ANALYSIS OF THE COGNITIVE DEMAND OF A GIFTED STUDENT'S STRATEGIES TO SOLVE GEOMETRIC PATTERNS PROBLEMS

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We present a teaching experiment based on a case of one mathematically gifted 9-year-old student solving geometric pattern problems, as a first step to introduce him to algebra and equations. We use the model of cognitive demand to assess student's outcomes. To get a fine analysis of the answers, we present a modification of the original characterization of the levels of cognitive demand to adapt them to the context of geometric patterns problems. We have identified and analyzed several types of answers typical of the different kinds of questions posed in the problems.

INTRODUCTION

Mathematically gifted students (gifted students hereafter) tend to show unusual paths of reasoning and ways to solve problems. Authors like Freiman (2006), Greenes (1981), Krutetskii (1976), and Miller (1990) suggest a number of characteristics of gifted students. Some of such characteristics are the abilities to: identify patterns and relationships among different elements, generalize and transfer mathematical ideas or knowledge from a context to another one, and to invert mental procedures in mathematical reasoning. These abilities are specially useful in some particular contexts, like the one we are dealing with in this paper, the use of geometric patterns problems to introduce students to algebraic language and equations.

The model of the *cognitive demand* was created to evaluate the intellectual effort required when students solve mathematics problems, and to decide on which problems are more adequate to pose to different students. In order to assess the power of tasks to help develop students' mathematical thinking, Stein, Grover and Henningsen (1996) analyzed the features of mathematical tasks (number of solution strategies, number and kind or representations, etc.), and the cognitive effort required from students to solve them, varying from tasks requiring just recall from memory to others requiring what could be characterized as doing mathematics. To allow teachers select tasks with an appropriate level of challenge for their pupils, Smith and Stein (1998) designed a set of criteria that classifies the tasks into those four levels of increasing required cognitive effort (the levels of cognitive demand). Up to now, the level of cognitive demand of a problem was decided analyzing the statement of the problem, but this method does not recognize that a problem may be solved correctly in several ways requiring different levels of cognitive demand. Instead, we assign levels of cognitive demand to students' outcomes to better understand their ways of reasoning and decide on the appropriateness of tasks.

A very fruitful way to introduce basic algebra to students is by solving geometric patterns problems (Cai, Knuth, 2011; Rivera, 2013). Literature has reported many teaching experiments based on students of different ages, from early Primary to lower Secondary, and different strategies used to solve the different questions asked in the geometric patterns problems. Specifically, Amit, Neria, (2008) focused on the generalization strategies used by able students, aged 11-13, in solving geometric patterns problems. They identified two approaches to generalization: recursive-local, when students' strategy to solve the next task is based on the task they had solved before, and functional-global, when they obtain a general method, usually multiplicative, to calculate any term of the pattern.

The context of geometric patterns problems seems especially useful for mathematically gifted to access pre-algebra concepts, but there are only a few publications reporting gifted students' behaviour when solving these problems. Amit, Neria (2008) confirmed that generalization via patterns problems is an adequate gateway to develop algebraic skills. Fritzlar, Karpinski-Siebold (2012) explored the algebraic abilities of a sample of primary school students, aged 9-10, of varying performance levels, which included gifted students, through the identification and generalization of patterns. As expected, the more able students got the better results, although none of them were able to answer adequately generalization questions (about the nth term).

A research question is how do gifted students solve geometric patterns problems and progress in learning more abstract strategies. Our research aims to contribute to answer this question. We have rephrased the levels of cognitive demand to adequate them to analyze students' outcomes when solving geometric patterns problems. The specific objectives of our research presented here are:

i) To analyze the relationships among the geometric patterns and the cognitive demand required by gifted students' ways of getting general rules, and

ii) To analyze the relationship among the complexity of the general rules obtained by gifted students and the cognitive demand required by their ways to answer the inverse relationship tasks.

THEORETICAL FRAMEWORK

Geometric patterns problems typically show a geometrical representation of the first terms of an increasing series of natural numbers (see some examples below), and pose students some questions about the series. Usual questions are (Amit, Neria, 2008) to calculate the values V_n of *immediate*, *near* and *far* terms of the series, to *verbalize a general rule* valid to calculate any specific term, and to *write an algebraic expression* $V_n = f(n)$ for such rule. Students may also be asked (Rivera, 2013) to calculate the *inverse relationship*, that is, to get the place n of a term given its value V_n (i.e., solve the equation $f(n) = V_n$).

The model of cognitive demand identifies four levels of complexity of the reasoning used to answer mathematical problems (Smith, Stein, 1998, p. 348):

- *Memorization* (low level): tasks that ask students to reproduce previously learned facts, rules, formulas or definitions.
- *Procedures without connections* (low-medium level): tasks that ask students to perform an algorithm in a routine manner, without connection to mathematical concepts.
- *Procedures with connections* (medium-high level): tasks that ask students to perform an algorithm presenting some ambiguity about what has to proceed, and having connection to mathematical concepts.

- *Doing mathematics* (high level): tasks that required a complex and non-algorithmic thinking.

The calculation of immediate and near terms of a pattern typically require only to continue the numeric or geometric structure of the given terms of the pattern, and students do not need to be aware of the implicit algebraic relationship among terms, so a low-medium level of cognitive demand is sufficient to make such calculations. On the contrary, to correctly calculate far terms or verbalize a general rule, students need to be aware of the implicit algebraic structure of the sequence, so a medium-high cognitive demand is necessary. To write a correct algebraic expression for a geometric pattern, there is not any algorithmic procedure to be applied; students have to analyze previous answers and connect relevant data from them, so this task requires from students a high level of cognitive demand. Respect to the inverse relationships, as we will se below (Table 1), the level of cognitive demand required to solve these tasks may vary depending whether the mathematical structure of the pattern is simple, just requiring arithmetic calculations (low-medium level), or complex, requiring to write and solve an equation (medium-high level).

To make such a broad description of the levels of cognitive demand useful to analyze specific students' answers to geometric patterns problems, we have adapted the theoretical characteristics of the levels specifically for this kind of problems. We present in Table 1 our characterization of the levels of cognitive demand particularized to geometric patterns problems (Benedicto, Jaime, Gutiérrez, 2015), that we have used to analyze the student's outcomes shown below.

Table 1. Characterization of the cognitive demand of answers to geometric patterns problems.

Levels of C. D.	Categories	Characteristics of the task
LOW (memorization)	Resolution procedure	• Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.
	Objective	• Involve either reproducing previously learned facts, rules, formulas, or definitions or committing facts, rules, formulas or data taken from the statement.
	Cognitive effort	• Are not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated.
	Implicit content	• Have no connection to the concepts or meaning that underlie the facts, rules, formulas, or definitions being learned or reproduce.
	Explanations	• Does not require explanations.
	Representation of solution	• The statement uses a geometric representation and the resolution will be represented by an arithmetic representation.
	Resolution procedure	• Is algorithmic. A procedure is evident from the geometric pattern, or it is obtained by trial and error. The inverse relationship is based on a single arithmetic operation, on a learned sequence of arithmetic operations, or on checking possible answers by trial and error.

Levels of C. D.	Categories	Characteristics of the task
LOW-MEDIUM (procedure without connection)	Objective	• Focused on producing correct answers instead of on developing mathematical understanding.
	Cognitive effort	• Solving it correctly requires a limited cognitive effort. Little ambiguity exists about what has to be done and how to do it.
	Implicit content	• Little ambiguity exists about what has to be done and how to do it.
	Explanations	• Does not ask for explanations, or the explanations consist only on describing the procedure used to solve the task.
	Representation of solution	• May be represented in multiple ways (visual diagrams, manipulative, symbols, and problem situations), but usually the easiest is chosen.
MEDIUM-HIGH (procedure with connections)	Resolution procedure	• Previous tasks suggest implicit general procedures closely connected to the underlying algebraic structure. The inverse relationship is based on solving the equation of the general procedure previously obtained.
	Objective	• Directs students' attention to the use of general procedures aiming to deepen their understanding of the underlying algebraic structure.
	Cognitive effort	• Solving it correctly requires some degree of cognitive effort. When students use a general procedure, they need to have some understanding of the algebraic structure of the pattern.
	Implicit content	•When students use a general procedure, they need to have some understanding of the algebraic structure of the pattern.
	Explanations	• Use particular examples (specific terms of the pattern) to refer to algebraic structures or relationships that underlie the procedures.
	Representation of solution	• The resolution connects several representations, but usually is used them which develop an abstract reasoning.
HIGH (doing mathematics)	Resolution procedure	• Require complex and no algorithmic thinking; a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example.
	Objective	• Require students to explore and understand the nature of mathematical concepts, processes, or relationships.
	Cognitive effort	• Require considerable cognitive effort and may involve some level of anxiety for the student because of the un-predictable nature of the solution process required. Demand self-monitoring or self-regulation of one's own cognitive processes.
	Implicit content	• Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task.
	Explanations	• Explanations are the proof of the general term of the geometric pattern.
	Representation of solution	• The resolution is represented by an algebraic representation.

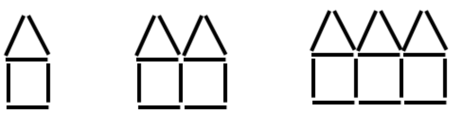
THE RESEARCH

We present results from a case study research based on data from a gifted student who solved several geometric patterns problems. Cai, Knuth (2011) described learning trajectories of students solving geometric patterns problems, but we found that our student solved correctly the problems from the very beginning, and he used consistently the same solving strategies along the experiment. The analysis we present here answers the above mentioned specific objectives.

Methodology

The subject for this study was Juan, a gifted student aged 9 that had finished grade 4 when the experiment began. He participated in 10 individual interviews, conducted by the second author. Sessions were conducted by means of Skype, and were video-recorded. Through these sessions, the student solved 19 geometric patterns problems, all including the same tasks: immediate, near and far generalizations, and two inverse relationship tasks. Next, we show one of the 19 geometric patterns problems as an example of the tasks posed:

Marc and his friend want to make an urbanization with sticks. They draw it as follows:



1 house 2 houses 3 houses

a) How many sticks will they need to draw 6 houses? How do you know it?

b) How many sticks will they need to draw 11 houses? How do you know it?

c) Would you know any way to calculate how many sticks will they need to draw 44 houses? How do you know it?

d) If there are 51 sticks, how many houses will they draw?

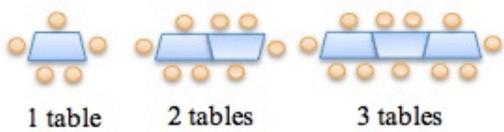
e) If there are 98 sticks, how many houses will they draw?

In each of this tasks, Juan had ever to explain his answers. His explanations for the direct tasks were to verbalize the general rule and for the inverse tasks were to verbalize the inversion of the general rule or his trial and error calculation. Below, we analyze the diverse Juan's strategies, which correspond to different levels of cognitive demand.

Analysis of the cognitive demand of student's answers

Although all the problems could be solved with strategies of medium-high level of cognitive demand, when the difficulty of a problem increased, Juan needed to use strategies of low-medium level to solve it. Also, as he still had not learned algebra, he could only solve the inverse relationships by using strategies of low-medium level of cognitive demand. We show below the different types of strategies used by Juan, with examples:

1) *Decomposition of the pattern.* Most geometric patterns show ways to split the figure, making it easy to find a general procedure to calculate the terms in the series. Juan explained his way to calculate the number of chairs around the tables:

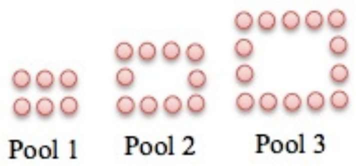


J: For one table, I added the numbers [of chairs] above and below, 1 and 2, are 1x3=3. For two tables, 3 above and 3 below, 2x3=6 and 3+3=6. And we have to add those two [the chairs on the sides]. For three tables, 5+4=9 and 3x3=9.

Juan was in the medium-high level of cognitive demand, since his general procedure (3xnumber of tables+2), derived from the geometric pattern, is related to the algebraic structure of the pattern. The figures given can be easily split, so they implicitly suggest general procedures having close connections to the underlying algebraic structure. This allowed Juan understand the algebraic structure of the pattern and find the general functional relationship.

2) *Counting from a drawing.* In some problems, Juan counted the total number of objects in each term, to get a general arithmetic procedure from those numbers. For example, in the task of the pools, he said that the pool 5 had 22 tiles around it because he looked at the examples and:

J: I discovered that it is 4 times the size [position] of the pool plus 2. ... The first size has 6, and 4x1+2=6. The second has 10, and 4x2+2=10. And the third has 14, and 4x3+2=14.



Juan either was not able to find an adequate decomposition of the geometric pattern or he, directly, got the numeric values of the given terms and looked, by trial and error, for a way to relate it to the position of the term. The strategy of resolution chosen is algorithmic, typical of a low-medium level of cognitive demand, since it consisted on counting the number of elements of each terms and looking for and arithmetic relationship between them. The objective of this solution was focused on producing a correct result. Despite of the procedure used by Juan, consisting on finding a general formula, he might not be aware of the underlying algebraic structure, since he only described the procedure used to solve the task but he did not explain a reason of it.

In another problem, asking for the number of sticks to make a series of laces, Juan explained the way he had proceeded to get a formula to calculate the number of sticks:

J: I think this is okay. I made a way. A triangle, I multiply it by two and I add one. A triangle, 1x2+1=3. Two triangles, 2x2=4 and 4+1=5. Three triangles, 3x2=6 plus… 3x2=6 and 6+1=7. And six [triangles]… 6x2=12 and 12+1=13.

Researcher: Ok. How did you get this formula?

J: I first thought that each time was different, but finally I got it by trial.

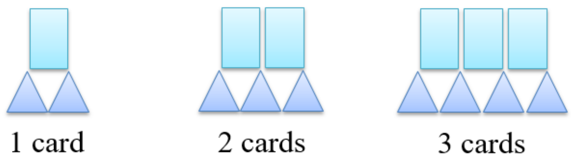
R: Did you divide the triangles in some way or did it just come up with you?

J: I tried, ok? I first tried to get how many sticks [are necessary to make each lace] by counting, but I said it would be boring, too long. Then I thought in making a formula and I tried some ones. Finally I got this one.

This trial and error arithmetic strategy focused on getting a correct answer, but it did not develop mathematical understanding, since it hid the understanding of the algebraic structure under the pattern. So this solution required a low-medium level of cognitive demand. Although Juan tried to use a decomposition of the geometric pattern to obtain a relationship, he was obstinated in getting a correct solution and did not pay enough attention to understanding the underlying algebraic structure, and he followed an algorithmic strategy with a limited cognitive effort.

Strategies to calculate inverse relationships are conditioned by students' (lack of) awareness of algebra and their (in)ability to solve equations. Juan had not had any previous contact with algebra. He showed three types of strategies to calculate inverse relationships:

3) *Correct inversion of the order of operations.* When the general procedure found for the direct tasks was the types $y = ax$ or $y = x \pm a$, Juan applied correctly the inverse arithmetic operation to get the position of the term. For example, a problem asked for the number of triangles under each row of cards. Juan explained his direct operations:



> J: If there are 5 cards, at the bottom we have always to add a triangle. Below there is always one [triangle] more.

Next, for the inverse relationship question, Juan had to calculate the number of cards on the top when there were 23 triangles below:

> R: What if we do it the other way around? How many cards are there on 23 triangles?
>
> J: 22.
>
> R: Very good. What have you done now?
>
> J: Subtract one.

Applying the rule of inverse arithmetic operations is not just a matter of memory, although it is algorithmic and requires a very limited cognitive effort, since he only need to apply a basic arithmetic operation. This strategy has a low-medium level of cognitive demand. The aim is to get a correct solution and the explanation was only a description of the procedure used to solve the task.

4) *Wrong inversion of the order of operations.* When the general procedure found for the direct tasks was the type $y = ax \pm b$, Juan knew that he had to use the inverse operations, but he was not aware of the relevance of the order of calculations. For instance, in the task of the laces (see the pattern above), the direct operations made by Juan were $V_n = 2n + 1$:

> J: The number of sticks to make a lace is 2 times the position plus 1.

Juan was first asked to calculate the position of a lace made with 20 sticks:

> J: I believe it is a number between 9 and 10. I did 20 divided by 2 minus 1.

After this wrong answer, the researcher guided Juan to consider the order of calculations in his procedure. Next, Juan had to calculate the position of a lace made with 31 sticks:

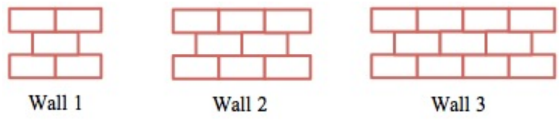
> J: It is 14 and a half. ... No, it's wrong, it is 15. ... I subtract 1 to 31 and get 30, and 30 divided by 2 is 15.

Juan decided the order of inversion of calculations in an in consistent way, not connected to the algebraic structure of the pattern. This solution required a low-medium level of cognitive demand, since the student did not understand the algebraic structure of the pattern, which induced him to use an incorrect procedure. He tried to give a correct answer, but he did not pay attention to the general procedure and made limited cognitive effort to give a solution.

5) *Trial and error direct calculations.* When the general procedure found for the direct tasks was the types $y = ax+b(x \pm c) \pm d$, or the quadratic types $y = x^2$ or $y = (x \pm a)(x \pm b)$, Juan was blocked because he was unable to invert such complex procedures (he still had not studied equations nor

square roots) and he resorted to trial and error, checking different values for *n* until the correct value was found. For instance, the task of the walls asked about the number of bricks necessary to build a wall. The direct operations made by Juan were $V_n = (n+1)\cdot 2 + n$:

J: The number of bricks in a wall is the position-plus-1 times 2, plus the position.



Then, he had to calculate the position of a wall made with 38 bricks:

J: I did 38 divided by 2 minus 2. But it does not work. ... I believe it is 13. ... I checked the numbers [positions] ... No, sorry, it is 12. Because 12+1=13; 13x2=26; 26+12=38.

Trial and error is an algorithmic process that does not connect to the algebraic structure of the pattern, and it is only aimed to get the correct answer. Therefore, this type of strategy needs a low-medium level of cognitive demand. As for the previous example, Juan did not understand the algebraic structure and he made limited cognitive effort to get this (incorrect) answer.

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