Demostración automática de teoremas geométricos con GeoGebra

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SEIEM 2021 https://seiem2021.uv.es
10 de septiembre 2021

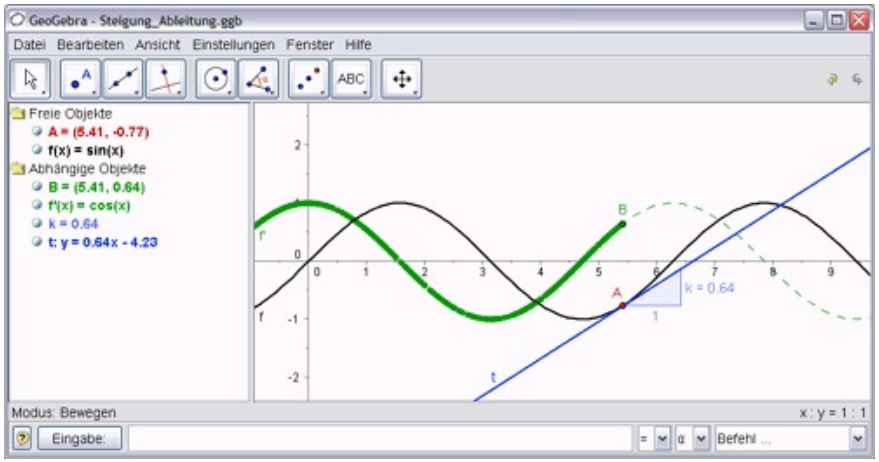
GeoGebra

Dynamic Mathematics for Everyone

GEOGEBRA

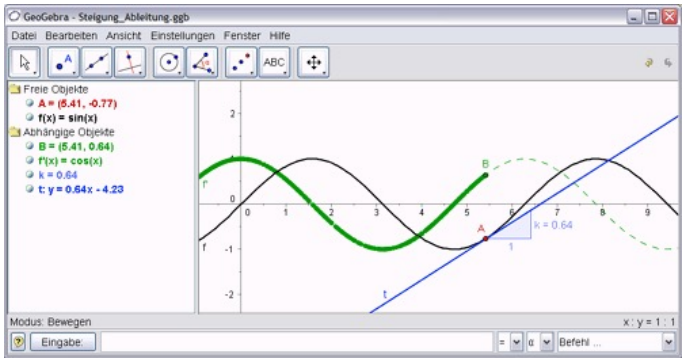
THE GRAPHING CALCULATOR FOR FUNCTIONS, GEOMETRY, ALGEBRA, CALCULUS, STATISTICS AND 3D MATH!

DYNAMIC MATHEMATICS FOR LEARNING AND TEACHING



GeoGebra

Dynamic Mathematics for Everyone



GEOGEBRA

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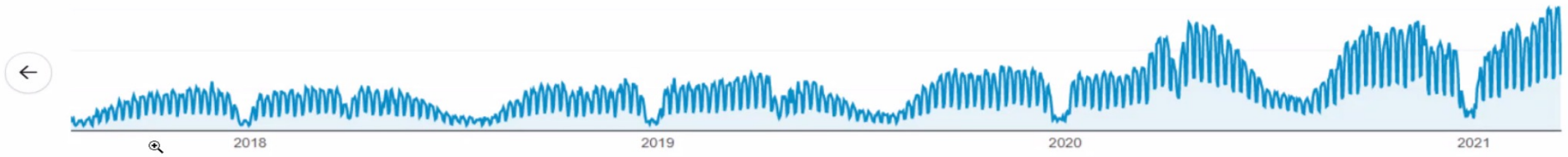
DYNAMIC MATHEMATICS FOR LEARNING AND TEACHING



De Markus Hohenwarter a Todos 5:47

Past year: March 2020 - March 2021
pageviews: 435m (+88% prev year)
users: 77m (+49% prev year)

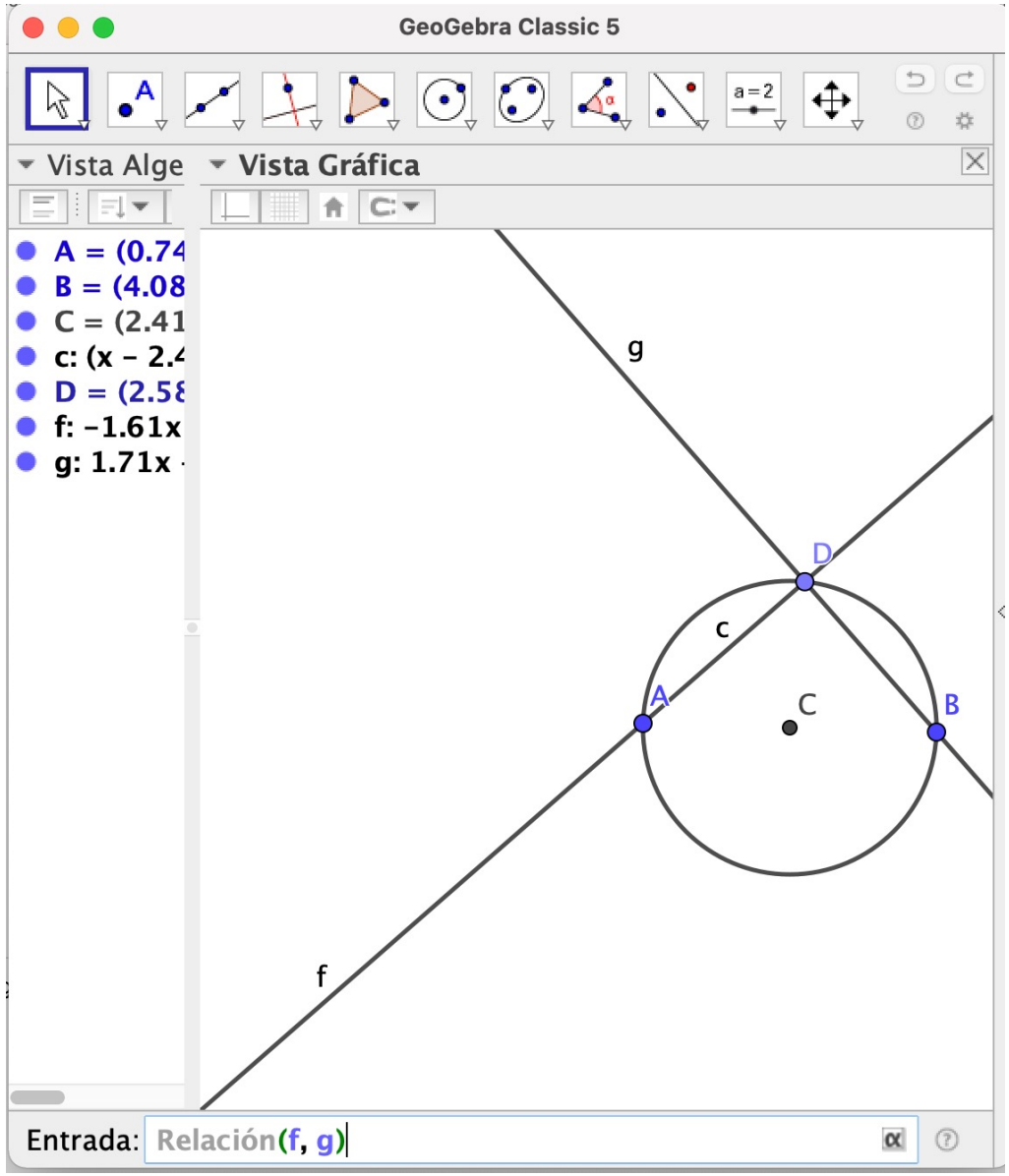
Continents: Americas 44%, Europe 35%, Asia 18%

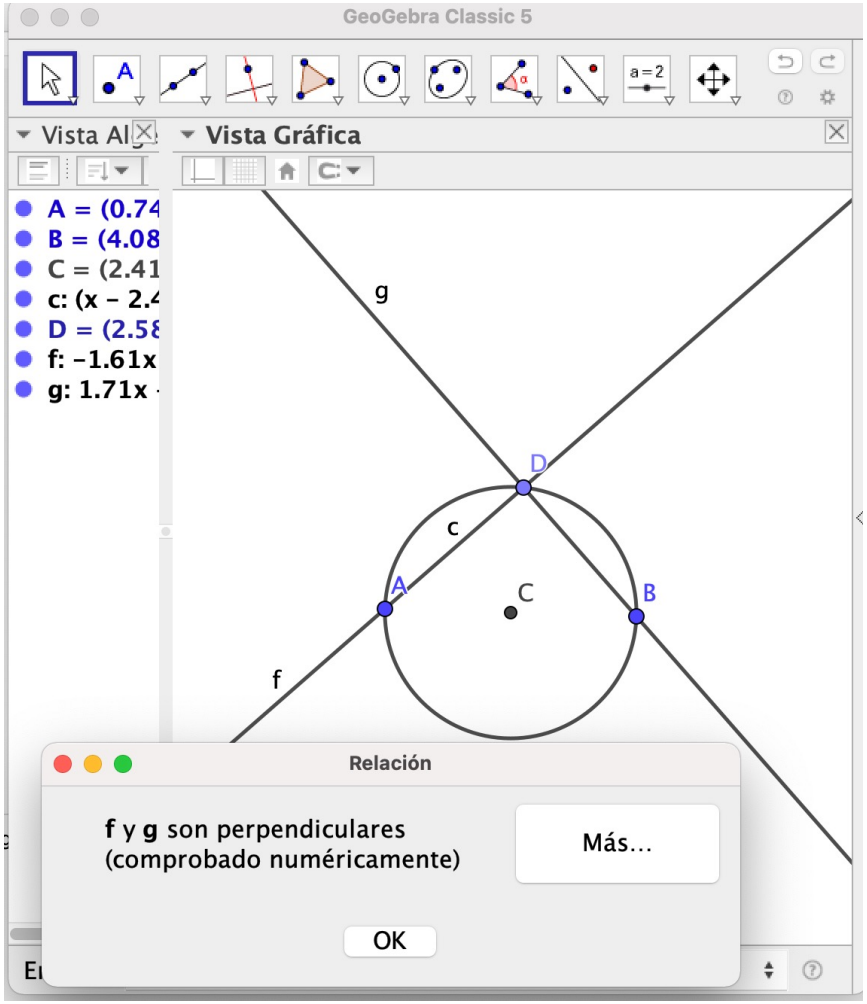
Languages: Spanish 30%, Englisch 29%, German 7%, French 6%, Chinese 5%, Italian 4%, Portugese 3%

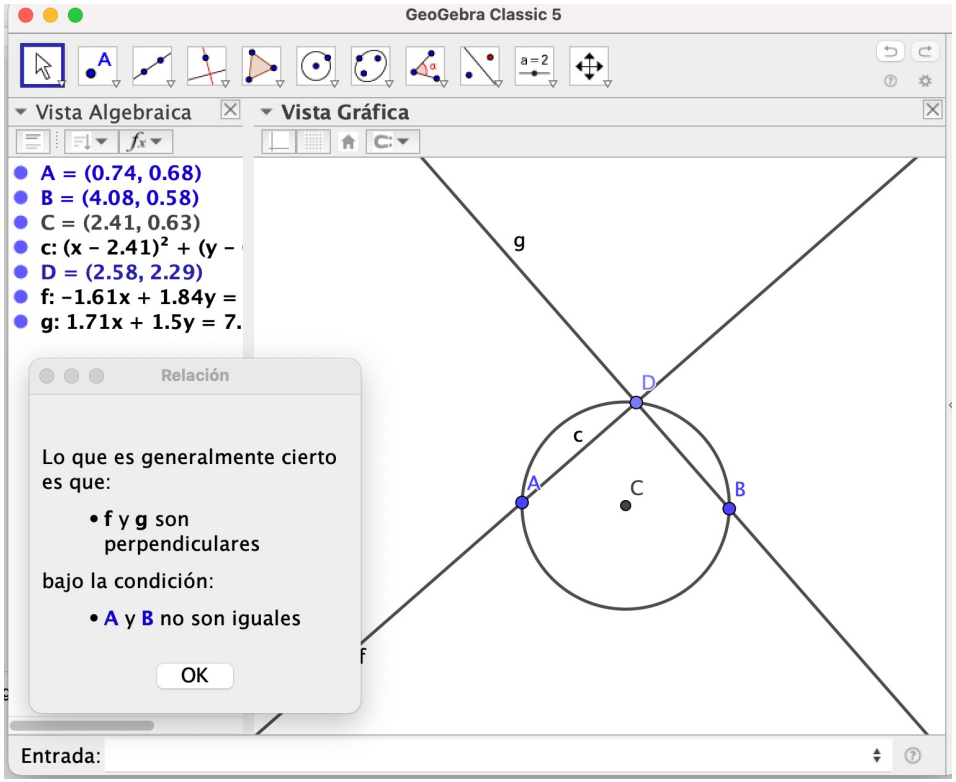
Top 10 Countries: USA 13%, Mexico 10%, Germany 6%, Colombia 5%, Italy 4%, France 4%, China 4%, Spain 4%, Argentina 3%, Brazil 3%

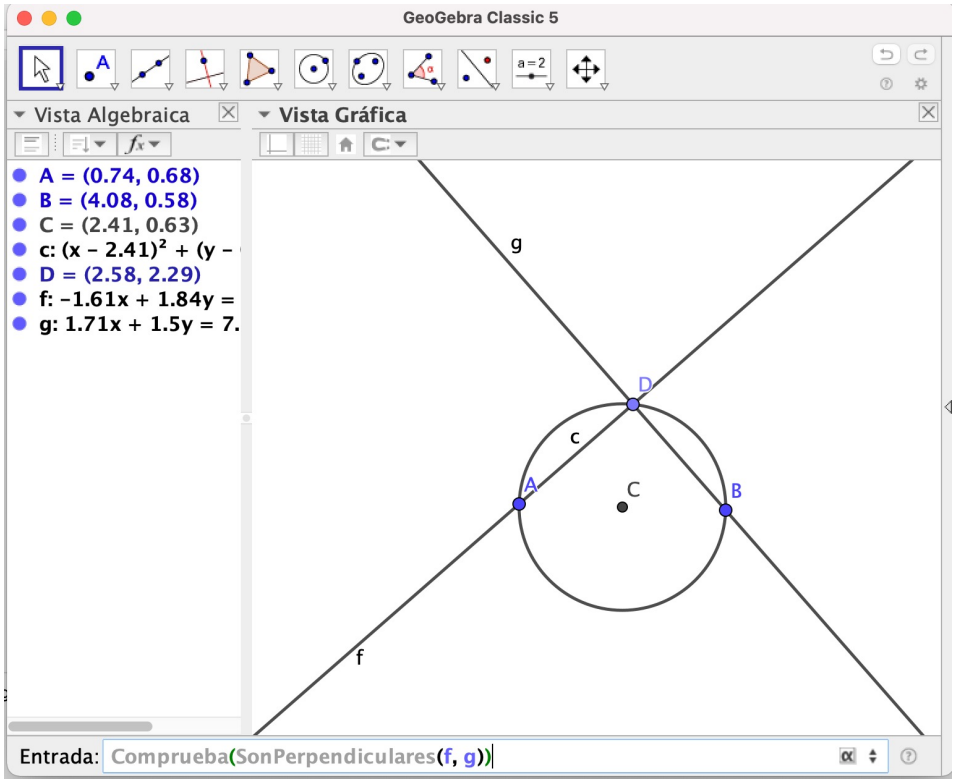
GeoGebra Community Gathering, March 25, 2021

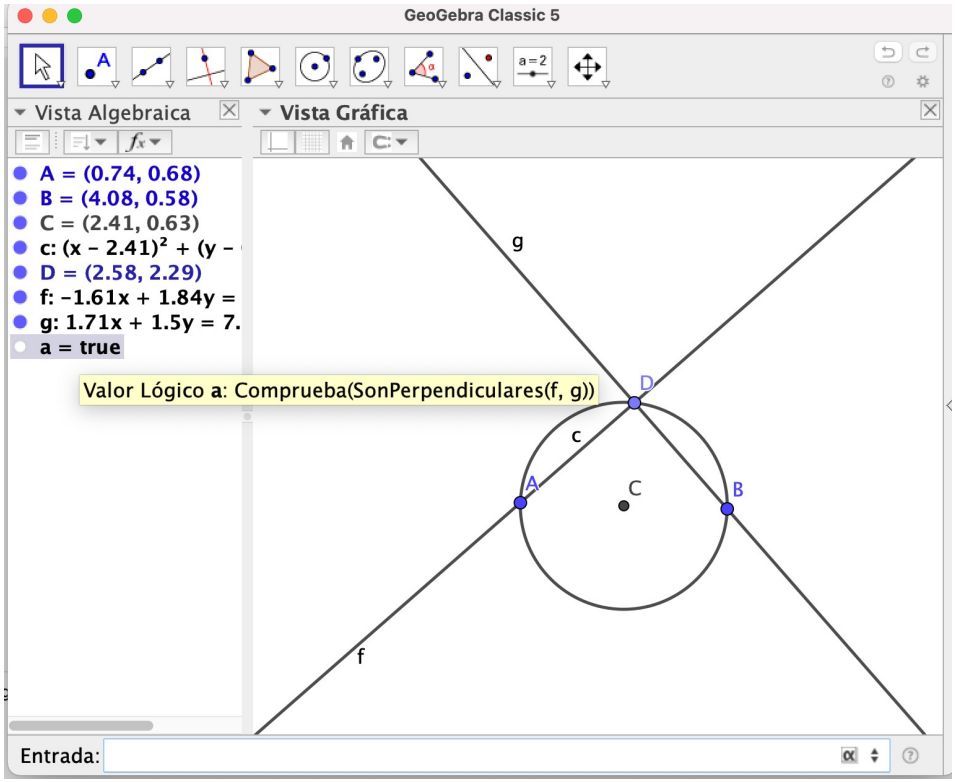
https://geogebra.zoom.us/rec/play/y26WqWb4FIfDXg94MvQ3qEraAI7SkseD8E8g6sjn7qV3ICxUNXBV5ah72l-r3EzljDXpQQ3b7aW402s8.qpSjiuB1vSl9J-jO

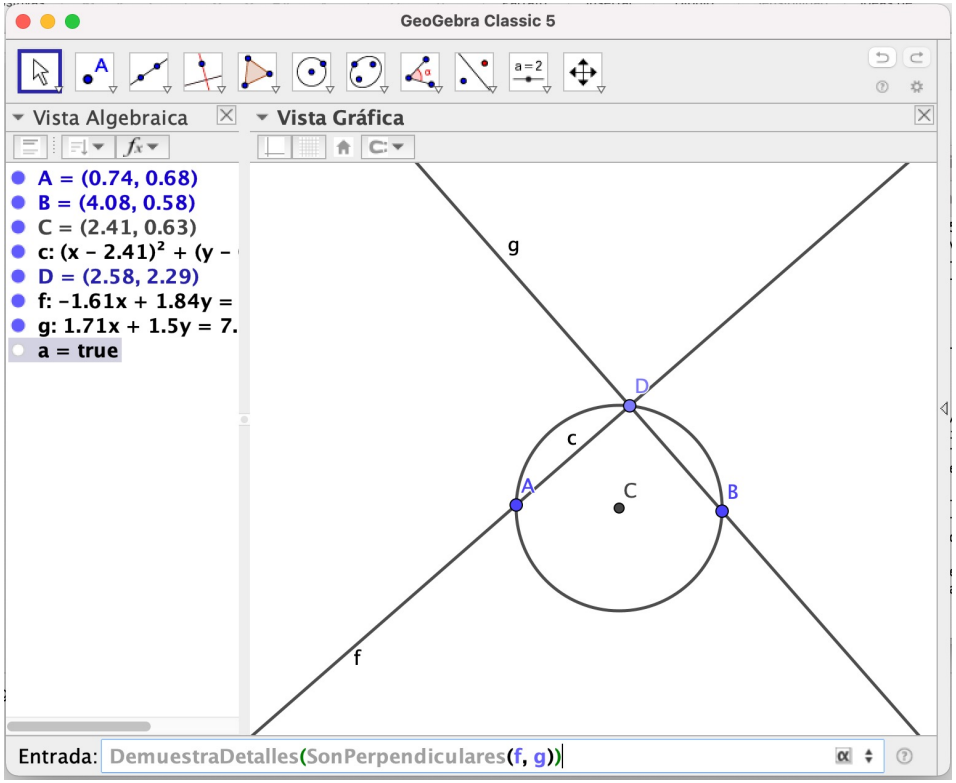

GeoGebra Classic 5
Vista Alge
Vista Gráfica
A = (0.74
B = (4.08
C = (2.41
c: (x – 2.4
D = (2.58
f: –1.61x
g: 1.71x
g
D
c
A
C
B
f
Entrada:
Relación(f, g)

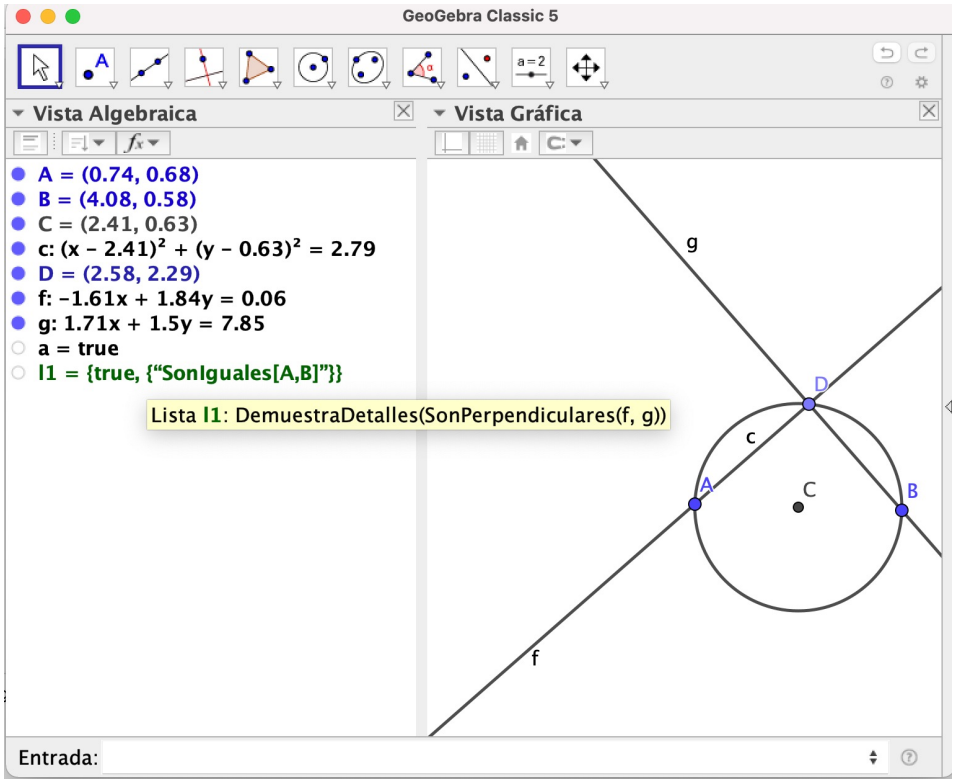

GeoGebra Classic 5
Vista Gráfica
A = (0.74
B = (4.08
C = (2.41
c: (x – 2.4
D = (2.58
f: –1.61x
g: 1.71x
g
D
c
A
C
B
f
Relación
f y g son perpendiculares
(comprobado numéricamente)
Más...
OK

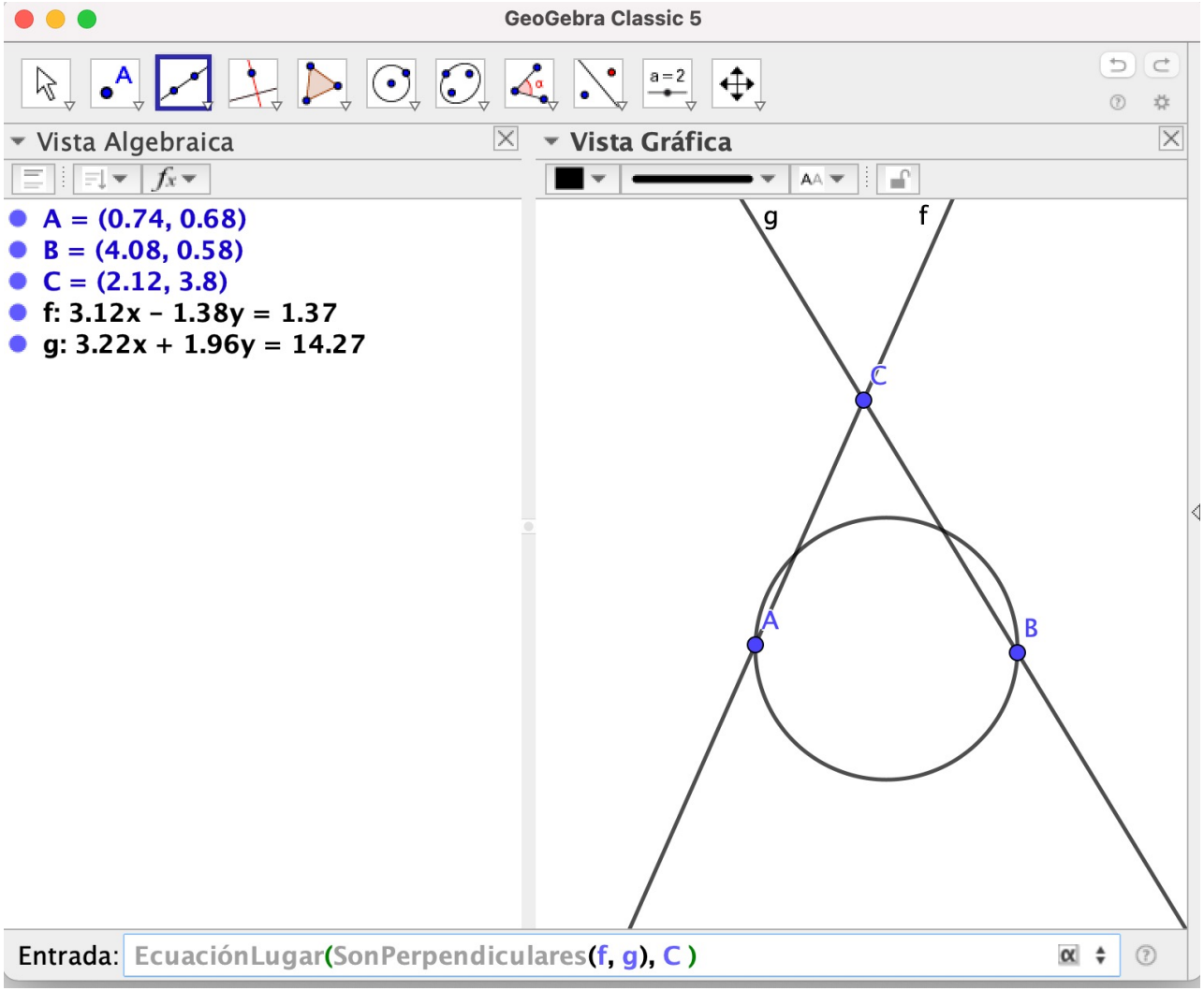

GeoGebra Classic 5
Vista Algebraica
Vista Gráfica
A = (0.74, 0.68)
B = (4.08, 0.58)
C = (2.41, 0.63)
c: (x – 2.41)² + (y –
D = (2.58, 2.29)
f: –1.61x + 1.84y =
g: 1.71x + 1.5y = 7.
g
D
c
A
C
B
f
Relación
Lo que es generalmente cierto es que:
• f y g son perpendiculares
bajo la condición:
• A y B no son iguales
OK
Entrada:

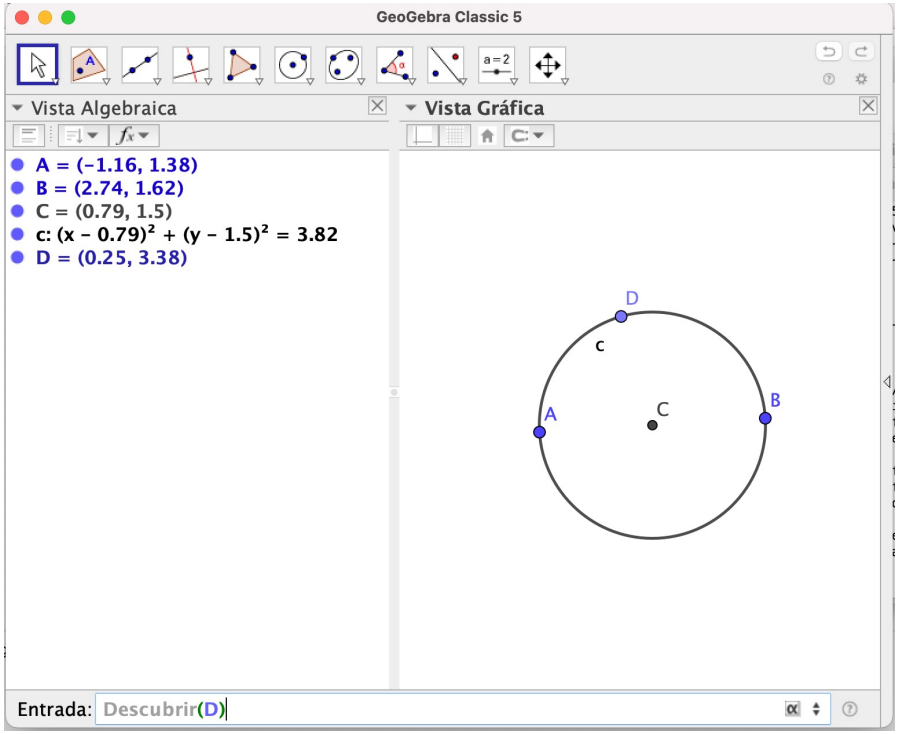

GeoGebra Classic 5
Vista Algebraica
Vista Gráfica
A = (0.74, 0.68)
B = (4.08, 0.58)
C = (2.41, 0.63)
c: (x – 2.41)² + (y –
D = (2.58, 2.29)
f: –1.61x + 1.84y =
g: 1.71x + 1.5y = 7.
g
D
c
A
C
B
f
Entrada: Comprueba(SonPerpendiculares(f, g))

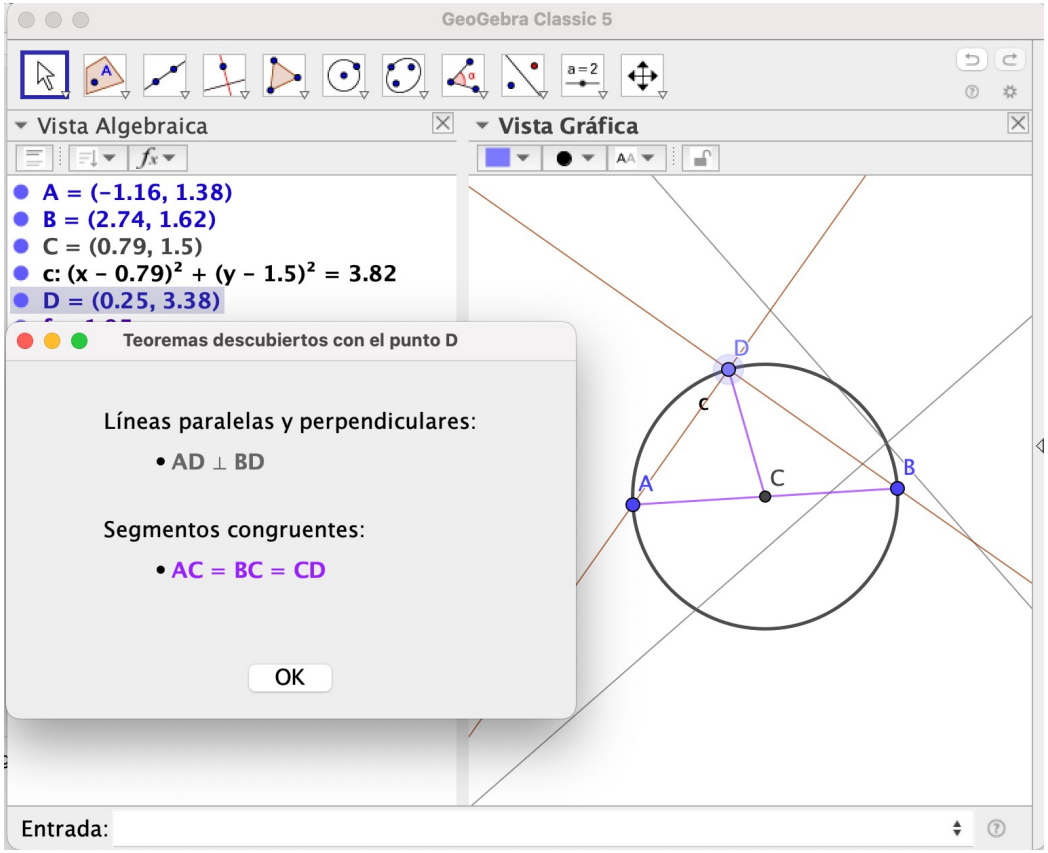

GeoGebra Classic 5
Vista Algebraica
Vista Gráfica
A = (0.74, 0.68)
B = (4.08, 0.58)
C = (2.41, 0.63)
c: (x – 2.41)² + (y –
D = (2.58, 2.29)
f: –1.61x + 1.84y =
g: 1.71x + 1.5y = 7.
a = true
Valor Lógico a: Comprueba(SonPerpendiculares(f, g))
g
D
c
A
C
B
f
Entrada:

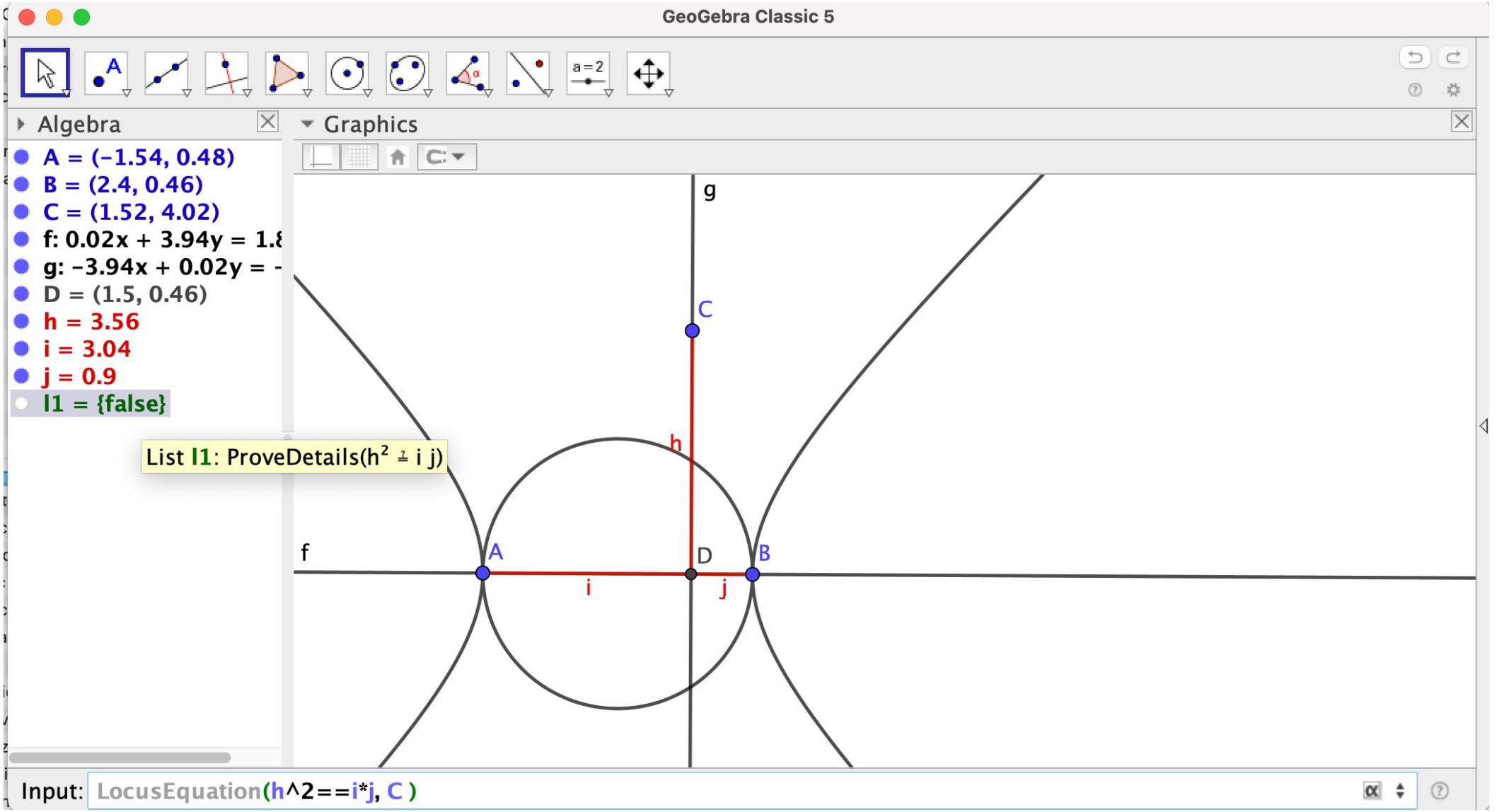

GeoGebra Classic 5
Vista Algebraica
Vista Gráfica
A = (0.74, 0.68)
B = (4.08, 0.58)
C = (2.41, 0.63)
c: (x – 2.41)² + (y –
D = (2.58, 2.29)
f: –1.61x + 1.84y =
g: 1.71x + 1.5y = 7.
a = true
g
D
c
A
C
B
f
Entrada: DemuestraDetalles(SonPerpendiculares(f, g))

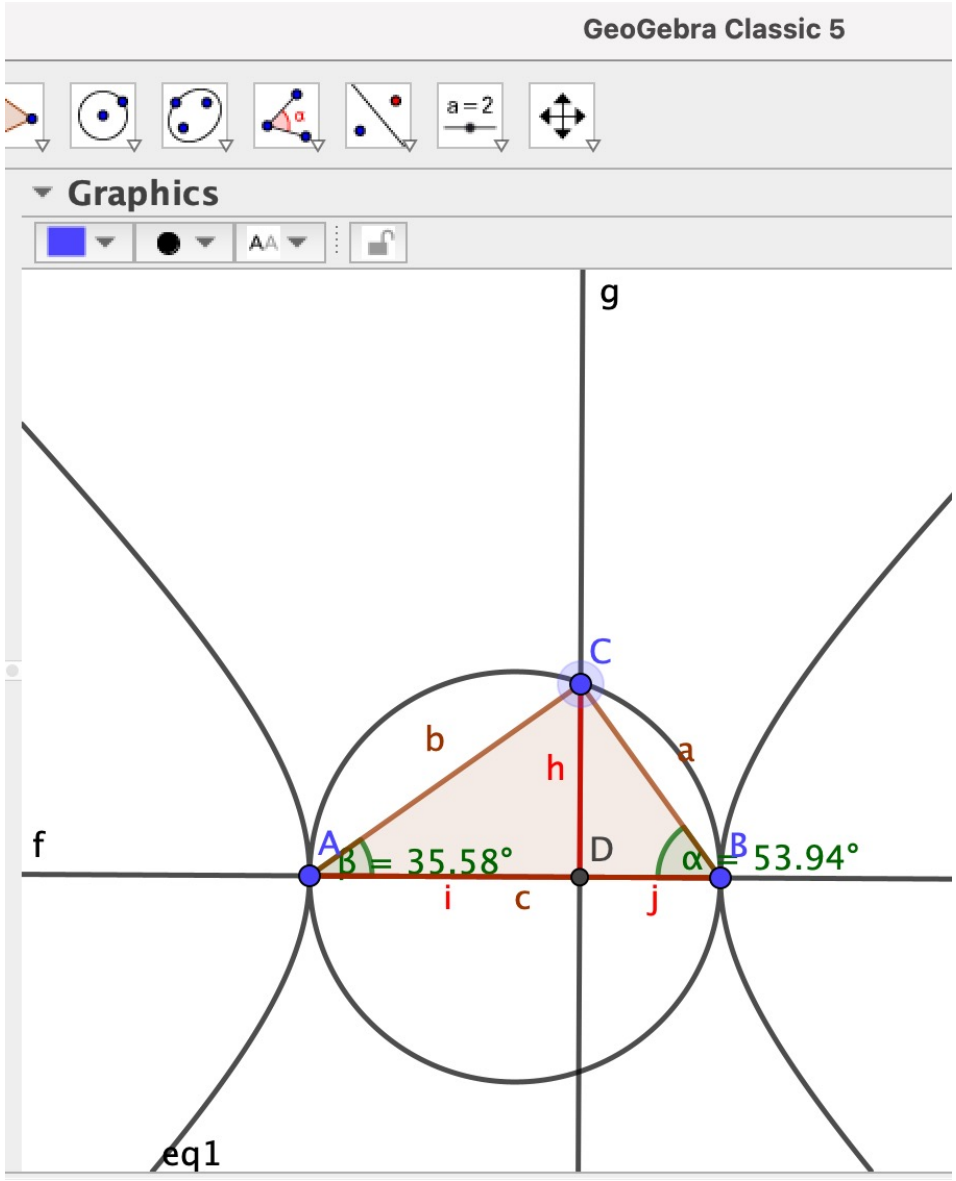

GeoGebra Classic 5
Vista Algebraica
Vista Gráfica
A = (0.74, 0.68)
B = (4.08, 0.58)
C = (2.41, 0.63)
c: (x – 2.41)² + (y – 0.63)² = 2.79
D = (2.58, 2.29)
f: –1.61x + 1.84y = 0.06
g: 1.71x + 1.5y = 7.85
a = true
l1 = {true, {“SonIguales[A,B]”}}
Lista l1: DemuestraDetalles(SonPerpendiculares(f, g))
g
D
c
A
C
B
f
Entrada:

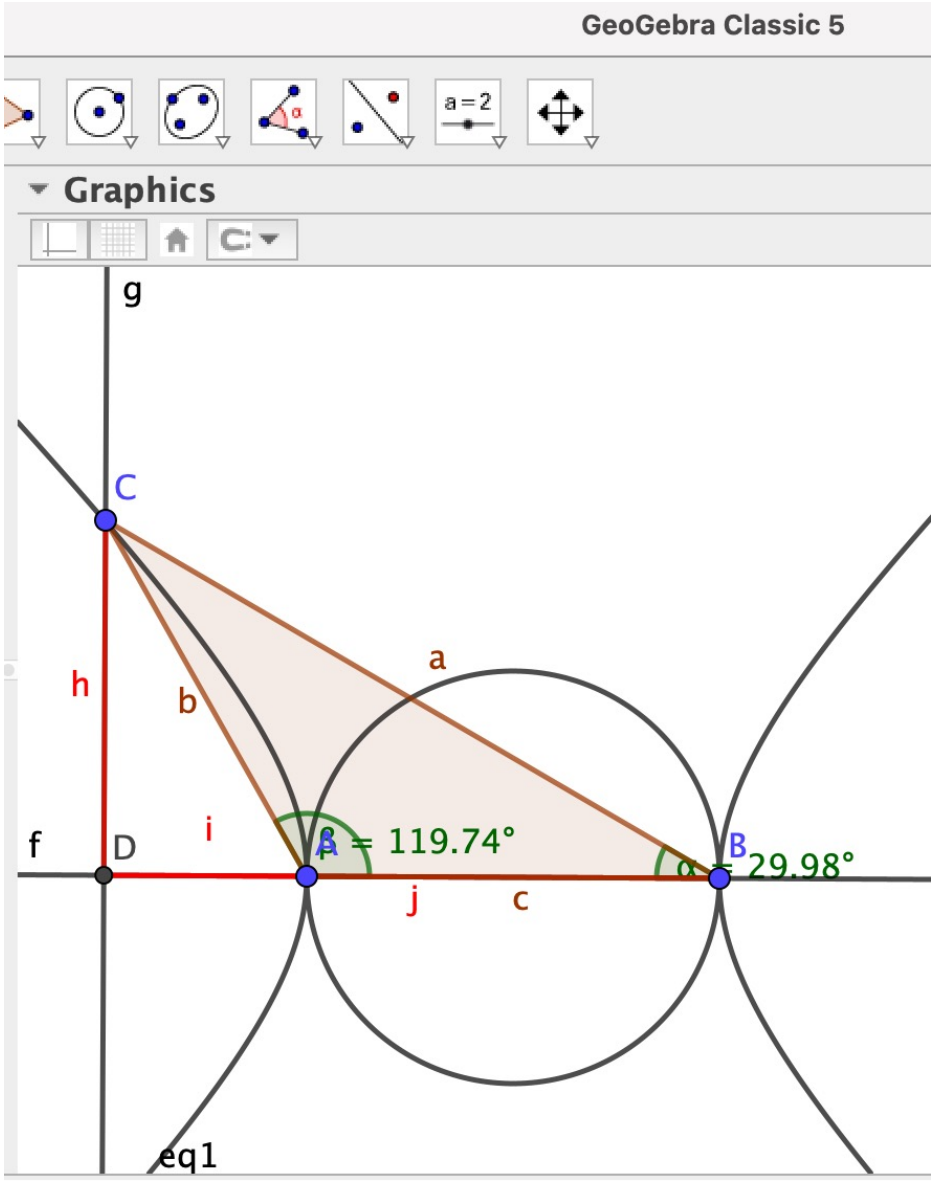

GeoGebra Classic 5
Vista Algebraica
A = (0.74, 0.68)
B = (4.08, 0.58)
C = (2.12, 3.8)
f: 3.12x − 1.38y = 1.37
g: 3.22x + 1.96y = 14.27
Vista Gráfica
g
f
C
A
B
Entrada: EcuaciónLugar(SonPerpendiculares(f, g), C)


GeoGebra Classic 5
Vista Algebraica
Vista Gráfica
A = (−1.16, 1.38)
B = (2.74, 1.62)
C = (0.79, 1.5)
c: (x − 0.79)² + (y − 1.5)² = 3.82
D = (0.25, 3.38)
D
c
A
C
B
Entrada: Descubrir(D)


GeoGebra Classic 5
Vista Algebraica
Vista Gráfica
A = (−1.16, 1.38)
B = (2.74, 1.62)
C = (0.79, 1.5)
c: (x − 0.79)² + (y − 1.5)² = 3.82
D = (0.25, 3.38)
Teoremas descubiertos con el punto D
Líneas paralelas y perpendiculares:
• AD ⊥ BD
Segmentos congruentes:
• AC = BC = CD
OK
D
c
A
C
B
Entrada:


GeoGebra Classic 5
Algebra
Graphics
A = (-1.54, 0.48)
B = (2.4, 0.46)
C = (1.52, 4.02)
f: 0.02x + 3.94y = 1.8
g: -3.94x + 0.02y = -
D = (1.5, 0.46)
h = 3.56
i = 3.04
j = 0.9
l1 = {false}
List l1: ProveDetails(h² = i j)
g
C
h
f
A
D
B
i
j
Input: LocusEquation(h^2==i*j, C)


GeoGebra Classic 5
Graphics
g
C
b
h
a
f
A
β = 35.58°
D
α = 53.94°
B
i
c
j
eq1

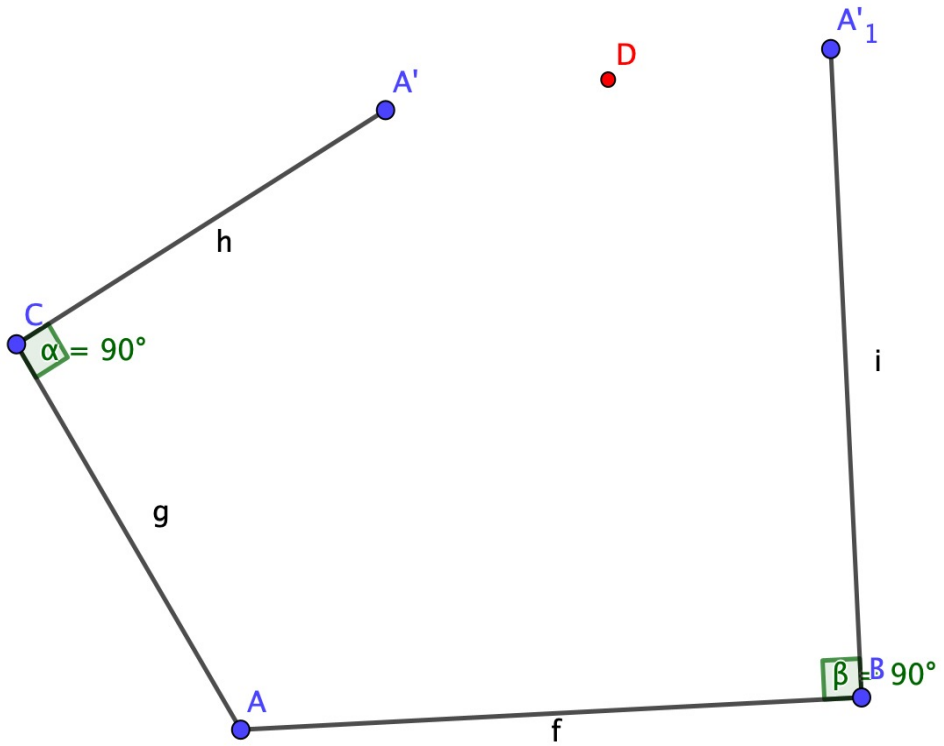

GeoGebra Classic 5
Graphics
g
C
h
b
a
f
D
i
β = 119.74°
B
α = 29.98°
j
c
eq1

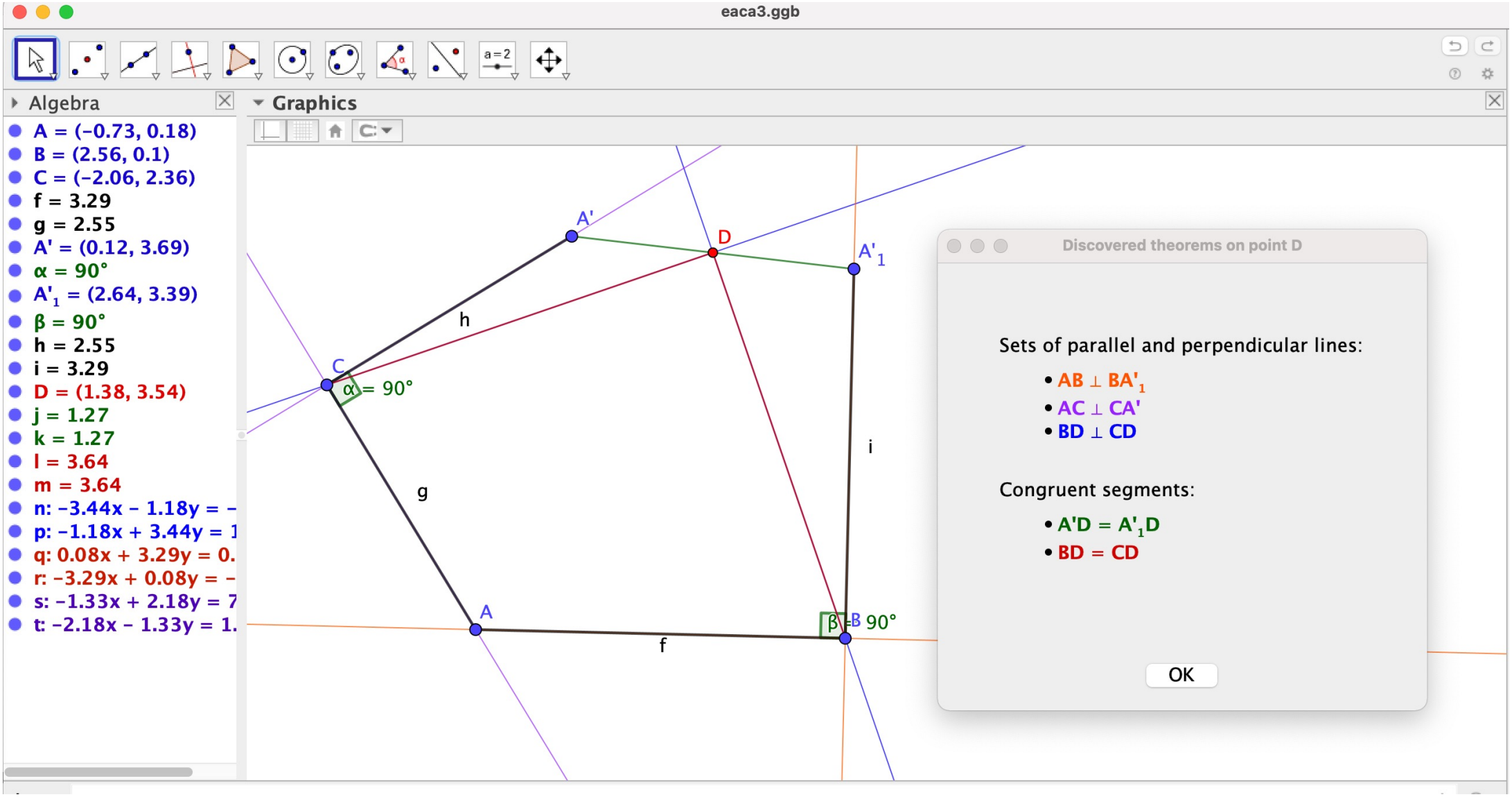
eaca3.ggb

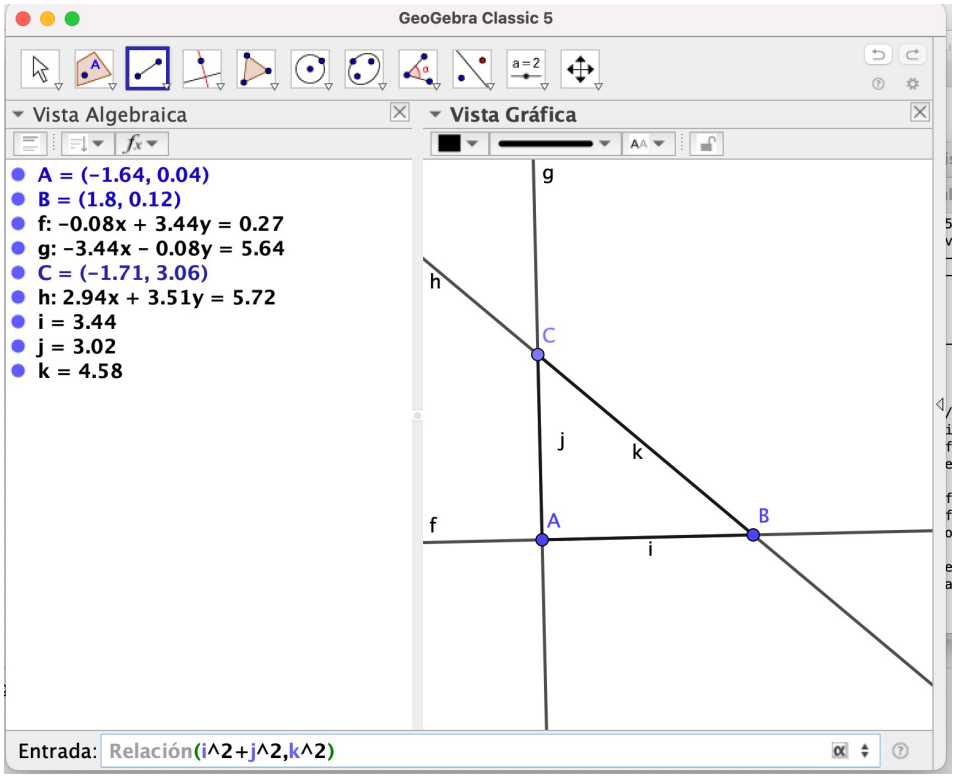
Algebra

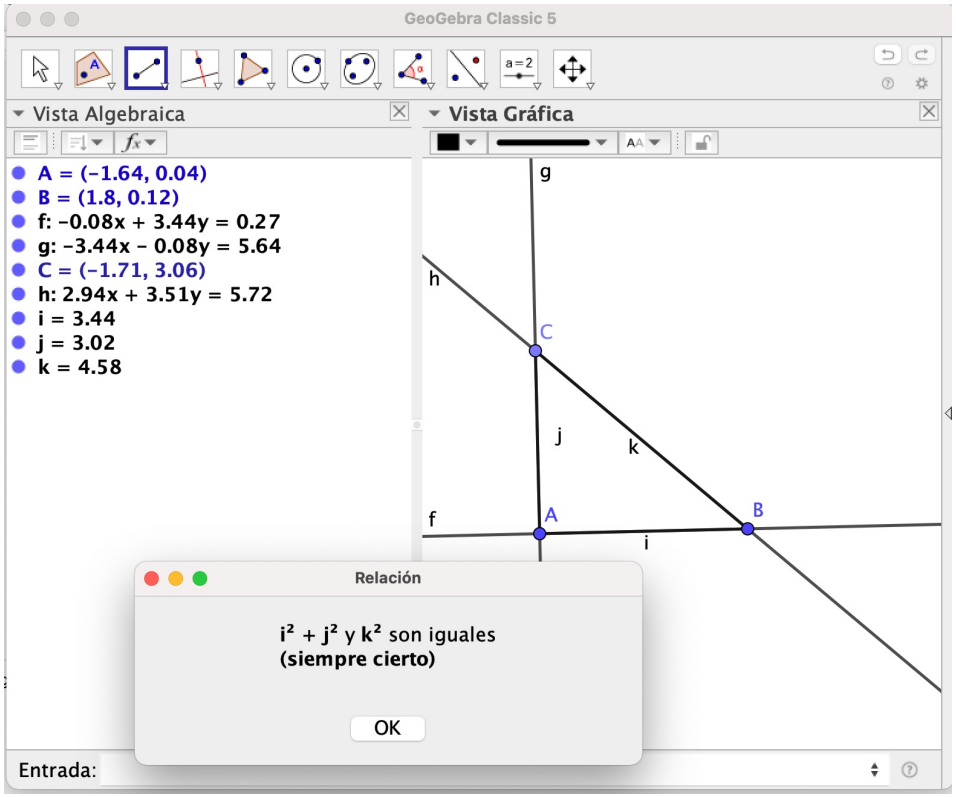
- A = (-0.73, 0.18)
- B = (2.94, 0.36)
- C = (-2.06, 2.36)
- f = 3.68
- g = 2.55
- A' = (0.12, 3.69)
- α = 90°
- A'_1 = (2.76, 4.03)
- β = 90°
- h = 2.55
- i = 3.68
- D = (1.44, 3.86)
- j = 1.33
- k = 1.33
- l = 3.81
- m = 3.81
- n: -3.5x - 1.5y = -10
- p: -1.5x + 3.5y = 11.
- q: -0.18x + 3.67y = (
- r: -3.67x - 0.18y = -
- s: -1.33x + 2.18y = 7
- t: -2.18x - 1.33y = 1.
- a: -5x + 2y = 0.53
- E = (0.44, 1.36)
- c: $(x - 0.44)^2 + (y - 1$

Graphics




eaca3.ggb
Algebra
A = (-0.73, 0.18)
B = (2.56, 0.1)
C = (-2.06, 2.36)
f = 3.29
g = 2.55
A' = (0.12, 3.69)
α = 90°
A'1 = (2.64, 3.39)
β = 90°
h = 2.55
i = 3.29
D = (1.38, 3.54)
j = 1.27
k = 1.27
l = 3.64
m = 3.64
n: -3.44x - 1.18y = -
p: -1.18x + 3.44y = 1
q: 0.08x + 3.29y = 0.
r: -3.29x + 0.08y = -
s: -1.33x + 2.18y = 7
t: -2.18x - 1.33y = 1.
Graphics
A'
D
A'1
h
C
α = 90°
i
g
A
β = 90°
f
Discovered theorems on point D
Sets of parallel and perpendicular lines:
AB ⊥ BA'1
AC ⊥ CA'
BD ⊥ CD
Congruent segments:
A'D = A'1D
BD = CD
OK


GeoGebra Classic 5
Vista Algebraica
Vista Gráfica
A = (−1.64, 0.04)
B = (1.8, 0.12)
f: −0.08x + 3.44y = 0.27
g: −3.44x − 0.08y = 5.64
C = (−1.71, 3.06)
h: 2.94x + 3.51y = 5.72
i = 3.44
j = 3.02
k = 4.58
g
h
C
j
k
f
A
B
i
Entrada: Relación(i^2+j^2,k^2)


GeoGebra Classic 5
Vista Algebraica
Vista Gráfica
A = (−1.64, 0.04)
B = (1.8, 0.12)
f: −0.08x + 3.44y = 0.27
g: −3.44x − 0.08y = 5.64
C = (−1.71, 3.06)
h: 2.94x + 3.51y = 5.72
i = 3.44
j = 3.02
k = 4.58
g
h
C
j
k
f
A
B
i
Relación
i² + j² y k² son iguales
(siempre cierto)
OK
Entrada:

GEOMETRIC INEQUALITIES

BY

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THE SIDES AND THE RADII OF A TRIANGLE

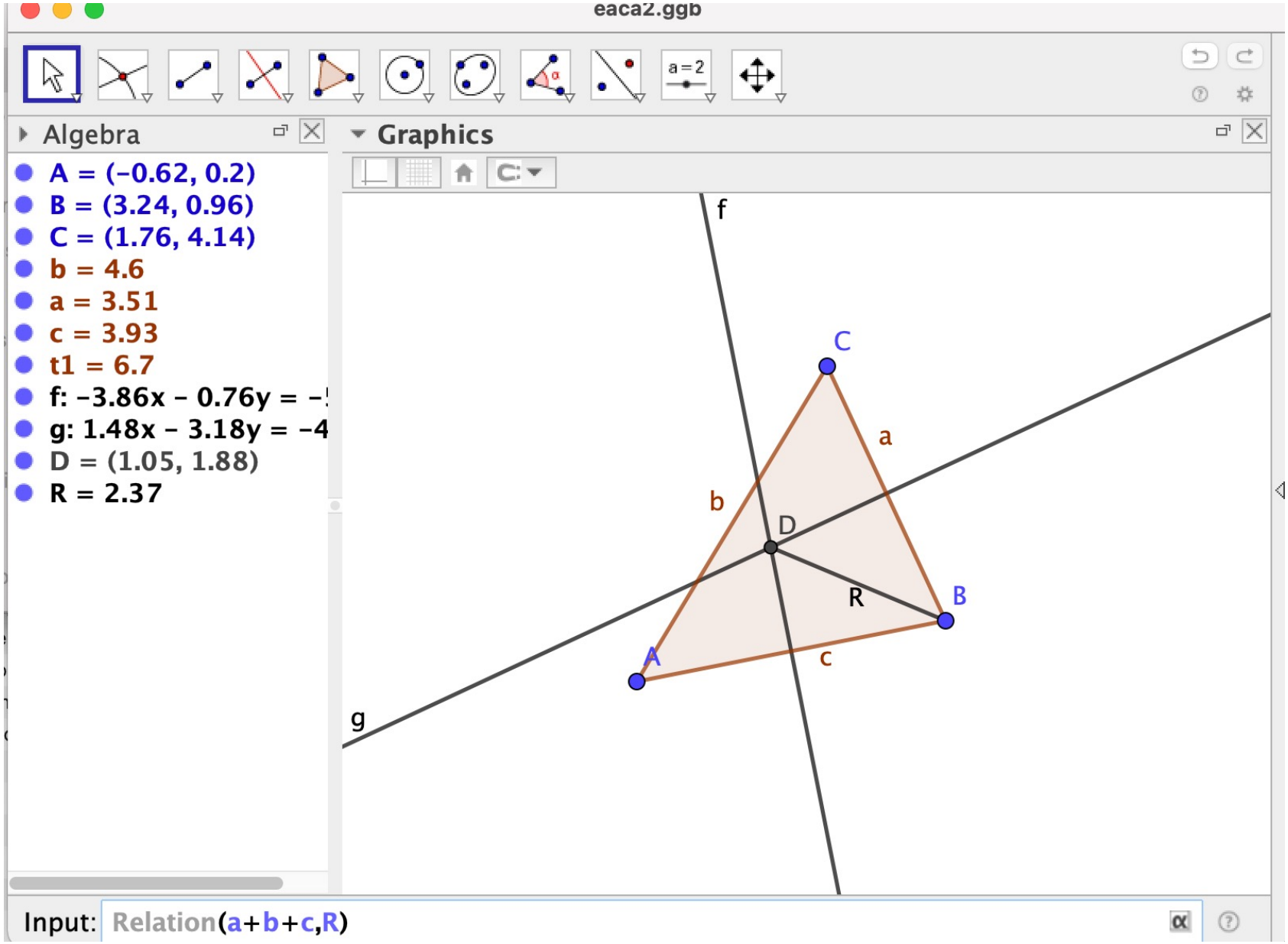
5.3 $a+b+c \leqslant 3R\sqrt{3}$.

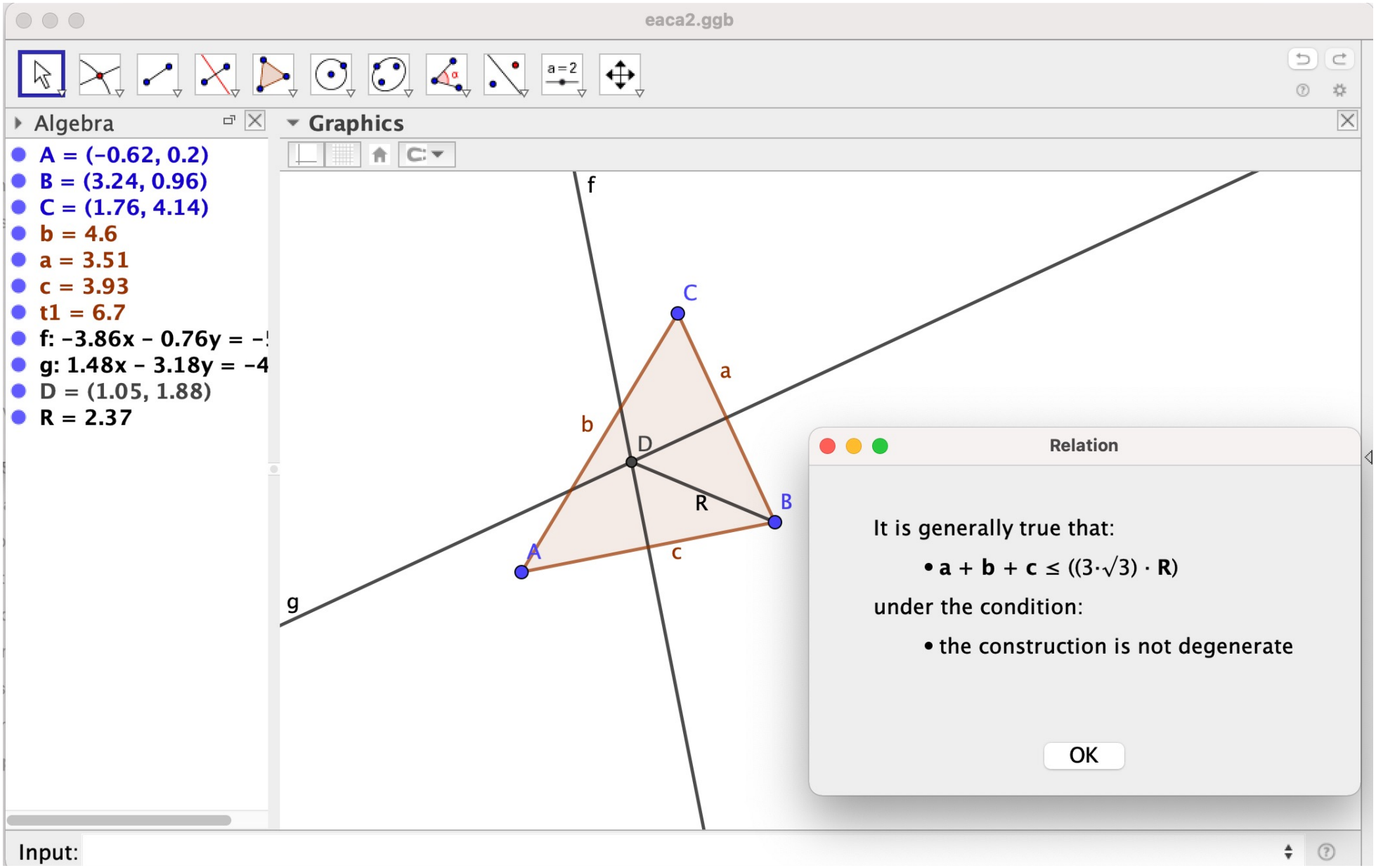
Equality holds if and only if $a = b = c$.

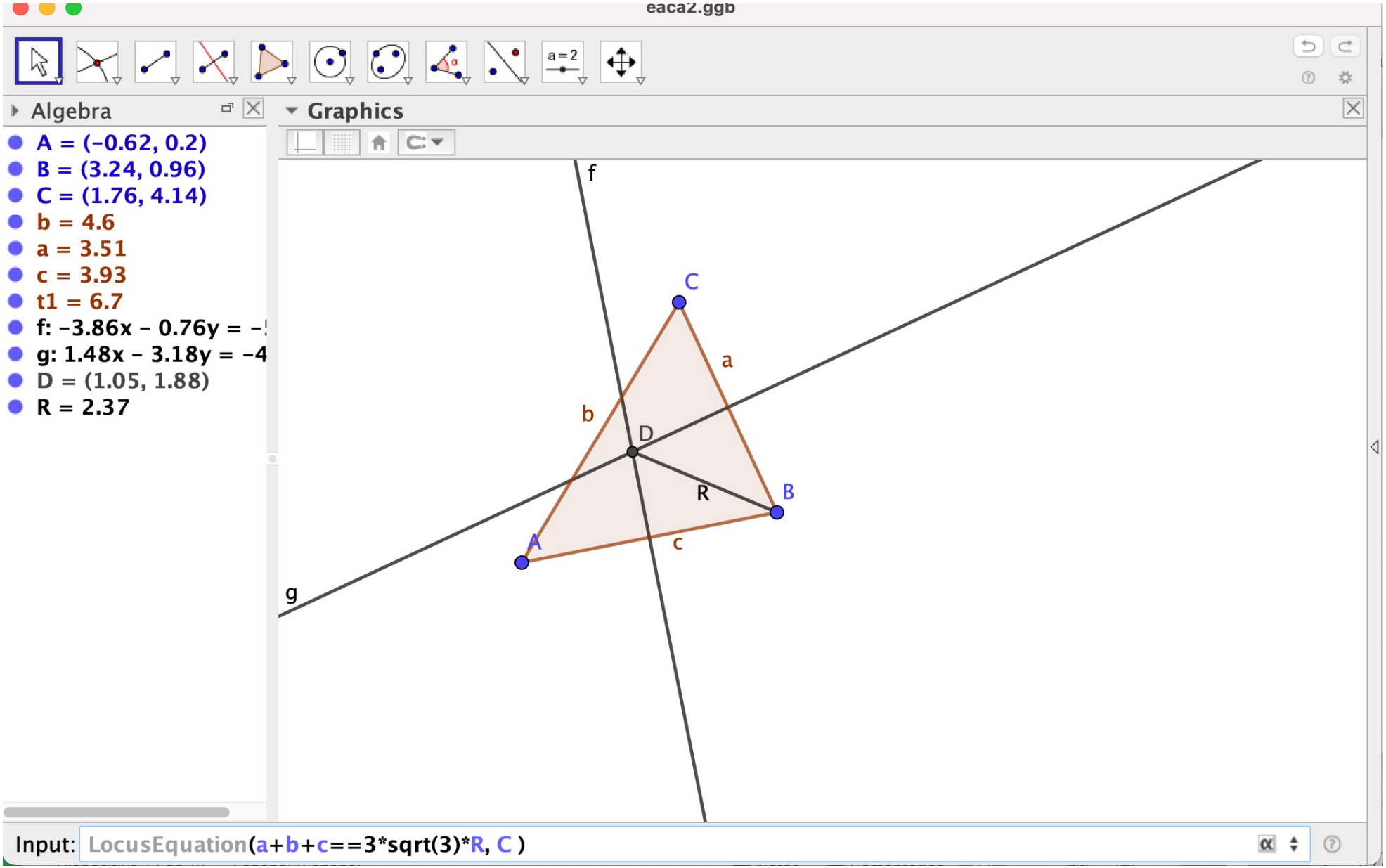
S. Nakajima, Tôhoku Math. J. 25 (1925), 115–121

A. Padoa, Period. Mat. (4) 5 (1925), 80–85.

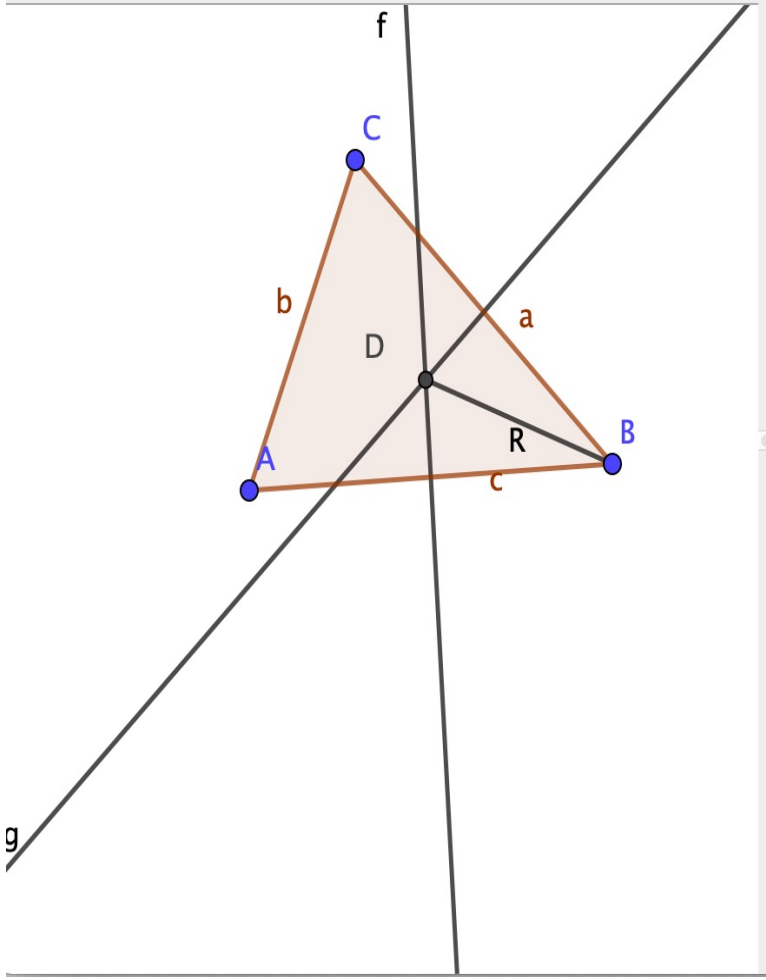
https://www.isinj.com/mt-usamo/Geometric%20Inequalities%20-%20Bottema,%20et.%20al.%20(1968).pdf


eaca2.ggb
Algebra
A = (-0.62, 0.2)
B = (3.24, 0.96)
C = (1.76, 4.14)
b = 4.6
a = 3.51
c = 3.93
t1 = 6.7
f: -3.86x - 0.76y = -
g: 1.48x - 3.18y = -4
D = (1.05, 1.88)
R = 2.37
Graphics
f
g
C
a
b
D
R
B
A
c
Input: Relation(a+b+c,R)


eaca2.ggb
Algebra
A = (-0.62, 0.2)
B = (3.24, 0.96)
C = (1.76, 4.14)
b = 4.6
a = 3.51
c = 3.93
t1 = 6.7
f: -3.86x - 0.76y = -
g: 1.48x - 3.18y = -4
D = (1.05, 1.88)
R = 2.37
Graphics
f
g
C
a
b
D
R
B
A
c
Relation
It is generally true that:
• a + b + c ≤ ((3·√3) · R)
under the condition:
• the construction is not degenerate
OK
Input:


eaca2.ggb
Algebra
A = (-0.62, 0.2)
B = (3.24, 0.96)
C = (1.76, 4.14)
b = 4.6
a = 3.51
c = 3.93
t1 = 6.7
f: -3.86x - 0.76y = -
g: 1.48x - 3.18y = -4
D = (1.05, 1.88)
R = 2.37
Graphics
f
g
C
a
b
D
R
B
A
c
Input: LocusEquation(a+b+c==3*sqrt(3)*R, C)

GeoGebra Classic 5



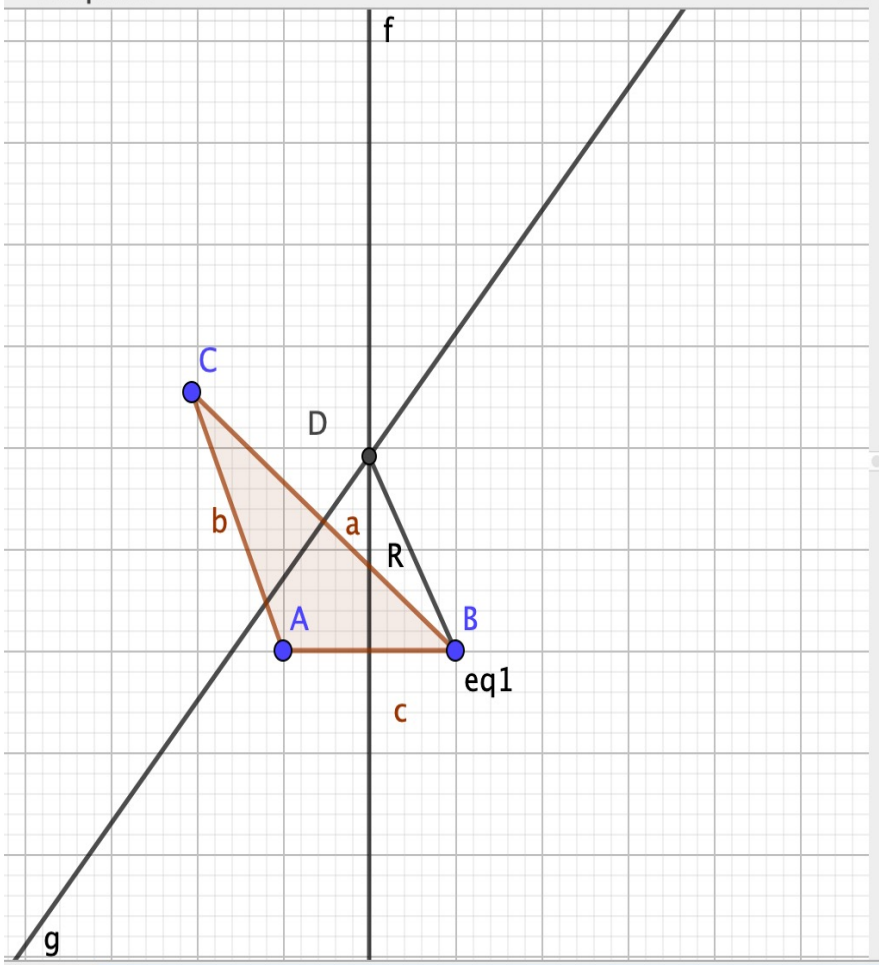
Graphics

Algebra

- A = (-1.36, 1.54)
- B = (2.82, 1.8)
- C = (-0.14, 4.78)
- b = 3.46
- a = 4.2
- c = 4.19
- t1 = 6.61
- f: -4.18x - 0.26y = -3.49
- g: 2.96x - 2.98y = -5.84
- D = (0.67, 2.63)
- R = 2.3
- eq1: 1865294940438712000000000000000000000000000000000000000000x^{16} + 2745849777294

Implicit Curve **eq1**: LocusEquation(a + b + c = 3sqrt(3) R, C)

GeoGebra Classic 5



Algebra

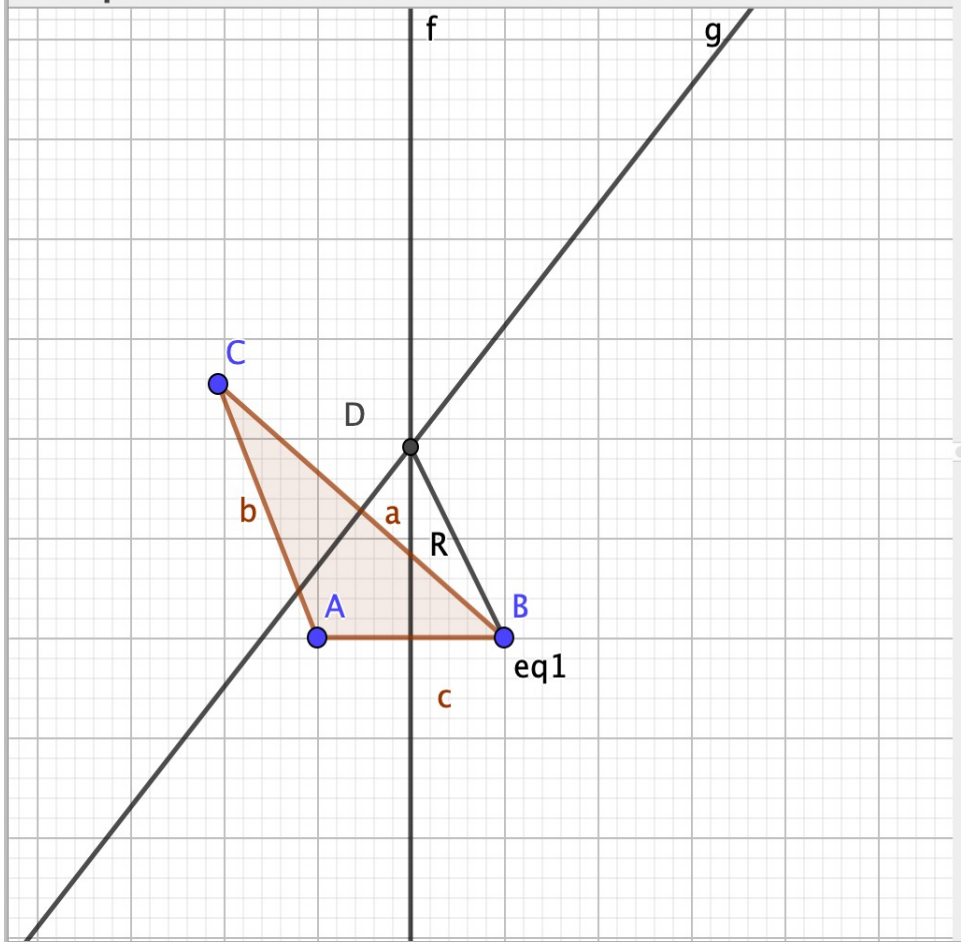
- A = (−1, 0)
- B = (1, 0)
- C = (−2.06, 2.54)
- b = 2.75
- a = 3.98
- c = 2
- t1 = 2.54
- f: x = 0
- g: 3.06x − 2.54y = −4.85
- D = (0, 1.91)
- R = 2.15
- eq1: $531441 + 2361960y^2 + 4592700y^4 + 4660632y^6 + 2106054y^8 - 341928y^{10} - 71$

Implicit Curve **eq1**: LocusEquation(a + b + c ≟ 3sqrt(3) R, C)

Input:

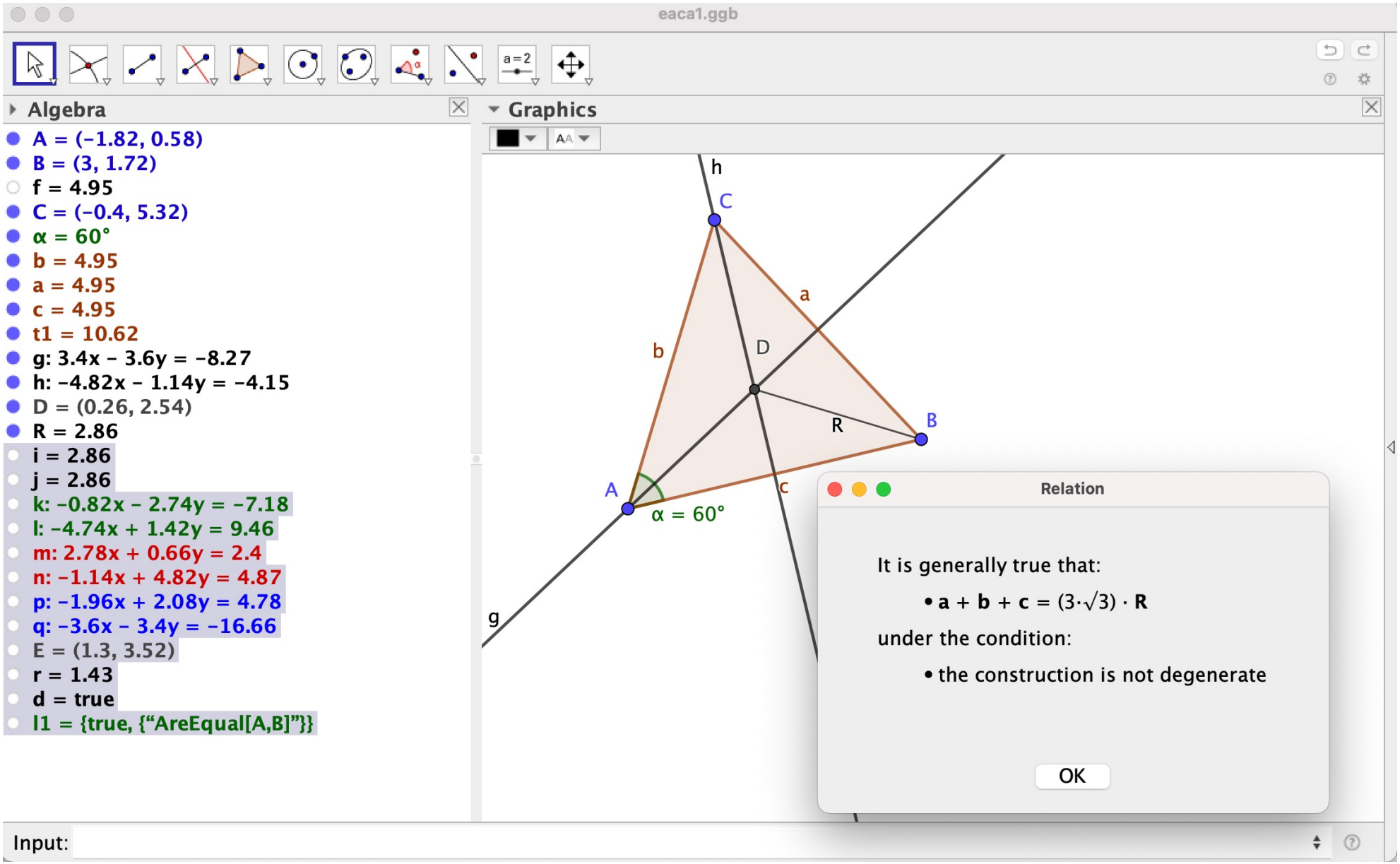
GeoGebra Classic 5

Graphics



CAS

1	eq2:=Substitute(eq1,x = 0) → $\mathbf{eq2 : 88209\,y^{16} - 145368\,y^{14} - 717572\,y^{12} - 341928\,y^{10} + 21}$
2	Solve(eq2,y) → $\{y = -\sqrt{3}, y = \sqrt{3}\}$
3	Factor(eq2) → $(y^2 - 3)^2\,(27\,y^4 + 38\,y^2 + 27)^2\,(121\,y^4 + 186\,y^2 + 81) = 0$
4	Solve(27*y^4+38*y^2+27=0) → {}
5	Solve(121*y^4+186*y^2+81=0) → {}
6	


eaca1.ggb
Algebra
A = (-1.82, 0.58)
B = (3, 1.72)
f = 4.95
C = (-0.4, 5.32)
α = 60°
b = 4.95
a = 4.95
c = 4.95
t1 = 10.62
g: 3.4x - 3.6y = -8.27
h: -4.82x - 1.14y = -4.15
D = (0.26, 2.54)
R = 2.86
i = 2.86
j = 2.86
k: -0.82x - 2.74y = -7.18
l: -4.74x + 1.42y = 9.46
m: 2.78x + 0.66y = 2.4
n: -1.14x + 4.82y = 4.87
p: -1.96x + 2.08y = 4.78
q: -3.6x - 3.4y = -16.66
E = (1.3, 3.52)
r = 1.43
d = true
l1 = {true, {"AreEqual[A,B]"}}
Graphics
h
C
a
b
D
R
B
A
c
α = 60°
g
Relation
It is generally true that:
• a + b + c = (3·√3) · R
under the condition:
• the construction is not degenerate
OK
Input:

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A. Montes (UPC)

M.A. Abánades (URJC)

Zoltán Kovács (The Private University College of Education of the Diocese of Linz)

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C. Abar (Pontificia Universidade Sao Paulo)

S. Van Vaerenbergh (UC)

…….

MTM2017-88796-P

GeoGebra como EPO

- https://github.com/kovzol/geogebra/releases

Versiones GeoGebra 5 Discovery y GeoGebra 6 Discovery off-line

- http://www.autgeo.online/geogebra-discovery/

GeoGebra 6 Discovery on-line

- http://www.autgeo.online/ag/automated-geometer.html?offline=1

Automated Geometer

¿PARA QUÉ?

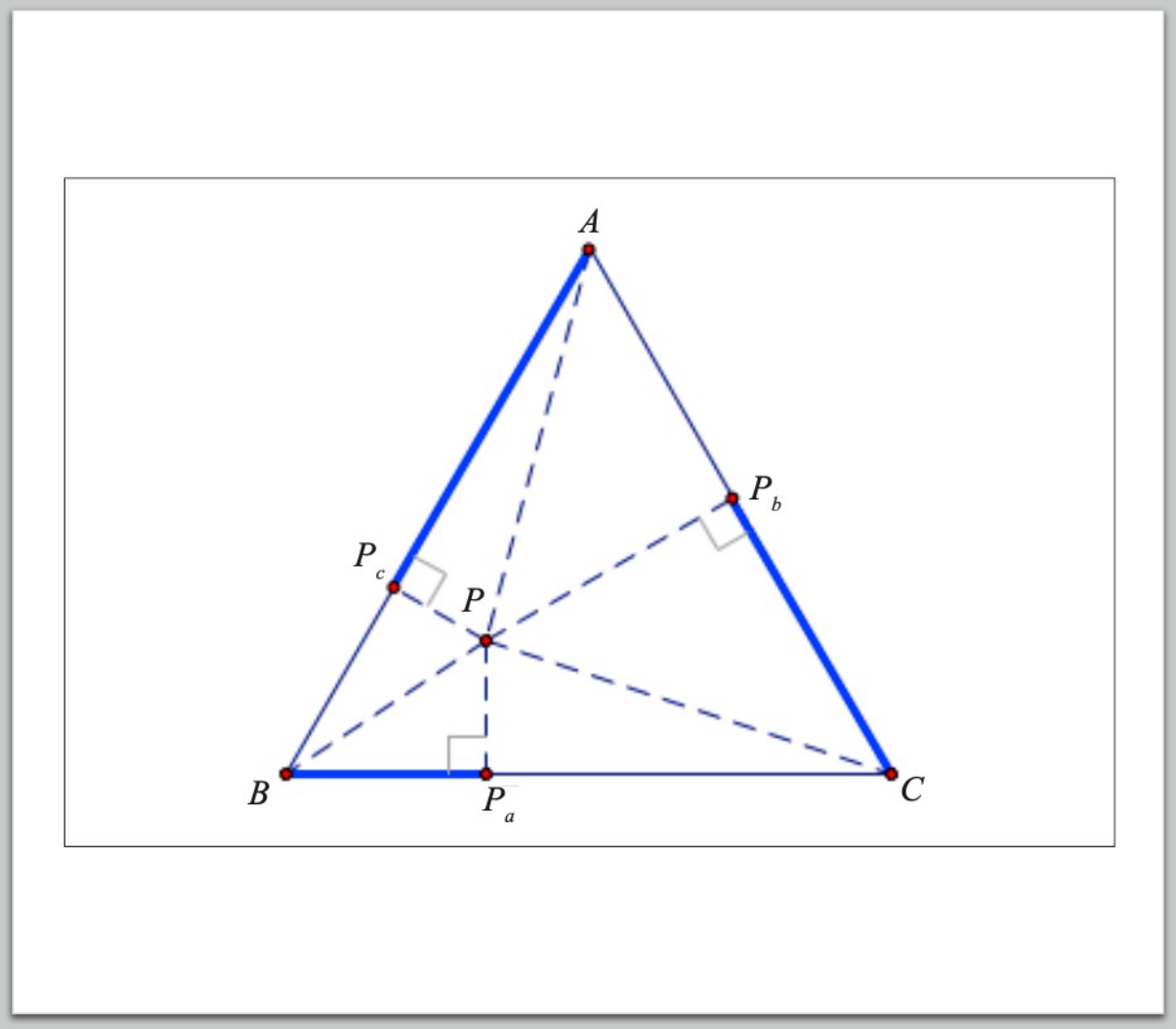
Razonamiento *geométrico*

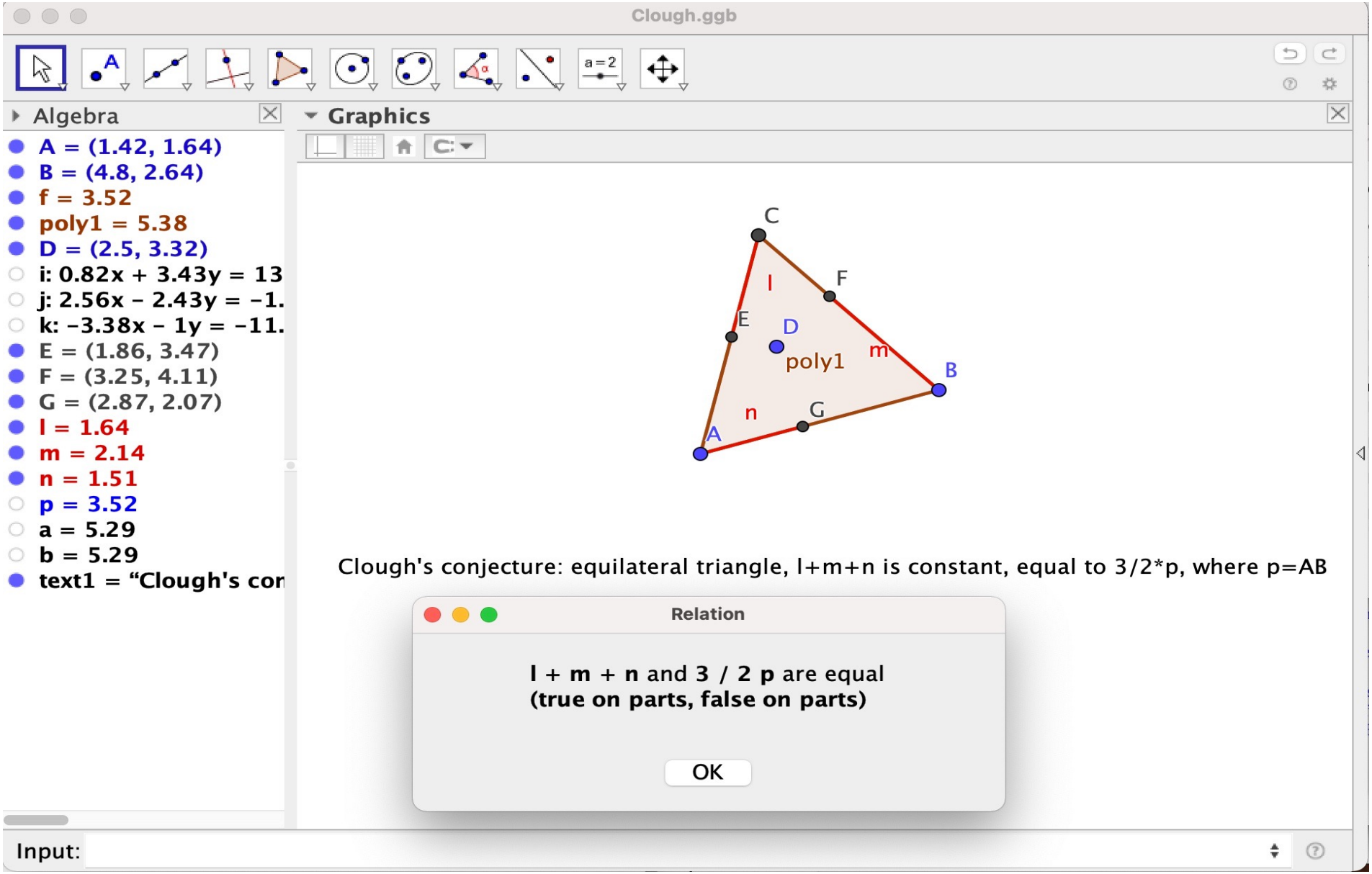
Falta enseñanza geometría euclídea (cuando se habla de razonamiento deductivo…no de geometría)

Enfasis: competencia, IBL, PBL, ámbitos, STEAM…

De Villiers, M., "An illustration of the explanatory and discovery functions of proof". Pythagoras, (2012), 33(3), Art. 193, 8 pages. http://dx.doi.org/10.4102/pythagoras. v33i3.193

The main purpose of this article is to contribute further to the theoretical aspects of the role of proof by providing a heuristic description of some of my personal experiences of the *explanatory and discovery functions of proof* with a geometric conjecture made by a Grade 11 student.




Clough.ggb
Algebra
A = (1.42, 1.64)
B = (4.8, 2.64)
f = 3.52
poly1 = 5.38
D = (2.5, 3.32)
i: 0.82x + 3.43y = 13
j: 2.56x – 2.43y = –1.
k: –3.38x – 1y = –11.
E = (1.86, 3.47)
F = (3.25, 4.11)
G = (2.87, 2.07)
l = 1.64
m = 2.14
n = 1.51
p = 3.52
a = 5.29
b = 5.29
text1 = "Clough's cor
Graphics
C
F
l
E
D
poly1
m
B
n
G
A
Clough's conjecture: equilateral triangle, l+m+n is constant, equal to 3/2*p, where p=AB
Relation
l + m + n and 3 / 2 p are equal
(true on parts, false on parts)
OK
Input:

Gila Hanna and Xiaoheng (Kitty) Yan: **Opening a discussion on teaching proof with automated theorem provers,** For the Learning of Mathematics, Nov. 2021.

- ***GeoGebra's automated proving tools***

GeoGebra …has gained in popularity over the last twenty years and is now widely used… GeoGebra has recently added an Automated Reasoning Tool (ART) to help students conjecture that a given property holds for a specific geometric object and then to find a proof that their conjecture is true. If that is not the case and the property does not hold, ART can also help students make the necessary changes to the original conjecture (Hohenwarter, Kovács, & Recio, 2019, p. 216).

Since the developers of GeoGebra added reasoning tools to their software, they have published a large number of papers in scholarly journals describing the potential of those tools for secondary- school learning…These additions appear to benefit students at both the undergraduate and the secondary level.

It is perhaps too early for empirical studies of classroom experience using the enhancements to GeoGebra… While it is reasonable to expect proof technology to foster students' proving abilities, and there is certainly supporting anecdotal evidence, its potential advantages have not yet been systematically assessed.

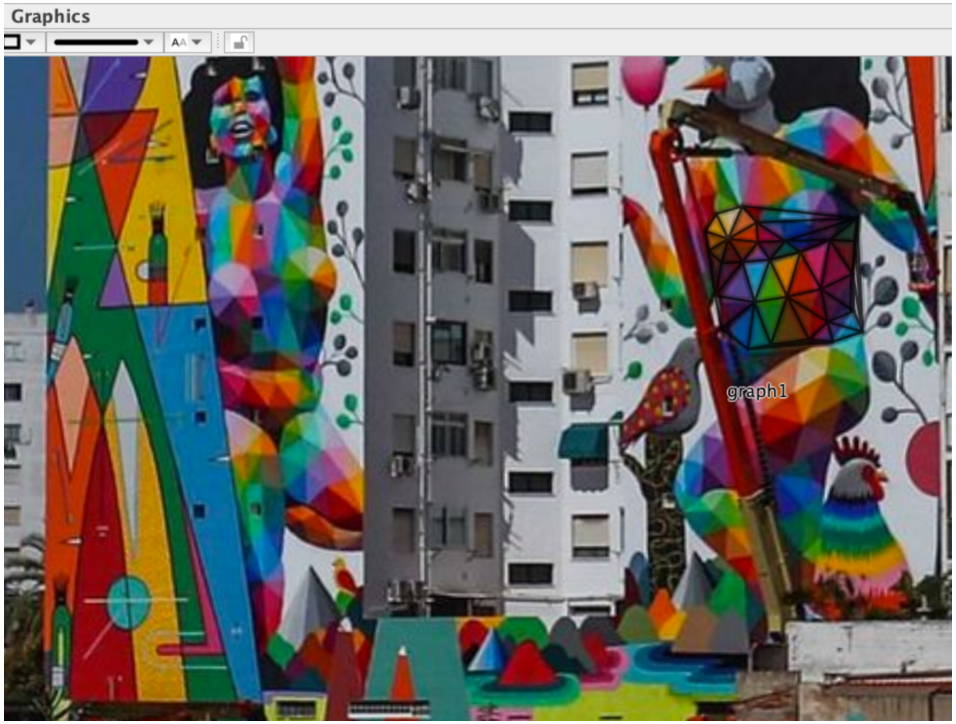
Proof assistants that meet the requirements of these stakeholders

(*the curriculum decision makers (who specify the standard of mathematical validation at a given grade), the teachers (who orchestrate learning and decide what counts as a proof in relation to a standard), and the learners (who are simultaneously constructing an understanding of proof and of the related content*) Balacheff & Boy de la Tour

will never be developed in the absence of initiative on the part of mathematics educators and a demonstrated demand fuelled by increased use. Secondly, success also requires new and effective teaching strategies. These two efforts stand in a reciprocal relationship, so that the full benefit of proof assistants will be seen only over time as new teaching strategies effect the demand for new tool features and vice versa. The responsibility for both efforts rests squarely on the shoulders of educators

The key is to make a start, beginning with exploratory studies of the potential of these new tools at both the secondary and post-secondary levels.

https://en.wikipedia.org/wiki/Gila_Hanna


Graphics
graph1


graph2

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GRACIAS

http://www.recio.tk