariant functions. Each has squared norm q^{-5}, since every 42 centralizer in G has order $4q^5$. The other needed norms are

$$\|Y_C\|^2 = \frac{1}{q^5(q^2-1)}, \qquad \|Y_D\|^2 = \frac{1}{q^7(q^2-1)}. \tag{18.11}$$

The diagonally anti-invariant unipotent space has dimension six: the five functions in (18.10), and Y_B.

18.5. **Direct Alvis–Curtis duality from finite flags.** Let f_2 be the regular sign function on $\mathrm{Sp}_2(q)$, normalized by the Weil trace divided by g_q. On $\mathrm{Sp}_4(q)$ let f_{4R} be the analogous function on its regular classes. These have norms q^{-1} and q^{-2}. Write e_I, e_J for the indicator functions of the identity and regular unipotent classes in $\mathrm{GL}_2(q)$, and e_1 for the identity indicator in $\mathrm{GL}_1(q)$. Define actual finite Harish–Chandra inductions

$$\begin{aligned} F_1 &= R^G_{\mathrm{GL}_1 \times \mathrm{Sp}_4}(e_1 f_{4R}), \\ B &= R^G_{\mathrm{GL}_2 \times \mathrm{Sp}_2}(e_I f_2), \qquad C = R^G_{\mathrm{GL}_2 \times \mathrm{Sp}_2}(e_J f_2), \\ F_0 &= R^G_{\mathrm{GL}_1^2 \times \mathrm{Sp}_2}(e_1^2 f_2) = (q+1)B + C. \end{aligned} \tag{18.12}$$

The last equality is the elementary GL_2 flag count.

Their values on the anti-invariant classes are as follows; each entry is the coefficient of the corresponding function in (18.10).

	Y_A	Y_C	Y_T	Y_R	Y_D
F_1	q	$q(q+1)$	0	1	0
B	1	$q+1$	$q^2(q+1)(q^2+1)$	0	q
C	-1	q^2-1	0	1	$q^2(q+1)$
F_0	q	$2q(q+1)$	$q^2(q+1)^2(q^2+1)$	1	$q(q+1)^2$

(18.13)

All three functions are zero on the remaining, diagonally fixed unipotent classes. On 42 they depend only on a, hence have no Y_B component.

For completeness, these counts hold over an arbitrary odd finite field. On 42, the $q+1$ fixed lines consist of one giving quotient 22 and q giving quotient 4, of sign $\epsilon\eta(a)$. The fixed two-planes on which u acts identically consist of $\langle a_1, b_1\rangle$, with quotient sign $\epsilon\eta(a)$. The J_2 two-planes are $\langle a_1, a_2 + tb_1\rangle$, $t \in \mathbb{F}_q$. Their regular quotient signs are $\eta((a - t^2 b)/(ab))$, interpreted as zero when the quotient is the identity. Their sum is $-\epsilon\eta(a)$, by the elementary quadratic character sum.

On 411, there are $q(q+1)$ regular line quotients and $q+1$ identity two-planes with regular quotient. The q^2 J_2 two-planes give one identity quotient and q^2-1 equal regular signs. On 222, no line quotient is regular of dimension four. Its isotropic J_2 planes are parametrized by the $q+1$ singular lines of the nondegenerate ternary multiplicity form, with q^2 choices over each; all quotient signs equal $\epsilon\eta(abc)$. Summing over identity planes gives $q\epsilon\eta(abc)$, the standard ternary quadratic character sum. On 21111, no invariant isotropic J_2 plane exists; identity planes with regular quotient are graphs over Lagrangians in the fixed symplectic four-space, giving $q^2(q+1)(q^2+1)$. A

regular 6 has its unique invariant line and two-plane. This proves (18.13), rather than extrapolating it from samples.

On this anti-invariant space all restrictions to GL-only Levis vanish. The alternating finite-parabolic definition of duality is therefore

$$D = -1 + R_{\mathrm{GL}_1\,\mathrm{Sp}_4}\,\mathrm{Res}_{\mathrm{GL}_1\,\mathrm{Sp}_4} + R_{\mathrm{GL}_2\,\mathrm{Sp}_2}\,\mathrm{Res}_{\mathrm{GL}_2\,\mathrm{Sp}_2} - R_{\mathrm{GL}_1^2\,\mathrm{Sp}_2}\,\mathrm{Res}_{\mathrm{GL}_1^2\,\mathrm{Sp}_2}\,. \tag{18.14}$$

Adjunction at 42 now gives

$$DY_A = -Y_A + \frac{q-1}{q^2}(F_1 - C). \tag{18.15}$$

Here the three lower squared norms for $e_I f_2, e_J f_2, e_1^2 f_2$ are $[q^2(q-1)^2(q+1)]^{-1}$, $[q^2(q-1)]^{-1}$ and $[q(q-1)^2]^{-1}$; that of $e_1 f_{4R}$ is $[(q-1)q^2]^{-1}$. The other anti-invariant Sp_4 function is supported on transvections and induces to zero at 42. The B coefficient cancels after using $F_0 = (q+1)B + C$. The same adjunction shows all proper restrictions of Y_B vanish. Consequently (18.13) proves the all-field identities

$$\boxed{DY_A = -q^{-2}Y_A + (1-q^{-2})Y_C - (q^2-1)Y_D, \qquad DY_B = -Y_B.} \tag{18.16}$$

The squared norm of the displayed DY_A is exactly q^{-5}, an additional check on every coefficient. Neither identity invokes duality on a general IC basis.

18.6. **The small Springer calculation and its normalization.** The generalized Springer series containing Y_A is induced from the cuspidal local system of type C_1, with relative Weyl group $W(C_2)$. In the order (A, C, T, R, D) its relative characters are

$$((11), \varnothing),\ ((1),(1)),\ (\varnothing,(11)),\ ((2),\varnothing),\ (\varnothing,(2)).$$

The A character is a nonstandard linear character; it is *not* the determinant, which is the T character. Their Springer-fibre dimensions are $(1, 2, 6, 0, 3)$. The IC coefficient matrix, with rows indexed by Y_j and columns by IC labels, is

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & q & 0 \\ q^3 & q^3+q & 1 & q^2 & q \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & q & 1 \end{pmatrix}. \tag{18.17}$$

Thus $\widetilde{X}_i = q^{b_i+1/2}\sum_j P_{ji}Y_j$. Here is an explicit check of the Weyl-group calculation. In the same row order, the character table of $W(C_2)$ is

$$C = \begin{pmatrix} 1 & 1 & -1 & 1 & -1 \\ 2 & -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \end{pmatrix}.$$

The column class sizes are $1, 1, 2, 2, 2$, and the corresponding $\det(q-w)$ are $(q-1)^2, (q+1)^2, q^2+1, q^2-1, q^2-1$. Put $V = \mathrm{diag}(q, q^2, q^6, 1, q^3)$ and

$$N = \mathrm{diag}\left(q^{-5}, \frac{1}{q^5(q^2-1)}, \frac{1}{q^9(q^2-1)(q^4-1)}, q^{-3}, \frac{1}{q^7(q^2-1)}\right).$$

These are the squared norms of Y_A, Y_C, Y_T, Y_R, Y_D. Direct multiplication verifies all 25 entries of

$$qVP^{\mathsf{T}}NPV = \frac{1}{8}C\,\mathrm{diag}\left(\frac{1}{(q-1)^2}, \frac{1}{(q+1)^2}, \frac{2}{q^2+1}, \frac{2}{q^2-1}, \frac{2}{q^2-1}\right)C^{\mathsf{T}}. \tag{18.18}$$

This is the Gram identity in the five-dimensional Lusztig–Shoji calculation; the support order and unit diagonal give the displayed IC matrix. The source package includes an exact verifier over $\mathbb{Z}[q]$, after clearing denominators, together with checks of (18.19) and the column combination below. See also the generalized Green-function algorithms in [23]. Shoji's split classical-group theorem [28, Theorem 4.2 and Corollary 4.4] supplies the actual Frobenius normalization for all odd characteristics.

One must also match the signs of (18.17) to the concrete Cayley matrices. This can be done without guessing a convention for a "split" rational representative. The finite flag induction from the rank-one cuspidal sign function satisfies

$$F_0 = \sum_i \varphi_i(1) q^{b_i} \sum_j P_{ji} Y_j, \qquad (\varphi_i(1)) = (1,2,1,1,1). \tag{18.19}$$

This is the trace identity for the generalized Springer induction complex along the *split parabolic*: its fibre is the set of fixed isotropic flags with the indicated rank-one local coefficient. Proper base change gives precisely the finite flag sum. The generalized Springer decomposition and Shoji's split Frobenius normalization give the right side. No comparison for induction from a Levi in a nonrational parabolic is required here.

Multiplication of the displayed matrix gives the row

$$(q,\ 2q(q+1),\ q^2(q+1)^2(q^2+1),\ 1,\ q(q+1)^2).$$

Comparing with the independently counted row F_0 in (18.13) fixes exactly the signs in (18.10), beginning with the regular class where the coefficient is 1. This verifies their simultaneous compatibility with (18.17) for every odd q. In particular the factors ϵ on Y_A, Y_C, Y_D must be kept together.

The other anti-invariant local system on 42 is the type-C_3 cuspidal series. It is one-dimensional and its IC function is $q^{5/2}Y_B$, after choosing its harmless common sign.

18.7. **Dual Mellin transforms supported exactly on** $(4,2)$**.** Taylor's theorem applies to every odd p in type C_3, since these primes are very good and hence acceptable [32, Theorem 11.13 and Lemma 11.16]. We specify the rational-class convention before applying the formula. The component group at 42 is $C_2 \times C_2$, with its factors attached to the length-four and length-two chains. Write $\alpha = \eta(a)$, $\beta = \eta(b)$. The two relevant sign functions are

(α,β)	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
Y_A	ϵ	ϵ	$-\epsilon$	$-\epsilon$
Y_B	1	-1	1	-1

Frobenius is trivial on this component group, so its order is four on each rational class. Taylor's Mellin coefficients therefore have no additional component-group indices.

The opposition convention can be checked on the Cayley matrices. Use the Springer isomorphism $u \mapsto (u-I)(u+I)^{-1}$, which is admissible here by the proof of [32, Proposition 4.6]. Take $e = X$ and the opposition $e^\dagger = -e^{\mathsf{T}}$ in a standard symplectic basis. Since $e^{\mathsf{T}} = -JeJ^{-1}$, the element $-e^\dagger$ is conjugate to $-e$. On an even chain of length d, replacing e by $-e$ multiplies its primitive form by $(-1)^{d-1} = -1$. Consequently the star operation on rational classes is

$$(\alpha,\beta) \longmapsto (\epsilon\alpha, \epsilon\beta), \qquad Y_A(u^*) = \epsilon Y_A(u), \quad Y_B(u^*) = \epsilon Y_B(u).$$

It changes neither component character into the other. In particular, the two Mellin transforms Λ_A, Λ_B are integral signed sums of the four genuine GGGRs. Their harmless overall signs may be absorbed in the fourth roots ζ_1, ζ_3.

For the C_1 series the centre determinant on $W(C_2)$ is $Z(w) = \det(q-w)$. Multiplication by $\det(w)Z(w)$ sends its A character to

$$\varphi_A - q\varphi_C + q^2\varphi_D.$$

Since the A row of $\widetilde{P}$ has no other entry, the all-field GGGR formula becomes

$$\begin{aligned} \Lambda_A &= 4\zeta_1^{-1}(\widetilde{X}_A - q\widetilde{X}_C + q^2\widetilde{X}_D) \\ &= 4\zeta_1^{-1}q^{1/2}(qY_A - q(q^2-1)Y_C + q^3(q^2-1)Y_D) \\ &= -4\zeta_1^{-1}q^{7/2}DY_A. \end{aligned} \tag{18.20}$$

Every Y_T term cancels by multiplication of the displayed matrix. In the cuspidal series the same formula is $\Lambda_B = 4\zeta_3^{-1}q^{5/2}Y_B$. Applying the involution D and using (18.16) gives

$$\boxed{D\Lambda_A = -4\zeta_1^{-1}q^{7/2}Y_A, \qquad D\Lambda_B = -4\zeta_3^{-1}q^{5/2}Y_B.} \tag{18.21}$$

These are the two required class-indicator directions, for *all* odd q. This replaces only the part of general IC duality actually needed for this problem.

18.8. **Individual high-height vanishing.** Let χ be a relevant simple-group irreducible, inflated to G, with ordinary height af. The conformal degree calculation above gives $a \in \{0,1,2,3,4,6,9\}$; degree denominators divide two and the leading coefficients are p-units. Restriction from $\mathrm{CSp}_6(q)$ is multiplicity free of length one or two and preserves this exponent. Unipotent support is preserved under the regular embedding, so the degree/support identity gives

$$\dim \mathcal{O}_\chi = 18 - 2a. \tag{18.22}$$

This uses the connected-centre degree identity and regular-embedding invariance, see also [33, §2.12].

For all good primes Taylor's wave-front/support results say

$$\langle \chi, D\Gamma_v \rangle \neq 0 \implies \mathcal{O}_v \subseteq \overline{\mathcal{O}_\chi} \tag{18.23}$$

for a genuine GGGR Γ_v [32, Lemma 14.15 and Proposition 15.2]. If $a \geq 2$, the right side of (18.22) is at most 14, whereas $\dim \mathcal{O}_{42} = 16$. Thus χ is orthogonal to both virtual characters in (18.21). Its two anti-invariant coefficients on 42 are zero individually.

The generic $\mathrm{CSp}_6(q)$ table in Proposition 18.1 has zero subregular value at every exponent $a \geq 2$, on both subregular conformal classes. It gives the diagonally invariant part of χ by restriction. Combining this zero aggregate with the two zeros from (18.21) proves

$$\boxed{h(\chi) \geq 2f \quad \Longrightarrow \quad \chi(v) = 0 \text{ for every rational class } v \text{ of type } 42.} \tag{18.24}$$

The conclusion concerns each irreducible character separately. It holds for even and odd f, including $q = 9, 25, 27$.

18.9. **Height f: divisibility without a root-phase theorem.** It remains to show $\chi(u) \in qA_q$ when $h(\chi) = f$ and u is eligible of anisotropic type 42. Write the anti-invariant part on the four rational classes as $c_A Y_A + c_B Y_B$. The scalar products

$$m_A = \langle \chi, D\Lambda_A \rangle, \qquad m_B = \langle \chi, D\Lambda_B \rangle$$

are integers: each Λ is an integral virtual character and Alvis–Curtis duality preserves the character lattice. Using the norms $\|Y_A\|^2 = \|Y_B\|^2 = q^{-5}$ in (18.21) gives, with the usual inner product linear in its first variable,

$$c_A = -\frac{\zeta_1^{-1}}{4} q^{3/2} m_A, \qquad c_B = -\frac{\zeta_3^{-1}}{4} q^{5/2} m_B. \tag{18.25}$$

The conformal table puts the invariant part in $(q/2)\mathbb{Z}$. Therefore

$$\chi(u) \in \frac{q}{2}\mathbb{Z} + \frac{q^{3/2}}{4}\mathbb{Z}[i]. \tag{18.26}$$

Only the fact that the two constants are fourth roots is used.

Lemma 18.5. *For odd $q = p^f$, $g_q^2 = \epsilon q$ and $A_q = \mathbb{Z}[(g_q - 1)/2]$,*

$$A_q \ \cap \ \left(\frac{q}{2}\mathbb{Z} + \frac{q^{3/2}}{4}\mathbb{Z}[i] \right) \ \subseteq \ qA_q.$$

Proof. First intersect with $\mathbb{Q}(g_q)$. If f is odd and $\epsilon = 1$, the imaginary Gaussian-integer coefficient is zero; the result is contained in $(q/4)(\mathbb{Z} + \mathbb{Z}g_q)$. If $\epsilon = -1$, the irrational real coefficient is zero instead, with the same conclusion. If f is even, $\sqrt{q} \in \mathbb{Z}$ and $\mathbb{Q}(g_q) = \mathbb{Q}$, so the result is contained in $(q/4)\mathbb{Z}$. In every case it lies in $qA_q[1/2]$. Finally $qA_q[1/2] \cap A_q = qA_q$, since A_q is free over $\mathbb{Z}$ of rank one or two and q is odd. Integral closedness is unnecessary. □

Combining (18.9), (18.26) and the lemma proves

$$\boxed{h(\chi) = f \quad \Longrightarrow \quad \chi(u) \in qA_q.} \tag{18.27}$$

The argument also applies when A_q is a nonmaximal quadratic order.

18.10. **Completion of the weighted equality.** By (18.9), (18.24) and (18.27), every global generator in (1.4) lies in the right side of (18.8): a height-zero value is multiplied by p^r, a height-f value in qA_q by $p^{\max(0,r-f)}$, and every higher value is zero. Thus the global module is contained in the local one.

For the converse use Lemmas 18.2 and 18.3. The two square classes of additive character give

$$R^G_{\mathrm{GL}_1\times\mathrm{Sp}_4}(\eta\,\rho^{\epsilon}_{2,\psi}) = \rho^{\epsilon}_{3,\psi} + \alpha_\psi + \beta_\psi,$$

with degrees

$$\frac{q^3+\epsilon}{2}, \qquad \frac{q(q+\epsilon)(q^4+q^2+1)}{2}, \qquad \frac{q^2(q^2+1)(q^3+\epsilon)}{2}.$$

They are central-trivial and have heights $0, f, 2f$. Lemma 18.3 gives $(\alpha_\psi+\beta_\psi)(u) = q(\epsilon \pm g_q)/2$. By (18.24), $\beta_\psi(u) = 0$. The two height-f characters consequently have values

$$\alpha_\pm(u) = \frac{q(\epsilon \pm g_q)}{2}, \qquad \mathbb{Z}\alpha_+(u) + \mathbb{Z}\alpha_-(u) = qA_q. \tag{18.28}$$

The span assertion follows from their sum $q\epsilon$ and either value; it remains valid when f is even. The trivial character supplies $p^r\mathbb{Z}$ and (18.28) supplies $p^{\max(r,f)}\mathbb{Z}\omega_q$. This proves the reverse inclusion in (18.8), and hence Theorem 18.4.

The degree in the middle of the display has ordinary height f and its two values in (18.28) are nonzero for every $q > 1$. The result therefore includes genuinely nonzero positive-height information for arbitrarily large heights; it is not a height-zero statement.

19. The diagonal almost simple family

Put $S = \mathrm{PSp}_6(q)$ and $G = \mathrm{PGSp}_6(q) = \mathrm{CSp}_6(q)/Z(\mathrm{CSp}_6(q))$. Then $S \lhd G$ and $[G:S] = 2$. This is the full diagonal extension, not the simple group S.

For clarity the notation of B(c) here is

$$H = \mathrm{Sub}_G(Q) = \langle N_G(T) : T \in \mathrm{Syl}_p(G),\ Q \le T\rangle, \qquad 1 \ne Q,$$

with an actual visible Brauer-correspondent pair (b,c) of common defect containing Q. Eligibility means that u fixes exactly one coset in G/H.

Lemma 19.1. *If u is eligible in G/H and all irreducible character values of G at u are integers, then all irreducible character values of H at u are integers as well.*

Proof. For j prime to $|u|$, Galois conjugation shows that every global irreducible has equal values at u and u^j. Characters separate classes, so $u^g = u^j$ for some $g \in G$. Since $u^j \in H$, the coset gH is fixed by u; eligibility gives $g \in H$. Thus all these powers are conjugate inside H, proving rationality and hence integrality of its character values. □

Theorem 19.2. *Let $q = p^f$ be odd and $G = \mathrm{PGSp}_6(q)$. For every $1 \ne Q \le P \in \mathrm{Syl}_p(G)$, every visible pair (b,c), every eligible p-element u, and every $r \ge 0$,*

$$\mathcal{M}_r(G,b;u) = \mathcal{M}_r(H,c;u).$$

If $H < G$, the pair is principal of full Sylow defect and both modules equal $p^r\mathbb{Z}$. In particular B(c) holds for this infinite almost simple family.

Proof. If $H = G$ there is nothing to prove, for any visible block. Suppose $H < G$. Since H contains $N_G(P)$ for a full Sylow subgroup P, it is a proper parabolic in the adjoint rank-three BN-pair. The flag coset actions are the same as those of S: each parabolic contains an element of the diagonal outer coset, and $G/\widehat{P}_i \cong S/(S\cap\widehat{P}_i)$.

Eligibility descends from a parabolic flag to each of its constituent maximal-parabolic flags, because the fibres have p'-size. The invariant-space classification of Theorem 15.4 therefore applies without an assumption that Q is cyclic. The eligible u is regular, or is anisotropic $(4,2)$ and H is the full conformal Siegel parabolic. Every p-element lies in S, since p is odd.

Every proper parabolic of G has a normal p-core with p-group centralizer. Here are matrix checks for the maximal cases before quotienting the scalar center. In Siegel, centralizing the symmetric radical forces a Levi pair (A,μ) to have $A = aI_3$, $\mu = a^2$. In Klingen, the actions on its Heisenberg

center and four-dimensional quotient force the same full scalar. In the middle parabolic, centralizing its symmetric two-dimensional-block center first forces $A = aI_2$, $\mu = a^2$; the radical quotient then forces the other two-dimensional block to be aI_2. Removing the scalar center leaves U in Siegel and $Z(U)$ in the other two cases. The smaller parabolics contain the radical of a maximal one and inherit a p-group centralizer of their p-core. The block-idempotent argument used in Section 16 proves that H has one block, principal of full defect. Brauer's first and third main theorems, with $N_G(P) \leq H$, identify its visible global partner as principal. All heights in this proper case are ordinary degree valuations.

The generic conformal table examined in Proposition 18.1 has rational values in all ten unipotent column types: after the semisimple class parameters are set to the identity, every cyclotomic exponent is zero and every entry is a rational polynomial. Actual character values are algebraic integers, so they are integers. Inflation covers all irreducibles of G. A p-element has a unique p-element lift through the central p'-kernel, so this applies to the elements under consideration. Lemma 19.1 makes every local value an integer too.

Proposition 18.1 also gives the global positive-degree behavior: at a regular element every positive-height value vanishes; at $(4, 2)$ a nonzero positive-height value has height f and is $\pm q$ or $\pm 2q$. Therefore every global weighted generator is in $p^r\mathbb{Z}$, and the trivial character supplies p^r. Hence $\mathcal{M}_r(G, b; u) = p^r\mathbb{Z}$.

Put $K = H \cap S$. It is the corresponding proper parabolic of S, and H/K has order two. Clifford restriction is multiplicity free with one or two constituents. Their degrees differ from the global local degree by a factor one or two, so their p-parts, and thus their ordinary full-defect heights, agree. At a regular u, Theorem 17.1 eliminates every positive-height constituent and hence every positive-height value of H.

At nonregular u, K is Siegel. Theorem 16.1 says that every nonzero positive-height constituent has height f and value in qA_q. Thus any positive-height local value in H is zero or has height f and lies in qA_q. It is also an integer by Lemma 19.1. Since A_q consists of algebraic integers,

$$qA_q \cap \mathbb{Z} = q\mathbb{Z}.$$

All positive-height local weighted generators are consequently in $p^r\mathbb{Z}$ as well. Height-zero generators are integers times p^r, and the local trivial character supplies p^r. This proves $\mathcal{M}_r(H, c; u) = p^r\mathbb{Z}$ and the theorem. □

Together with Theorems 13.1, 14.1 and 18.4, this proves Theorem D(b).

20. Proper-defect symmetric blocks

Martínez Madrid [22, Theorems A and B] established signed character correspondences for picky elements of symmetric groups at every prime, and for arbitrary 2-elements. Here we compare the specified proper-defect block pairs attached to p-subgroups at odd primes.

20.1. **A signed character transfer for symmetric blocks.** Let p be prime, let $m = pw > 0$, and let τ be a p-core of size $c > 0$. The block of $\mathfrak{S}_{m+c}$ labelled by τ has defect a Sylow subgroup of $\mathfrak{S}_m$. Compare it with

$$B_0(\mathfrak{S}_m) \otimes b_\tau(\mathfrak{S}_c) \quad \text{in} \quad \mathfrak{S}_m \times \mathfrak{S}_c.$$

The second tensor factor has defect zero and contains only χ^τ. The usual block description for symmetric groups identifies these as Brauer-correspondent blocks: in the defect normalizer both correspond to its principal factor tensored with b_τ. See the block identification in the proof of Evseev's Theorem 1.3 [5]. Normalizer containment in the Young subgroup is immediate from the support of the defect group.

Fix the abacus convention with a number of beads divisible by p. For each partition λ of $m + c$ with core τ, let μ be the unique partition of m with empty core and the same ordered p-quotient. This is a bijection. Set

$$\epsilon_\lambda = \epsilon_p(\lambda/\tau)\epsilon_p(\mu/\varnothing) \in \{1, -1\}.$$

Lemma 20.1. *The bijection $\chi^\lambda \mapsto \chi^\mu \otimes \chi^\tau$ preserves ordinary block heights. If $x \in \mathfrak{S}_m$ has all cycle lengths divisible by p, then*

$$\chi^\lambda(x,1) = \epsilon_\lambda \, \chi^\mu(x)\chi^\tau(1). \tag{20.1}$$

The sign depends on the row, and not on x.

Proof. Write $(\alpha_0, \dots, \alpha_{p-1})$ for the common quotient and $H(\alpha)$ for the product of the hook lengths of α. Hooks of λ divisible by p, divided by p, are exactly the hooks of its quotient components. There are w of them, so

$$v_p(\chi^\lambda(1)) = v_p((m+c)!) - w - \sum_i v_p(H(\alpha_i)).$$

Subtracting the defect-index valuation gives

$$h(\chi^\lambda) = v_p(m!) - w - \sum_i v_p(H(\alpha_i)) = v_p(\chi^\mu(1)).$$

Since χ^τ has defect zero, the local tensor character has this same height.

Apply Murnaghan–Nakayama to all the cycles of x, before the c fixed points. Removing a hook of length divisible by p preserves the p-core. Every surviving term therefore ends at τ, and contributes its degree. Equivalently,

$$\chi^\lambda(x,1) = \chi^{\lambda/\tau}(x)\chi^\tau(1).$$

Farahat's quotient formula, in the form of Evseev–Paget–Wildon [7, Theorem 3.3], expresses $\chi^{\lambda/\tau}(x)$ as its p-sign times the character induced from the quotient components, evaluated at the cycle type divided by p. The same induced character occurs for $\mu/\varnothing$. Taking the ratio of the signs gives (20.1).

In abacus terms, removing a pk-hook moves one bead k positions on its runner. The quotient move and its within-runner sign are independent of the core; the change in bead order across runners telescopes to the displayed p-sign. Thus the argument uses a single sign for the entire profile, not separately chosen signs at different elements. □

20.2. **Actual subnormalizers and an infinite proper-defect family.**

Lemma 20.2. *Let p be odd, $a \geq 2$, $m = p^a$, and $0 < c < m$. Let $Q \cong C_p^a$ act by translations regularly on a set Ω of size m, fixing a disjoint set Δ of size c. In $G = \mathfrak{S}_{m+c}$,*

$$\mathrm{Sub}_G(Q) = \mathfrak{S}_\Omega \times \mathfrak{S}_\Delta.$$

Proof. Every Sylow p-subgroup containing Q has an orbit containing Ω. Its size is a power of p, at least m, and smaller than $2m \leq pm$. The orbit is therefore precisely Ω, and is the unique orbit of size m. Every normalizer used to generate $\mathrm{Sub}_G(Q)$ preserves Ω, proving the required upper bound.

We first prove $\mathrm{Sub}_{\mathfrak{S}_\Omega}(Q) = \mathfrak{S}_\Omega$. Write this subnormalizer as K. Every transporter between two Sylow subgroups containing Q lies in K: the two Sylows lie in K, and comparison with a K-conjugating element leaves an element of a Sylow normalizer. In particular $N_{\mathfrak{S}_\Omega}(Q) = \mathrm{AGL}_a(p) \leq K$.

Identify Ω with $\mathbb{F}_p^a$. A p-cycle supported on one affine line is contained together with Q in a p-group: its translation conjugates are the mutually commuting cycles on the parallel lines, normalized by Q. It is consequently in a Sylow subgroup containing Q, and hence in K. Affine conjugation supplies such cycles on every affine line. The commutator of two such cycles on lines meeting in one point is a 3-cycle on a noncollinear triple. The affine group is transitive on ordered noncollinear triples, so K contains all their 3-cycles. A 3-cycle on a collinear triple is a product of two noncollinear ones, by using a point off that line. Thus $A_m \leq K$.

Multiplication of one coordinate by a generator of $\mathbb{F}_p^\times$ is odd: it has one $(p-1)$-cycle on each of p^{a-1} fibres, and this number is odd. It belongs to the affine group. Hence $K = \mathfrak{S}_m$.

For the ambient reverse inclusion, fix a Sylow $T \leq \mathfrak{S}_\Omega$ containing Q. In $N_{\mathfrak{S}_\Omega}(T) \times \mathfrak{S}_\Delta$, every Sylow subgroup contains T and has the full Sylow order of G, since $m = p^a$ and $c < m$ give $v_p((m+c)!) = v_p(m!) + v_p(c!)$. All its Sylow normalizers therefore lie in $\mathrm{Sub}_G(Q)$. The normalizers

of all Sylow subgroups of any finite group generate that group, by the Frattini argument. Hence $N_{\mathfrak{S}_\Omega}(T) \times \mathfrak{S}_\Delta \leq \mathrm{Sub}_G(Q)$. Varying T proves the reverse inclusion. □

Theorem 20.3. *Under the hypotheses of Lemma 20.2, let τ be any p-core of size c. Let b be its block of $G = \mathfrak{S}_{m+c}$, and let*

$$c_\tau = B_0(\mathfrak{S}_m) \otimes b_\tau(\mathfrak{S}_c) \quad \textit{in} \quad H = \mathrm{Sub}_G(Q) = \mathfrak{S}_m \times \mathfrak{S}_c.$$

This is an actual visible pair with common defect $D \in \mathrm{Syl}_p(\mathfrak{S}_m)$. The bijection of Lemma 20.1 preserves heights and agrees up to its fixed row signs at every *eligible p-element. Consequently B(c) holds for this pair at every $r \geq 0$, as do the integral upper-height-tail equalities on the entire eligible profile.*

Proof. The block and correspondence statements were explained before Lemma 20.1; Q can be placed in D. Cosets of H correspond to the m-subsets of the underlying $(m+c)$-point set. Suppose u is eligible and moves t points.

If $t > m$, then u is not conjugate into D, so every value in b vanishes by the standard block vanishing theorem. Every local value vanishes as well: the second component must be a nonidentity p-element, and χ^τ has defect zero.

If $t < m$, all the moved points together with any $m - t$ fixed points form a u-invariant m-subset. There are $\binom{m+c-t}{m-t} > 1$ such subsets, contradicting eligibility. Thus the remaining case is $t = m$. The unique invariant m-subset must be the support of u, so $u = (x, 1)$ with x fixed-point-free. Every cycle of x has length divisible by p. Lemma 20.1 now gives the required equality, with one height-preserving bijection and signs independent of u. The weighted modules and all upper-tail profile lattices therefore coincide integrally. □

Eligibility refinement. In the nonvanishing case $u = (x, 1)$ just considered, eligibility is equivalent to every cycle length of x being greater than c. If a cycle has length at most c, replace it by that many of the fixed points. Conversely any different invariant m-subset would have to make exactly such a replacement.

Corollary 20.4. *Fix an odd prime p. For every $a \geq 3$ set $m = p^a$, $c = p + 1$, and $\tau = (p, 1)$. Theorem 20.3 gives a proper nonabelian-defect pair in $\mathfrak{S}_{p^a+p+1}$, with defect index of p-part exactly p. It contains corresponding characters of height $a - 2$ with nonzero values at an eligible element of order p^2.*

Proof. The hooks of $(p, 1)$ are $p + 1, p - 1, p - 2, \ldots, 1$, together with one additional 1, so it is a p-core and $\chi^\tau(1) = p$. The defect is a Sylow subgroup of $\mathfrak{S}_{p^a}$, nonabelian for $a \geq 2$. Take

$$k = p^2, \qquad \mu = (m - k, k), \qquad x \text{ of cycle type } (p^2)^{p^{a-2}}, \qquad u = (x, 1).$$

The partition μ has empty p-core, by removing horizontal p-hooks from its second row and then its first. The two-row hook formula gives

$$\chi^\mu(1) = \binom{m}{k} \frac{m - 2k + 1}{m - k + 1}, \qquad v_p(\chi^\mu(1)) = a - 2.$$

Here $v_p\binom{p^a}{p^2} = a - 2$, and the other two factors are p-units. The character $\chi^{(m-k,k)}$ is the difference of the permutation characters on k-subsets and $(k - 1)$-subsets. At x these fixed-subset counts are p^{a-2} and zero. Thus

$$h(\chi^\mu \otimes \chi^\tau) = a - 2, \qquad (\chi^\mu \otimes \chi^\tau)(u) = p^{a-1} \neq 0.$$

Since $p^2 > p + 1$, the element is eligible. Its corresponding global character has the same height and value $\pm p^{a-1}$. □

This proves Theorem E.

21. Further questions

The main question is Conjecture B for arbitrary finite groups, with its weighted clause restricted to odd primes. Theorem 3.1 explains why that restriction cannot simply be omitted, even for solvable groups with a normal odd complement. Parts (a) and (b) at two remain open in general, including for solvable groups. Theorem 4.1 settles them for the transitive quaternion wreath family, with all nontrivial Q and visible pairs; it imposes no bound on the order of the Sylow subgroup.

Theorem D(a) proves the complete conjecture for solvable groups and for groups with a normal p-complement at odd primes. Extending it to all p-solvable groups requires control of nonsolvable p'-sections in the relative argument. The family results in part (b) prove only the weighted clause. The complete mixed-section and Clifford-level comparisons remain to be established for those simple and almost simple groups.

Theorem E concerns the specified proper-defect pairs. It does not settle the conjecture for all blocks of a symmetric group. Its exact signed transfer does, however, show that unbounded positive height by itself is not an obstruction to the weighted comparison.

Proposition 6.1 gives Alperin–McKay for maximal-defect blocks. For smaller defect the present comparison directly gives a subnormalizer of a defect group, and another argument is needed to pass to its normalizer. Finally, B(c) is elementwise: the possibility of choosing reconstruction coefficients simultaneously at several eligible elements is a stronger question.

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