Strategies exhibited by good and average solvers of geometric pattern problems as source of traits of mathematical giftedness in grades 4-6

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We describe and analyse the strategies used by students in primary grades 4, 5 and 6 to solve linear and affine geometric pattern problems. Based on two problems posed in a teaching experiment, we have identified several profiles of strategies used by students to solve the problems. We consider the profiles of students who were good geometric pattern problem solvers as traits that may help identify mathematical giftedness. Our results show that average students used very often incorrect functional strategies and were consistent in using incorrect proportional strategies along the grades. On the other hand, good geometric pattern problem solvers tended to use correct functional strategies, although, when they had difficulties in identifying the structure of a pattern, they tended to switch to correct recursive strategies, because they are easier to apply and more reliable.

Keywords: Early algebra, geometric pattern problems, mathematical giftedness, primary school.

Introduction

Learning algebra is a very fruitful way to boost mathematical abilities of all primary school students, in particular mathematically gifted students (gifted students hereafter). Algebra is also an essential tool in secondary school, since it is needed to solve problems in the different areas of mathematics. However, most secondary school students have difficulties to understand and learn algebra, which hinder their mathematical progress. Some of those difficulties are understanding the meaning of letters and the equal sign, distinguishing between the notions of variable and unknown, and transforming word statements into algebraic expressions (Banerjee & Subramaniam, 2012; Jupri, Drijvers, & Van den Heuvel-Panhuizen, 2015).

In the early grades, algebra can be introduced through algebraic thinking that allows working and operating with variables and unknowns, avoiding the use of symbolic alphanumeric expressions (Radford, 2011a). The context of geometric pattern problems (gp problems hereafter) has proved to be successful in developing algebraic thinking in primary school (García-Reche, Callejo, & Fernández, 2015; Rivera, 2013).

Gp problems (Figures 1 and 2) present a graphical representation of the first few terms of an increasing sequence of natural numbers and ask for values (V) or positions (n) of specific terms of the sequence. These problems are especially useful to facilitate the access of gifted students to basic algebraic concepts. Furthermore, gp problems are an adequate context to identify possible gifted students, since they have to use their abilities for generalization, abstraction and symbolization. Those abilities are important components of mathematical reasoning, more developed in gifted students than in the majority of students of the same age or the same school grade (we refer to them as *average* students) (Krutetskii, 1976). Amit and Neria (2008) analysed strategies used by gifted students in grades 6 and 7 when solving gp problems, while Fritzlar and Karpinski-Siebold (2012) distinguished a set of five components of algebraic thinking by observing algebraic abilities of gifted and non-gifted primary school students when identifying and generalising patterns. However, there is little research reporting mathematically gifted students' behaviour when solving gp problems, so it is of interest to inform on traits of mathematical giftedness in the specific context of gp problems.

Teachers should provide all their pupils with opportunities to develop high order thinking. In particular, gifted students require more complex problems compared to average students; thus, it is important to provide teachers with tools to identify their gifted students. Students' solutions to gp problems may be very diverse, depending on the ways of reasoning and performing calculations used, since they reflect different levels of mathematical talent. Gifted students are unusually good problem solvers compared to average students, so a reliable way to identify gifted students, used by most researchers, is to observe their problem solving profiles.

In this context, we present results from a research project aimed to identify and analyse profiles of good gp problem solvers in grades 4, 5 and 6 when solving linear and affine gp problems, to characterise profiles that could differentiate gifted students from average students, and also gifted students in different school grades. The specific objectives of the part of the research presented here are:

- To identify differences between profiles of good gp problem solvers and average students in their solutions of gp problems, as possible traits of mathematical giftedness.
- To identify differences in students' strategies along 4, 5 and 6 primary school grades.

Theoretical framework

Gp problems present realistic contexts to help students give meaning to the pictorial representations of sequences and their numerical values and answer questions (Billings, Tiedt, & Slater, 2007). *Direct questions* ask for values of *immediate, near* and *far* terms (Stacey, 1989), i.e., the number of pieces in the graphical representation of those terms, and also ask to write a *general rule* and an *algebraic expression* to calculate the value of any term of the sequence. *Inverse questions* ask for the position of the term with a given value (Rivera, 2013). Here we focus on answers to direct questions.

Gp problems could have several levels of complexity depending on their characteristics. Friel and Markworth (2009) analysed 18 different geometric patterns and ordered them from less to more complex, the most basic patterns being those based on linear relationships $V=an$ (Figure 1). Patterns based on affine relationships $V=an\pm b$ (Figure 2) increase their difficulty. Lastly, patterns based on quadratic relationships $V=an^2\pm bn\pm c$ are more complex than the previous ones.

Several authors (García-Reche, Callejo, & Fernández, 2015; Radford, 2011b; Stacey, 1989) have identified different strategies used by students to solve direct questions in linear or affine gp problems. As some strategies are labelled differently by those authors, we have merged them into a single categorization of students' answers: the *counting* strategy is based on drawing the graphical representation of the term asked and counting its pieces. The *recursive* strategy uses the constant difference between the values of two consecutive terms to calculate the value of another term, by adding this difference to the value of a known term as many times as necessary. The *proportional* strategy assumes that there is a proportional relationship between the positions of a known term (n) and the asked term (m) and their values, $V(n)$ and $V(m)$: if $m=a\times n$, with $a\in\mathbb{N}$, then $V(m)=a\times V(n)$. The *functional* strategy consists of calculating the value of a term by using an arithmetic or algebraic expression based only on the position of the term in the sequence, and not on the value of a known term. The counting, recursive, and functional strategies give the correct answer if applied properly, while the proportional strategy is correct for linear problems but it is wrong for affine problems.

Gifted students are those who show “a unique aggregate of mathematical abilities that opens up the possibility of successful performance in mathematical activity” (Krutetskii, 1976, p. 77), their problem solving abilities being higher than those of average students. Greenes (1981), Krutetskii (1976), and Miller (1990) detailed several characteristics that gifted students can present, some of them being particularly important to solve gp problems, like identifying patterns and relationships among elements, generalising and transferring mathematical ideas or knowledge between numeric and algebraic contexts, locating the key of problems, abbreviating solution processes, or inverting mental procedures in mathematical reasoning.

The experimental part of our research took place in ordinary classrooms, where, quite often, there are gifted students who have not been identified by their teachers. We did not have any external way to identify gifted students in the sample of our research (e.g., their IQ), so we looked for *good gp problem solvers*. By comparing the solutions provided by the good gp problem solvers and the other students in their classrooms, we aim to identify specific good gp problem solvers’ solution profiles that could be considered as traits of mathematical giftedness and used, together with other procedures, to identify gifted students in ordinary classrooms.

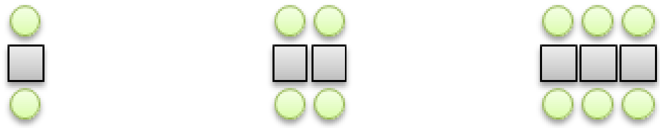
Research methodology

We present results from a study based on grades 4, 5 and 6. These are the last grades in Spanish primary schools, just before students start learning algebra, in grade 7. Two classroom groups in each grade followed an experimental teaching unit for early algebra based on gp problems. There were 43, 34, and 41 students in grades 4, 5, and 6, respectively.

The teaching unit took place during three 45-minutes ordinary mathematics classes. It was aimed to work on: i) the generalisation of functional relationships from the geometric representations, ii) the meaning and use of basic algebraic concepts and symbols, like letter notations and the translation of verbal expressions into algebraic ones, and iii) the reinforcement of the algebraic contents previously learned. The teaching unit was based on linear and affine gp problems similar to those shown in Figures 1 and 2. Students solved individually about 3 problems in each session, depending on their ability and quickness. The teacher (the first author) provided some guidelines to the students; after they had solved each problem, she collected the students’ answers, encouraging them to share on the blackboard their different strategies and debate whether they were correct or wrong. Finally, the teacher explained why the wrong strategies did not work. In this paper, we analyse two problems (Figures 1 and 2) posed in the third session of the experiment.

The two problems are based on the same well-known context of tables and chairs. Both have the same wording (Figure 1), with 2 direct questions (*a*, *b*), an algebraic generalisation question (*c*), another direct question (*d*) aimed to check their generalisation, and an inverse question (*e*). However, they have a difference: the first problem (Figure 1) presents tables (T) with chairs (C) only at the sides, so the sequence is linear ($C=2T$); the second problem (Figure 2) presents tables with chairs both at the sides and the ends, so the sequence is affine ($C=2T+2$). The problems were posed to induce students to generalise the relationships, write an algebraic expression, and use it.

María is organising her birthday party with her friends and relatives. She wants to calculate how many tables and chairs she needs to sit people as in the pictures:



1 table 2 tables 3 tables

a) How many people will sit around 6 tables? How do you know it?

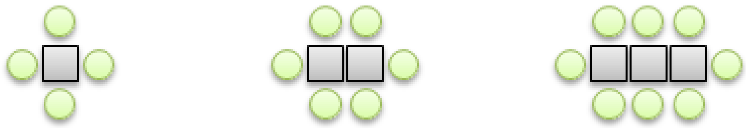
b) How many people will sit around 50 tables? How do you know it?

c) Explain to a friend how she can calculate how many people will sit depending on the number of tables. Write down the formula you have mentioned.

d) Use the previous formula to find how many people will sit around 15 tables.

e) If there are 22 people sitting, how many tables will be? How do you know it?

Figure 1: The linear gp problem



1 table 2 tables 3 tables

Figure 2: The affine gp problem, with the same wording as in Figure 1

After analysing the answers to the direct questions in the problems (*a*, *b*, and *d*), we have identified different profiles of students' solutions, depending on which strategies were used. For example, the *recursive-functional* profile identifies those students who used a recursive strategy in the linear gp problem and a functional strategy in the affine one.

Analysis of students' answers

We have analysed the types of students' strategies in the direct questions in both gp problems. We have considered as *good gp problem solvers* those students who solved correctly, or with minor errors, all the gp problems in the teaching unit. We identified as good gp problem solvers three students in grade 4, four in grade 5, and three in grade 6. Table 1 shows the number of good gp problem solvers and average students in each grade that used each profile to answer the direct questions. In the average students' profile functional-functional, we use brackets to show the number of their correct answers.

In grade 4, two out of the three good gp problem solvers based their solutions to both problems on functional strategies. On the other hand, 23 average students in grade 4 used functional strategies in both problems, but only 7 students solved them correctly. The other students used a diversity of combinations of functional and proportional strategies, all incorrect.

Linear problem	Affine problem
Student 5B-13: *6×2 = 12* *Because you must multiply the table x2.*	Student 5B-13: *14* *Because I've been adding.*

Figure 3: Functional and recursive strategies in a good gp problem solver's answers to question *a*

In grade 5, the good gp problem solvers provided solutions based on functional and recursive strategies, half of them changing from using the functional strategy in the linear problem to the less complex recursive strategy in the affine problem (Figure 3), or vice versa. On the other hand, average students mostly used the functional strategy in both problems, but only 8 students produced correct answers in both problems. The change of strategy made by the good gp problem solvers seems illogical, but it allowed them to succeed in solving the problem. On the contrary, the average students used the same strategy in both problems, maybe because they were comfortable using it or considered that both problems were analogous, but many average students produced incorrect answers (Table 1).

Profiles of solutions		Good gp problem solvers			Average students		
Linear	Affine	4th	5th	6th	4th	5th	6th
Recursive	Functional	0	1	0	0	0	0
Functional	Recursive	0	1	0	0	0	0
Functional	Functional	2	2	3	23 (7*)	20 (8*)	27 (22*)
Functional	Proportional	0	0	0	6	2	0
Proportional	Proportional	0	0	0	3	2	6
Other solutions		1	0	0	8	6	5
Total		3	4	3	40	30	38

(*) Number of average students in this profile producing correct solutions to both problems.

Table 1: Profiles of solutions to the direct questions (*a*, *b*, and *d*) in both problems

In grade 6, all good gp problem solvers and most average students used functional strategies in both problems, with a few average students using such strategies incorrectly. As in the other grades, some average students used proportional strategies. The fact that all good gp problem solvers in grade 6 used correct functional strategies, while a part of the average students used wrong proportional and functional strategies, points to traits of mathematical giftedness in the context of gp problems, namely identification of patterns and relationships among different elements, and generalising and transferring mathematical ideas from a numeric context to an algebraic one.

A profile typical of students when they start solving gp problems is that they tend to move from a strategy to another in the consecutive questions of the same problem (Gutiérrez, Benedicto, Jaime, & Arbona, 2018). However, as the problems we are analysing were posed in the third class session of the teaching experiment, the students had already learned that the final aim of the gp problems was to state a generalisation. Then, all students but one in the sample used the same strategy for all direct questions in each problem, although some students used different strategies for the linear and affine problems, showing diverse profiles in their solutions. Students had also learned that recursive strategies are efficient only for the immediate or near terms.

After comparing the data in Table 1 for the different grades, we get the following conclusions:

i) The good gp problem solvers used, in each problem, a strategy with which they felt confident and that they were sure it was correct, even using a different strategy in each problem (Figure 3). They were more successful than average students in using simpler recursive strategies, and avoided the proportional strategy even in the linear problems, were it provides a correct answer. The good gp problem solvers also became more efficient along the grades using functional strategies.

ii) Average students in all grades used mostly functional strategies, with an increase of the percentage of correct answers along the grades, but we do not observe a reduction in the use of (incorrect) proportional strategies in the affine problem.

Due to the relevant number of average students using the same strategy in the linear and affine gp problems, we have analysed the errors caused by this profile. We have identified three types:

Constancy of proportional relationship

Some average students used a (correct) proportional strategy in the linear problem and they used it again in the affine problem, although now it was wrong. Students did not analyse the whole pattern, but only one term: they calculated proportionally the value of the term requested, considering only the number of guests sitting around one table in the first or the second term of the pattern. Figure 4 shows the written answers of an average student who only considered the number of chairs around the table in the first term.

Linear problem	Affine problem
Student 5A-6: *50×2 = 100* *Because there are twice as many guests as tables.*	Student 5A-6: *50×4 = 200* *Because, if there are 50 tables, there are 4 guests for each table.*

Figure 4: Constancy of proportional relationship in an average student's answers to question *b*

Constancy of recursive relationship

Some average students identified the difference between the values of two consecutive terms and used it in a repeated addition or a multiplication. In Figure 5, the student did not pay attention to the chairs at the ends of the tables in the affine problem and used the increment of 2 chairs ("two more chairs") as proportional ratio.

Error of analysis of diagrams

Some average students did not analyse correctly the parts of the patterns. They identified a wrong difference between the linear and the affine patterns and used it to create a wrong formula. In Figure 6, the student did not identify correctly the chairs at the ends of the tables and, furthermore, he did not use correctly the number of chairs around the first table (4).

Linear problem	Affine problem
Student 5A-11: *50×2 = 100* *Because each time there are 2* [more] *chairs and 50 tables* [multiplied] *by 2 chairs too.*	Student 5A-11: *50×2 = 100* *I multiply the number of tables by 2 more chairs.*

Figure 5: Constancy of recursive relationship in an average student's answers to question *b*

Linear problem	Affine problem
Student 5A-1: *50×2 = 100* *Because, each time, 2 chairs are added with one table.*	Student 5A-1: *50×2 = 100; 100+3=103* *Because there are 4 guests in 1 table, but 2 guests are added each time.*

Figure 6: Error of analysis of diagrams in an average student's answer to question *b*

Conclusions

We have presented a comparative analysis of strategies of solution used by a sample of good gp problem solvers and average students in grades 4 to 6 when solving direct questions in a linear ($V=an$) and an affine ($V=an+b$) gp problems posed as part of an experimental teaching unit. The analysis of students' answers to those problems shows some significant and original findings: most students used mainly functional strategies, although good gp problem solvers showed a tendency to follow profiles using functional and recursive strategies, which were less complex but more successful than the profiles followed by average students, based on functional and proportional strategies. There is a tendency of average students in all grades to use wrong proportional strategies. According to Van Dooren, De Bock, Hessels, Janssens, and Verschaffel (2005), students are prone to apply proportional strategies when they should not be applied. The frequency of such strategies seems to be higher when students start learning proportional reasoning, which, in Spain, usually happens in grade 4.

The analysis of the data collected suggests a difference between the profiles of good gp problem solvers and average students in the use of recursive strategies of solution in the (more difficult) affine problem. Data also seem to show a clear difference between the profiles of both types of students in the use of proportional strategies, even when they were correct. Hence, a contribution of this research is the suggestion that a trait of giftedness in solving gp problems seems to be the use of correct recursive and functional strategies and the absence of proportional strategies to solve gp problems.

Respect to the differences between grades, we have observed an increment in the amount of solutions based on functional strategies, and an increment in the percentage of correct functional strategies. However, it is not apparent a (expected) reduction in the use of incorrect proportional strategies.

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