ANALYSIS OF A GIFTED PRIMARY SCHOOL STUDENT'S ANSWERS TO A PRE-ALGEBRA TEACHING UNIT

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Abstract. *We present a teaching unit aimed to introduce students to algebra and solution of linear equations through geometric pattern problems. We analyse the answers, to the teaching unit, made by a 9-years-old mathematically gifted student in grade 5. We have classified his answers in several styles of strategies of use of graphical information and different types of generalisation. We have also analysed the student's capacity to transfer algebraic knowledge to other algebraic contexts different from geometric pattern problems. Our results show that the student exhibited several characteristics of mathematical giftedness related to his strategies of solution of problems, which are notably different from those strategies that, according to the literature, are used by average students in the same grade.*

Key words: Geometric pattern problems, Giftedness, Pre-algebra, Primary education

INTRODUCTION

Algebra is a key topic in secondary mathematics education, since it is the tool to solve new kinds of problems that otherwise cannot be solved. However, most students have difficulties in learning algebra, related to the meaning of letters and the equal sign, the notions of variable and unknown, and transforming word statements into algebraic expressions (Banerjee & Subramaniam, 2012; Jupri, Drijvers & Van den Heuvel-Panhuizen, 2015), among others.

Geometric pattern problems are a context used to develop algebraic thinking in primary school (García-Reche, Callejo & Fernández, 2015; Merino, Cañadas & Molina, 2013; Rivera, 2013), a type of thinking that is based on dealing with variables and unknowns without operating symbolically with them (Radford, 2000). Some authors have reported that this context seems very useful and fruitful for mathematically gifted students in order to start learning pre-algebraic contents. Amit & Neria (2008) made a detailed analysis of the strategies used by gifted students of grades 6 and 7 to solve pattern problems aimed to generalisation. Fritzlar & Karpinski-Siebold (2012) differentiated five components of algebraic thinking by analysing the differences among the ways of thinking of grade 4 students, including gifted students.

Several manipulative models of equations are used by teachers and researchers. One of them is the *balance*, that allows to represent and manipulate the terms of an equation in a balance. This model helps students to conceptualize the solution of linear equations as a process of compensation to maintain the balance horizontal.

A research question is how design teaching units to introduce gifted students into algebra in a meaningful way. In this context, the specific objectives of the research we present are:

i) To analyse the strategies of solution and ways of generalisation used by a mathematically gifted student when solving geometric pattern problems.

ii) To analyse some aspects of the learning trajectory of a mathematically gifted student while working on a pre-algebra teaching unit.

THEORETICAL FRAMEWORK

Geometric pattern problems show (Figure 1) a graphical representation of the first terms of an increasing sequence of natural numbers, and pose students some questions about the terms of the sequence. These problems include different kinds of questions. The most common *direct questions* are to calculate the values (i.e., the number of pieces in the graphical representation) of *immediate*, *near* and *far terms* of the sequence (Stacey, 1989) and to *express a general rule* to calculate any term of the sequence. The *inverse questions* give the value of a term and ask for the position of that term in the sequence (Rivera, 2013).

A characteristic of geometric pattern problems is the graphical information in the statements. Students use it in different ways. Rivera & Becker (2005) distinguished between *figural* and *numerical* strategies. Figural strategies are those that derive generalisation from analysing and decomposing the graphical representation of the terms. Numerical strategies are those that use the numerical values of the terms, without paying attention to the graphical organization of the pieces.

Students also follow different strategies to calculate the numerical value of terms in the direct questions. García-Reche, Callejo & Fernández (2015) described the strategies *counting*, *recursive*, *functional* and *proportional*. The counting strategy consists of drawing the graphic of the term and counting its pieces. The recursive strategy is based on adding the numerical difference among consecutive terms to calculate the next term. The functional strategy uses the position of the term to calculate the number of pieces asked. The proportional strategy uses a supposed relationship of proportionality between the positions of two terms and their values to calculate the value of one of the terms.

A key component of geometric pattern problems is generalisation. Radford (2006) distinguished five levels of generalisation: *naïve induction* (trial and error guessing), *arithmetic* (recursive calculations), *factual algebraic* (a general rule expressed only by particular numbers), *contextual algebraic* (a general rule referring to any term, but expressed verbally and contextualized), and *symbolic algebraic* (a general rule expressed by using alphanumeric symbols).

Küchemann (1981) classified the meanings students assign to letters in algebraic expressions and the ways they use the letters into six different types: *evaluated*, *not used*, *object*¸ *specific unknown*, *generalised number* and *variable*.

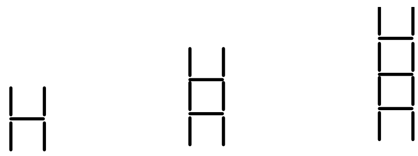
Mathematically talented students have some abilities especially useful to solve geometric pattern problems, like *locating the key of problems*, *identifying patterns and relationships*, *generalising* and *transferring ideas from a context to another*, *developing efficient strategies*, *abbreviating solution process* and *quickness of learning* (Freiman, 2006; Greenes, 1981).

RESEARCH METHODOLOGY

We present results from a case study based on a 9-years-old gifted student who followed an experimental teaching unit for pre-algebra based on geometric pattern problems. The student had been identified as gifted according to the usual criteria (IQ≥120). Furthermore, we had personal information about his problem solving behaviour in other areas of mathematics because he had attended a workshop for gifted students. The experiment was completed when the student was starting grade 5 of primary school. The sessions were conducted by the first author by videoconference (Skype) and were video-recorded.

We designed and implemented a teaching unit divided into three parts. The first part was aimed to teach the student to generalise functional relationships from geometric patterns and solve inverse questions. It consisted of 20 problems, with 15 linear functions (Figure 1) and 5 quadratic functions (Figure 2), all of them including three direct questions (a to c) and an inverse question (d).

We have to calculate the number of pieces of wood necessary to make a ladder with many steps.



1 Step 2 Steps 3 Steps

a) How many pieces are needed for a ladder with 4 steps? How do you know it?

b) How many pieces are needed for a ladder with 10 steps? How do you know it?

c) How many pieces are needed for a ladder with 45 steps? How do you know it?

d) If we need 26 pieces to make a ladder, how many steps does the ladder have?

Figure 1: A linear geometric pattern problem used in the first part of the teaching unit.

A snake grows in this way:

Day 1 Day 2 Day 3

a) How many triangles shall the snake have in day 5? How do you know it?

b) How many triangles shall the snake have in day 10? How do you know it?

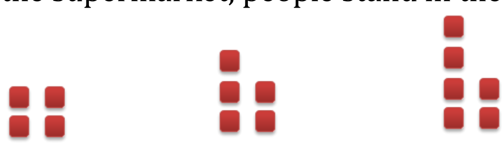
c) How many triangles shall the snake have in day 40? How do you know it?

d) If the snake has 37 triangles, which day will it be? How do you know it?

Figure 2: A quadratic geometric pattern problem used in the first part of the teaching unit.

The second part was aimed to teach basic algebraic concepts, like the meaning of letters, translation of verbal expressions into algebraic ones and solution of linear equations. It consisted of 6 problems (Figure 3), all of them including a direct question (a), an algebraic generalisation question (b) and an inverse question (c).

Waiting in the queue at the supermarket, people stand in the following way:



Minute 1 Minute 2 Minute 3

a) How many people will be waiting in minute 12? How do you know it?

b) Write down the formula you have used in the previous question.

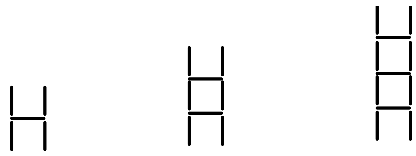
c) If there are 49 people waiting, how many minutes have passed?

Figure 3: A geometric pattern problem used in the second part of the teaching unit.

In this part, like other authors (for instance, Store, Richardson & Carter, 2016), letters were introduced as objects by using the first letter of the unknown. In addition, we used a virtual balance to introduce the student into linear equations.

The third part of the teaching unit focused on reinforcing the algebraic contents learned. We selected 7 problems from the first part, in which the student was only able to solve the inverse questions by trial and error, and we modified their statement asking (Figure 4) for an algebraic generalisation (*a*), to simplify this algebraic expression (*b*), and an inverse question (*c*). We also posed 6 algebraic word problems (Figure 5) and 4 linear equations to practice the routine of solving equations.

We have to calculate the number of pieces of wood necessary to make a ladder with many steps.



a) Write down an algebraic formula to calculate the pieces of wood necessary for any number of steps.

b) Do you believe that it is possible to get a simpler formula? If so, write it down.

c) If we need 41 pieces to make a ladder, how many steps does the ladder have?

Figure 4: A variation of the problem in Figure 1 used in the third part of the teaching unit.

Hector and Mary are counting how many color pencils has each one. Hector has twice as many pencils as Mary plus 7 pencils. If Hector has 33 color pencils, how many does Mary have?

Figure 5: An algebraic word problem used in the third part of the teaching unit.

ANALYSIS OF STUDENT'S ANSWERS

In the first part of the teaching unit, the student generally used visual and functional strategies to solve direct questions. Moreover, he did mainly factual algebraic generalisations and also contextual generalisations. Table 1 shows the number of times the student used each type of strategy to answer direct questions (he did not answer questions *b* and *c* in one problem).

	Use of graphical information		Calculations for direct questions		Ways of generalisation		
	Visual	Numerical	Recursive	Functional	Arithmetic	Factual	Contextualized
a	12	8	5	15	5	8	7
b	14	5	1	18	1	12	6
c	14	5	0	19	0	12	7

Table 1: Strategies used to answer direct questions in the first part of the teaching unit.

To solve the inverse questions in the first part of the teaching unit, the student used different strategies depending on the type of generalised expression he produced in the direct questions (Table 2). If the generalisation was of type $y=ax\pm b$, the student was able to correctly get the answer by reversing the calculations, with only two exception. In contrast, if the generalisation was of types $y=ax+b(x\pm c)\pm d$, $y=x^2$ or $y=(x\pm a)(x\pm b)$, he was not able to calculate the answers because he still did not know how to solve equations, and he used trial-and-error direct calculations.

In the second part of the teaching unit, the student understood rapidly the meaning of letters in algebraic expressions, making use of the letters in a dual way: as an object (e.g., using *m* to express *minute*) and as a generalised number (he knew that letters represent a variety of values). He translated his verbal expressions into algebraic ones using the standard hierarchy of arithmetic operations and parenthesis. He meaningfully interiorized the balance model so that he was able to simulate it in the word processor. For that reason, he learned to solve equations by compensation (operating the same way in both sides) and made a great progress towards the algebraic syntax, being able to solve equations with a quasi-unnoticeable use of the balance.

	Trial and error	Correct inversion	Wrong inversion
$y = ax \pm b$	1	6	1
$y = ax \pm b(x \pm c) \pm d$	7	- -	- -
$y = x^2$	2	- -	- -
$y = (x \pm a)(x \pm b)$	3	- -	- -

Table 2: Types of generalisation and strategies for the inverse questions in the first part.

In the third part of the teaching unit, thanks to the algebraic contents taught in the second part, the student only used functional strategies, symbolic algebraic generalisations and letters as variables for the direct questions. He was also able to simplify his own algebraic expressions. For the inverse questions, he ever solved the equations in an algebraic syntactic way and he solved correctly all the algebraic word problems with little difficulty.

CONCLUSIONS

We have presented a synthesis of the development of a research experiment aimed to introduce a 9-year-old gifted student into algebra. The student learned to use algebra understanding the meaning of letters (as variables or unknowns), translating verbal expressions of the generalised geometric patterns into algebraic expressions, giving meaning to linear equations as a balance process and overcoming the difficulties showed in the first part of the unit.

The analysis of the student's answers, taking all the geometric pattern problems solved into consideration, shows that he used mainly visual functional strategies. This demonstrate a behaviour clearly different from that of average students in the same grade since, according to García-Reche et al. (2015) and Merino et al. (2013), average students in grade 5 solve direct questions mainly by using the counting strategy, even when they are asked to calculate a far term, and only a few of them try to solve inverse questions, ever by using trial and error strategies (Rivera, 2013). Furthermore, the student showed some characteristics of mathematically gifted students, like identification of patterns and relationships, generalization, development of efficient strategies to generalize the patterns, abbreviation of solution processes and quickness of learning.

Along the teaching experiment, in the resolution of the problems, the student showed often characteristic features of mathematically gifted students, mainly abilities to identify patterns and relationships, to generalise them, to develop efficient strategies, to locate the key of problems, to abbreviate solution processes, to transfer ideas from one context to another and, particularly, a high speed of learning.

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