IMPROVEMENT OF GIFTED STUDENTS' VISUALIZATION ABILITIES IN A 3D COMPUTER ENVIRONMENT

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We present the software Cubes & Cubes, designed to help students improve their visualization abilities. It presents tasks asking to draw the orthogonal projections of given sets of stacked cubes, or asking to build a set of stacked cubes corresponding to given orthogonal projections. This software allows teachers to pay differentiated attention to their pupils, in particular to mathematically talented students. We describe the strategies used by some mathematically talented students to solve tasks posed by Cubes & Cubes, and we analyze students' outcomes in terms of the amount of cognitive demand of their strategies.

Keywords: Spatial visualization; mathematically talented students; plane representations; 3d software.

INTRODUCTION

Research in mathematics education (summarized in Clements, 2013; Gutiérrez, Boero, 2006; Lester, 2007) has showed that the use of technology in mathematics classes facilitates the learning and improves the understanding of mathematical concepts, since technology offers opportunities to achieve better learning by engaging students in solving tasks. Furthermore, the suitable use of software can help teachers to organize a personalized learning in their classes. An aim of this paper is to present the software *Cubes & Cubes*, which helps students develop their visualization skills while solving 3-dimensional representation tasks. The controlled use of the software can be very helpful for teachers to pay individualized attention to their pupils in a class group, and to attend the learning necessities and develop deeper levels of understanding of all students.

Nowadays we can find students with different mathematical abilities in the same classroom. Teachers should take care of their pupils' different needs, but sometimes teachers do not have the necessary media to attend adequately their mathematically talented pupils. This can cause that those students do not develop their high mathematical capabilities as much as they could or, even, they could come to school failure. By *mathematically talented students* we mean those students having a mathematical ability clearly over the average students with their same age, school grade or learning experience. Gifted students are a extreme case of mathematically talented students. Authors like Freiman (2006), Greenes (1981) and Krutetskii (1976) have analyzed and described behaviour characteristics of mathematically talented students.

The other aim of this paper is to analyze different strategies used by mathematically talented students to solve visualization tasks with Cubes & Cubes. Namely, we aim i) to identify strategies used by students to solve different space visualization tasks, and ii) to analyze the tasks and those students' strategies to identify different levels of cognitive demand in their outcomes. We show different students' strategies to solve activities using the software Cubes & Cubes and we analyze the cognitive demand of those strategies. The results presented here are a part of a research project focused on the design of teaching units for ordinary classrooms that pay differentiated attention to mathematically talented students.

We use the model of *cognitive demand* (Smith & Stein, 1998) to organize tasks to teach mathematically talented students and to evaluate their problem solving outcomes. A mathematical task, activity or problem is classified into four levels of cognitive demand depending on the cognitive effort necessary for a student to solve it, which is narrowly related to the sophistication of the student's reasoning while solving the task. The levels, from the lowest to the highest, are labelled *memorization*, *procedures without connections*, *procedures with connections*, and *doing mathematics* (Smith & Stein, 1998). The cognitive demand model allows teachers and researchers understand students' answers from the viewpoint of the complexity of the mathematical knowledge used by students to solve tasks. For instance, it is possible to note that some tasks are solved by students experiencing a certain level of cognitive demand to get the answer, while students using lower levels of cognitive demand cannot solve those tasks.

This model allows teachers select tasks with an appropriate degree of challenge for their pupils. It has probed to be useful in analyzing mathematical problems by theoretically estimating the difficulty that supposes solving the problem for students (Stein, Grover & Henningsen, 1996). The model can also be used to evaluate the role of teachers selecting and implementing mathematical activities (Henningsen & Stein, 1997), as well as, to analyze the behaviour of students with different mathematical talent when they solve activities (García & Benítez, 2013).

CUBES & CUBES: A 3D SOFTWARE TO IMPROVE VISUALIZATION ABILITIES

The educational software *Cubes & Cubes* (Hoyos, Aristizábal & Acosta, 2014) aims to enhance the spatial visualization abilities of elementary and secondary school students. It allows them to handle solids made of unit-sized cubes (Figure 1). The solids can be rotated to visualize them from different positions, just by dragging the mouse/pad or pressing the key arrows, so the user can experience a tri-dimensional rotation of the solid. It also has a tools that automatically allows to observe the solid on the screen from its top, front and right side views (orthogonal projections), as showed in Figure 2, where the red arrow identifies the right side of the solid.

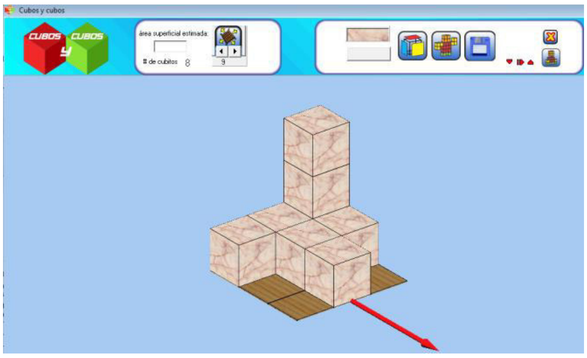


Figure 1. Cubes & Cubes environment to visualize objects constructed with unit sized cubes.

It is possible to build solids on different board sizes by adding and dropping cubes, and to paint the cubes with several colours and textures available. There is also the possibility of saving the solid on the screen and reload it, so teachers can prepare their own activities adapted to their specific pupils.

Cubes & Cubes offers several types of activities based on solids that can be built randomly by the software or loaded from files designed by the teacher. Those activities are specifically designed to help students understand important concepts related to spatial visualization, like orthogonal projection and orientation in space. In these types of activities the users have to rotate the solids to accomplish the task, helping them to develop their visualizations skills, to learn how to describe what a solid looks like from different views, and to learn how to get a 2-dimensional representation from a 3-dimensional solid in the space, and vice versa. The software also has the ability to evaluate the user's answer to all types of tasks. The different types of activities are:

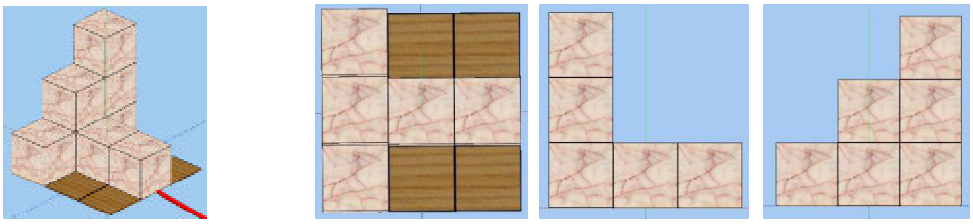
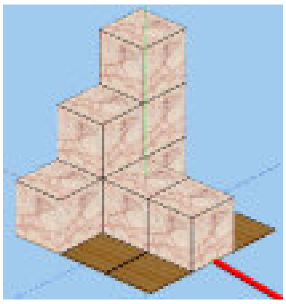
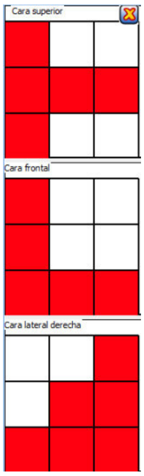


Figure 2. A solid and its views from top, front and right side in Cubes & Cubes.

- Draw on the screen the orthogonal projections (Figure 3a) or the numeric orthogonal projections (Figure 3b) of a given 3-dimensional solid. In numeric orthogonal projections, each cell of the projection shows the number of cubes the solid has in the row represented by that particular cell.



(a)



(b)

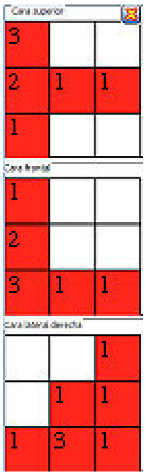
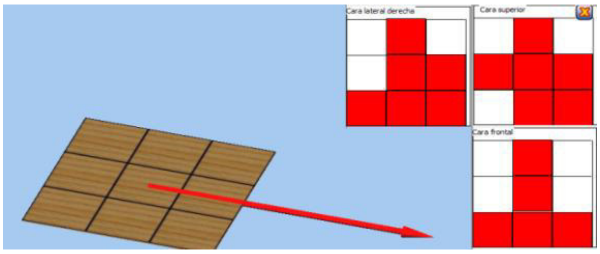


Figure 3. (a) Orthogonal and (b) numeric orthogonal projections of the solid from top, front and right side views.

- Build a 3-dimensional solid based on a given set of orthogonal projections (Figure 4a) or numeric orthogonal projections (Figure 4b). As the solution may not be unique in this type of tasks, this gives the teacher the possibility to open a discussion in the classroom, since different students may have built different solids from the same orthogonal projections.

(a)



(b)

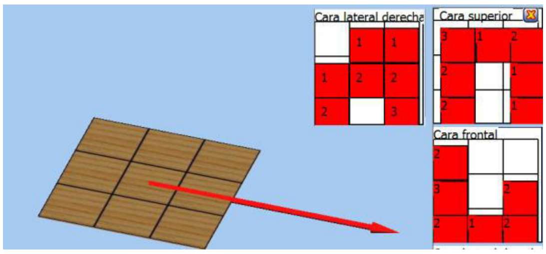


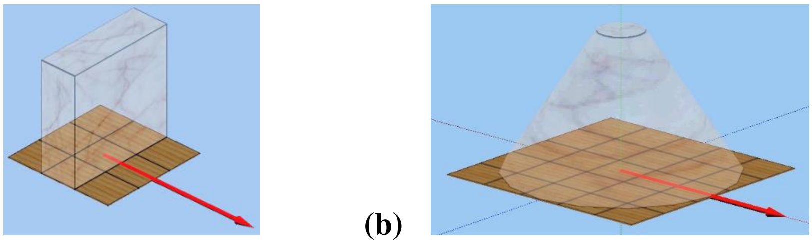
Figure 4. Build a solid represented by the given (a) orthogonal or (b) numeric orthogonal projections.

- Count the number of cubes used to build a given solid.

- Build a solid congruent to a given solid.

- Match the positions of two identical solids shown on the screen, so they look the same on the screen.

- Explore the concept of volume by calculating (Figure 5a) or estimating (Figure 5b) the number of unit-sized cubes necessary to fill in a solid.



(a) (b)

Figure 5. (a) Calculate or (b) estimate the volume of a given solid by filling it in with unit-sized cubes.

METHODOLOGY

Participants and context

The subjects for this study were 40 mathematically talented students aged 10 to 12 years participating in a special out-of-school workshop conducted by the researchers. The classroom was organized in pairs of students, with one computer for each pair. The introduction of the experimental environment to the students was limited to make a short presentation of orthogonal projections, since they did not know about this kind of plane representation in geometry, and to show them how to manage the software Cubes & Cubes. We described the types of tasks they were going to be posed and, occasionally, the whole class worked out an example. We never showed the students any procedure for completing the activities nor explained how to solve them.

Data gathering instrument

Our source of data are the videos recorded by a screen capture software that also recorded sound, so we can see all the actions made by students on the screen and hear their talks. These data allowed us to identify the reasoning under students' decisions when choosing strategies to solve the tasks.

Activities

From the tasks supported by the software, described in the previous section, we posed to students the following ones, in this order: Draw the orthogonal projections of a solid; draw the numeric orthogonal projections of a solid; build a solid from a set of numeric orthogonal projections; and build a solid from a set of orthogonal projections. Previous research has proved that those types of activities have different difficulties for students (Gutiérrez, 1996), so we posed the tasks from the easiest to the most difficult one. For each type of task, we stated several problems differing on the complexity of the solids. These activities do not require any specific mathematical knowledge, but visual or analytical reasoning (Krutetskii, 1976) and visualization abilities to create and manage adequate mental images (Presmeg, 1986).

RESULTS

We have noted different strategies to solve each type of activities posed, and we have analyzed the cognitive features of these strategies to identify their level of cognitive demand. Below we describe and analyze the most interesting strategies used by students to solve each type of task.

Task: Draw the orthogonal projections of a solid

Strategy: Reproduction or copy. Some students discovered that the software has an option to automatically show each orthogonal projection of the solid on the screen, so they used it to copy the image given by the computer in the grid (Figure 6). This strategy only involves the careful reproduction of the image, and it is unambiguous, clear and direct. Students do not use any procedure, nor they need to use the meaning of orthogonal projections, they just copy what they see on the screen. Then, this strategy is typical of the memorization level of cognitive demand. The students who chose this strategy had not any problem to solve the task correctly but they did not improve their visualization abilities.

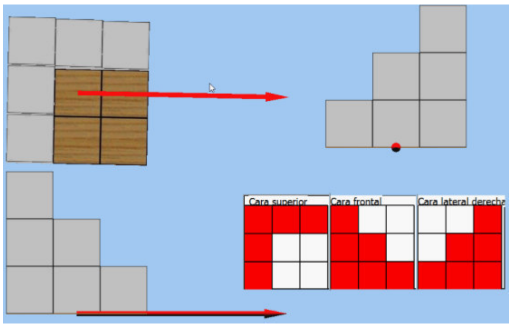
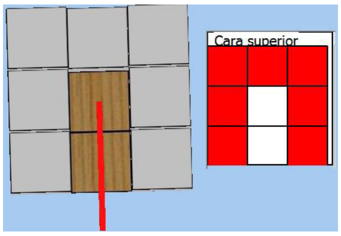


Figure 6. Task solved by reproducing the shapes of the solid's faces.

Strategy: Movement of the solid. Most students moved the solid on the screen to place it in a position they consider suitable to identify one of its orthogonal projections and draw it in the grid. Then, students moved the solid again, looking for another projection, and so on.

We have identified two mistakes made by students who used this strategy because they moved the solid to an inadequate position. An error was to show the top face placed in a wrong position (Figure 7a). The other error was to place the solid showing a wrong face (Figure 7b).

(a) 

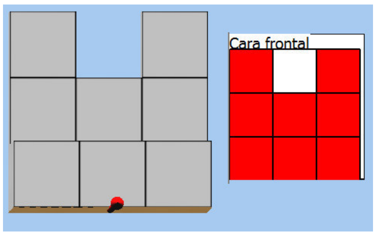
(b) 

Figure 7. (a) Incorrect position of the top face. (b) Mix-up between front and right side faces.

This strategy is algorithmic but it requires from the students to know the meaning of each orthogonal projection and its corresponding face of the solid, to place it correctly on the screen. It has a limited cognitive demand for a successful completion, since it is only necessary to know what is an orthogonal projection, and it does not require neither reasoning nor connections. This strategy is focused on producing correct answers instead of on developing mathematical understanding. Then, it is typical of the procedures without connections level of cognitive demand.

Strategy: Still solid. Some students were able to visualize the three orthogonal projections keeping the solid still in a specific position. The position chosen is very important because it must let students imagine all the orthogonal projections. The students can successfully solve the task in different ways depending on the position of the solid. In general, the arrow has to point to the right and the figure should be slightly inclined (Figure 8).

This strategy requires some degree of cognitive effort. Although it may be used by most students, it cannot be applied mindlessly, since students need to coordinate different faces and visualize the solid from positions different from their real one. The students using this strategy developed their visualization abilities more than students using the other strategies. This strategy is typical of the procedure with connections level of cognitive demand.

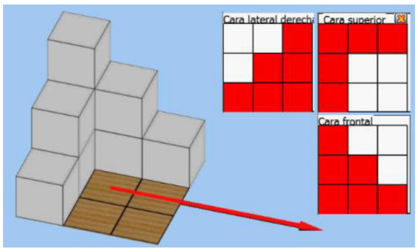


Figure 8. Orthogonal projections from a still solid.

Task: Draw the numeric orthogonal projections of a solid

The need to count the number of cubes in each row makes this type of tasks very different from the previous one. For instance, the first strategy used in the previous task is not useful now, since the orthogonal projections showed by the software do not allow count the number of cubes in each row.

We have found only two different strategies to solve this type of tasks, that are similar to the two last strategies described for the previous task. Some students moved the solid as many times as they considered necessary to see all the rows and count their cubes. This procedure is algorithmic and, if students follow it carefully, they do not have problems to solve the task. It does not require either reasoning or connections of elements, so it has a limited cognitive demand for successful completion. This strategy belongs to the procedure without connection level of cognitive demand.

Other students kept the solid still and used their visualization abilities to count the number of cubes in the rows. We have identified two different levels in this way of solution: Some students set the solid in only one position, while other students used several still position to complete each projection. This strategy requires a certain cognitive effort and it is necessary for the students to have developed their ability to visualize and understand the space. It is typical of the procedure with connection level of cognitive demand.

Task: Build a solid from a set of numeric orthogonal projections

Only 60% of the students were able to solve this kind of task, because it is more difficult than the previous tasks. The strategy used by most students consisted in building the solid observing first the top projection. Having made this step, we have found two different ways to follow up. Some students moved the solid to analyze the next orthogonal projection but forgetting that the solid has to fit both projections. This is an algorithmic strategy that does not establish the necessary connections between the projections and/or the solid's faces. The cognitive demand of this poor strategy belongs to the procedure without connections level.

Other students continued solving the task by moving the solid to see another projection and adding cubes to the solid while checking that it fitted both orthogonal projections. These students developed deep visualization abilities as they were able to connect the three numeric orthogonal projections and to solve the task correctly. This solving strategy corresponds to the procedure with connection level of cognitive demand.

Finally, a student used a great strategy, synthesized in Figure 9. The student analyzed the numbers in the orthogonal projections and identified analytic relationships that helped him to build the solid. This strategy required considerable cognitive effort, to establish non-algorithmic relationships among parts of the numeric projections, so it is typical of the doing mathematics level, the highest level of cognitive demands.

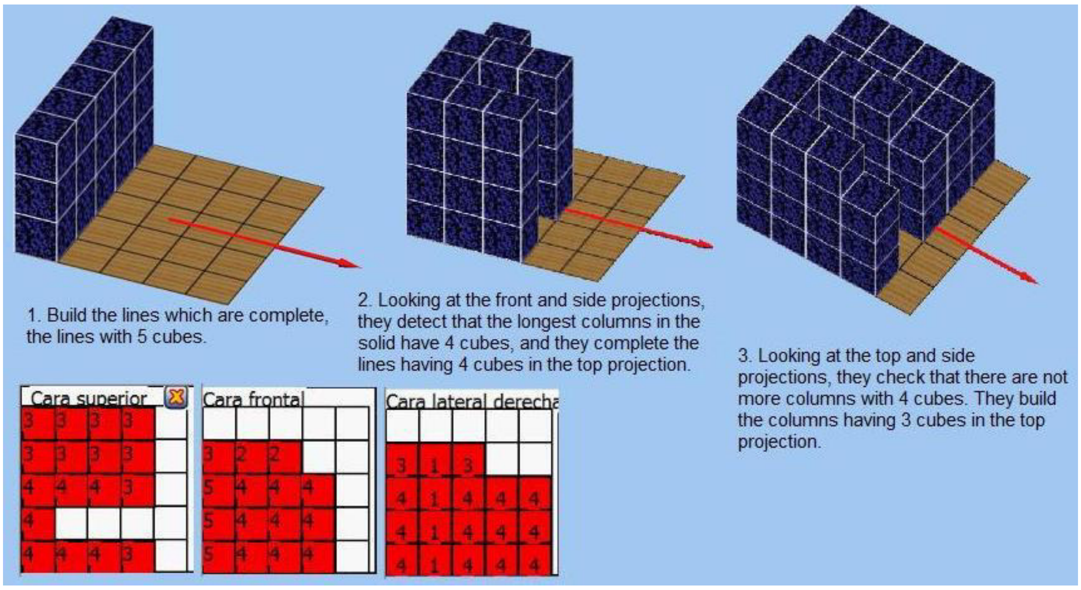


Figure 9. An optimal strategy to build a solid from a set of numeric orthogonal projections.

Task: Build a solid from a set of orthogonal projections

This task is the most difficult one. Only 40% of the students solved correctly this type of tasks. Similarly as for the first strategy showed for the previous task, there were students who tried to solve it by keeping in mind only one projection each time, so they were not able to build a solid fitting the three projections at the same time. Therefore, this strategy belongs to the procedure without connections level of cognitive demand.

Other students built first a solid looking at one projection. Then, they identified the cubes they had to add or remove to fit all the projections at the same time (Fig. 10). The students solved the task by a strategy requiring some degree of cognitive effort, since they had to link the different projections to the partially built solid, so its cognitive demand is in the procedure with connections level.

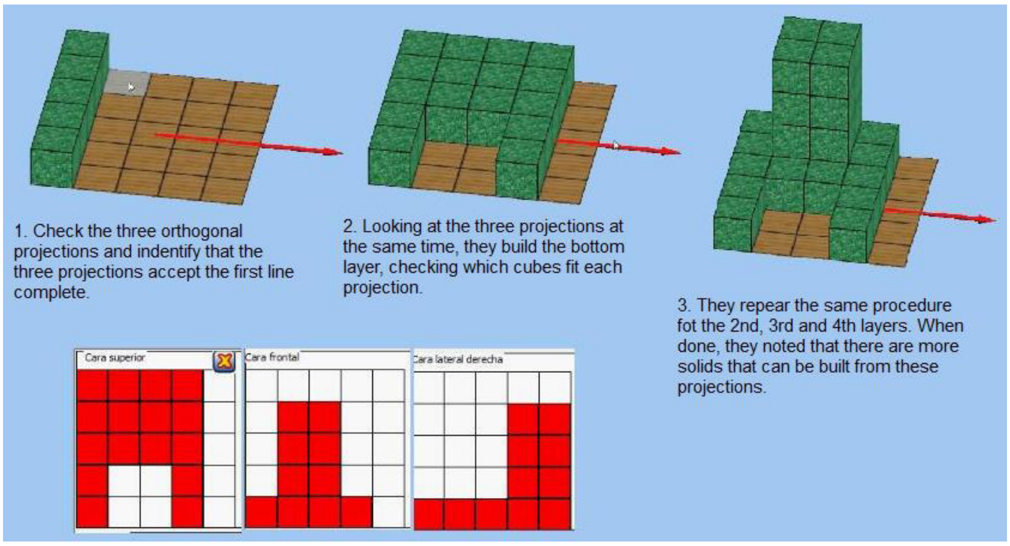


Figure 10. A strategy to build a solid from a set of orthogonal projections.

CONCLUSIONS

We have presented a software which is very useful for students to learn and improve visualization skills. It is easily adaptable by the teacher, allowing her to state easy tasks to students with more difficulties and, at the same time, to state challenging tasks to the mathematically talented students.

We have presented different mathematically talented students' answers classified according to their ways of reasoning and use of visualization abilities. We have analyzed these answers and identified styles of behaviour characteristic of the different levels of cognitive demand, proving that the model of cognitive demand is useful to discriminate among the different responses offered by the students.

Note:
The results reported are part of the R+D+I projects *Analysis of Learning Processes by Primary and Middle School Mathematically Talented Students in Contexts of Rich Mathematical Activities* (EDU2012-37259) and *Key Moments in the Learning of Geometry in a Technological and Collaborative Environment* (EDU2011-23240), funded by the Spanish Ministry of Economy and Competitivity.

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