When the theoretical model does not fit our data: A process of adaptation of the Cognitive Demand model

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Some theoretical models emerged from specific mathematical contents or educational levels, so they are well adapted to the context where they were created but, sometimes, they do not fit well in a different context. Related to the issue of the need to go beyond a specific theory when researching a phenomenon, to solve the tension between home-grown needs and borrowed theories, we present the reasons for and steps of an adaptation we have had to do of the Cognitive Demand model so that it fits the requirements of our research questions, methods and data. We systematized the definition of the model stated by its authors, and completed this definition with some necessary statements. Then, we re-stated some characteristics of the model to avoid inconsistencies when using it. Finally, we particularized the general model to several specific mathematical topics.

Keywords: Theoretical framework, adaptation, cognitive demand, mathematical problem solving.

Introduction

The choice of a specific theoretical framework is a key decision for researchers when they start working on a new research project. However, sometimes, researchers wish to use a theoretical model in a context that the model does not fit well, and they decide to modify the theoretical model to adapt it to the new requirements. We present our experience of fitting a theoretical model with flexibility so it can be adapted to contexts and used in ways the model had never been used.

We are developing a research project[1] aimed to better understand the cognitive processes of primary and lower secondary mathematically talented students (i.e., students showing mathematical abilities clearly over average students) when solving problems. A characteristic of those students is that they demand problems making them engage in high level of reasoning. A way for teachers to succeed in it is by posing them that kind of problems. Then, we had to find a way to determine the cognitive effort required by problems and done by students when solving them. The *Cognitive Demand model* (Smith & Stein, 1998) fitted our requirements, so we integrated it in our theoretical and methodological research frameworks. However, when we used this model to analyze our data, we found some difficulties in applying it due to ambiguity or inconsistency of results, so we decided to adapt it to our needs. We detail in next pages the steps in the process of adaptation, the reason for each step, and the resulting theoretical model. We present a reconstruction of the main parts of the real recurrent process, which has taken several years and still has not been finished.

The next section presents the characterization of the Cognitive Demand model as stated by its authors. Third to sixth sections present examples of the difficulties we found and the steps we followed to adapt the original model: organizing the original characterization, completing such charac-

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terization with some new statements, re-wording some characteristics of the model to avoid inconsistencies when using it, and particularizing the new general characterization of the model to different specific mathematical topics. Due to space limitations we cannot show the details for all levels of cognitive demand, but only an example of each step for certain levels.

The starting point: The levels of Cognitive Demand

The Cognitive Demand model resulted after a process of characterization of mathematical tasks according to their "potential to engage students in high-level thinking" (Smith & Stein, 1998, p. 344). It includes four *levels of cognitive demand* that assess the cognitive effort required from students to solve a mathematical task. These levels are labelled (Smith & Stein, 1998) as *memorization*, *procedures without connections* to concepts or meaning, *procedures with connections* to concepts and meaning, and *doing mathematics*, when complex mathematical thinking is required. Each level is defined by a set of characteristics paying attention to different aspects of the solutions of problems. We present in Table 1 the characteristics of two levels that are the ground for the rest of the paper.

Procedures without connections 2.1. Are algorithmic. Use of the procedure either is specifically called for or is evident from prior instruction, experience, or placement of the task. 2.2. Require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it. 2.3. Have no connection to the concepts or meaning that underlie the procedure being used. 2.4. Are focused on producing correct answers instead of on developing mathematical understanding. 2.5. Require no explanations or explanations that focus solely on describing the procedure that was used.
Procedures with connections 3.1. Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. 3.2. Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts. 3.3. Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations helps develop meaning. 3.4. Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.

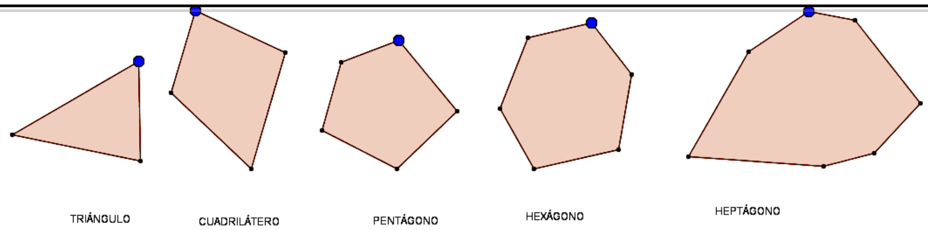
Table 1: Definition of the levels of cognitive demand of *procedures without connections* and *procedures with connections* (Smith & Stein, 1998, p. 348). Order numbers are added for easy reference

To attain our research objectives, we created *rich problems* intended to be posed to whole-class groups, consisting of several related questions of increasing complexity, in such a way that all students should be able to solve the first questions but only the more able students could solve all of them. The levels of cognitive demand allow us to classify the questions in a problem and decide whether each question is more appropriate for average students or for talented students.

The Cognitive Demand model was created after the analysis of problems that, most of them, were in the quite algorithmic areas of school arithmetic and algebra (Stein & Smith, 1998; Stein et al., 2009), but we are trying to use the model to analyze problems in mathematical topics very different from the previous ones, as are plane geometry, geometric patterns (pre-algebra) and visualization.

Identifying inconsistencies between the characteristics of the levels

After using the model to analyze different problems, a difficulty related to lack of consistency between some levels arose. We exemplify it by analyzing a problem consisting of several questions guiding students to discover and prove a formula to provide the number of diagonals of any polygon. The problem has two parts: the first one can be seen in Figure 1; the second part asks students to draw and count all the diagonals of the same polygons and to fill in a table, to calculate the number of diagonals of a 20-sided polygon, to generalize the procedure of calculation of the diagonals to any given polygon, and to prove this relationship. We present our analysis of question 1a (Figure 1) by considering typical average students' solutions that do not go further than what is asked to do.



1a) In each polygon, draw all the diagonals starting from the marked vertex. Change the shape of the polygons by dragging that vertex. Count the number of diagonals. Fill in the table below.

Polygon	Nº of sides	Nº of diagonals from one vertex
Triangle		
Quadrilateral		
Pentagon		
Hexagon		
Heptagon		

1b) What is the relationship between the number of sides of a polygon and the number of diagonals from one vertex? Why?

Figure 1: First part of a problem focused to discover the number of diagonals of any polygon

Regarding the level of *procedures without connections*, question 1a is algorithmic and the statement suggests the procedure to be used, namely to draw the diagonals from a given vertex and count them to fill in the table, so it fits characteristic 2.1 in Table 1. The polygons drawn in the statement guide students to draw the diagonals from a specific vertex, so there is little ambiguity about what they have to do and how to do it (it fits 2.2), and it does not require explanations (it fits 2.5). On the other hand, the procedure to be used has connections with the relationship between the number of sides and diagonals from a vertex of polygons (it does not fit 2.3), although question 1a is focused on producing correct answers, not on developing understanding of that relationship (it fits 2.4).

Regarding the level of *procedures with connections*, question 1a does not fit 3.1, since it is not focused to make students develop understanding of the underlying relationship. However, it fits 3.3, since, to answer it, students use geometric and numeric representations of information about polygons and diagonals: the numeric representation shows the general relationship between number of sides and diagonals, while the geometric representation may help students understand why it is true. So the question has the potential to let students connect both representations, which would help them develop the meaning for the relationship. Question 1a fits 3.2, since it explicitly suggests a procedure that is closely connected to the underlying concepts, the number of sides and diagonals from a vertex of polygons. Question 1a does not fit 3.4 since its procedure may be followed without need of being mindful, and a correct solution to it does not require understanding the underlying relationship between number of sides and diagonals from a vertex of polygons.

The epistemological conception of the levels of cognitive demand is that they are mutually exclusive. We see that question 1a fits several characteristics of each level *procedures without connections* and *procedures with connections*, so it is unclear to which level of cognitive demand should it be assigned. This happens because some characteristics of these levels, as stated in Table 1, are not precise enough, which can lead to errors when trying to assign a level of cognitive demand to some problems. The most evident vagueness, or contradiction, happens with characteristics 2.1 and 3.2.

Organizing the characteristics of the levels and filling their gaps

After having identified the difficulty analysed above, we made a detailed comparison of the characteristics of each pair of consecutive levels, to identify possible weaknesses and modify their wording to correct them. We noted that the characteristics of levels (see examples in Table 1) focused on six domains of objectives of a problem or its process of solution. These six domains of characteristics are: *Procedure of solution*, *objective* of the problem, required student's *cognitive effort*, mathematical *contents implicit* in the problem, kind of *explanations* required, and types of *representations* used in the solution. The domains helped us arrange the characteristics of the levels of cognitive demand provided by Smith and Stein (1998) and identify some gaps in the definitions of the levels.

Levels of cogn. demand / Domains	Memorization	Procedures without connections	Procedures with connections	Doing mathematics
Procedure of solution	1.2	2.1	3.2	4.1, 4.5
Objective	1.1	2.4	3.1	4.2
Cognitive effort	1.3	2.2	3.4	4.3, 4.6
Implicit contents	1.4	2.3	3.4	4.4
Explanations	--	2.5	--	--
Representations	--	--	3.3	--

Table 2: Domains of the characteristics of the levels of cognitive demand in Smith and Stein (1998)

Table 2 shows the assignation of the characteristics of the levels to the domains. It also shows that two domains are considered only in the definitions of a level, and that 3.4 includes references to two domains, while several characteristics of the level *doing mathematics* refer to a same domain.

Next step to improve the usability of the original definition of the Cognitive Demand model was to complete the definitions of the levels, by including characteristics referring to the missed domains, taking care that each new characteristic is consistent with the corresponding characteristics of the other levels. Table 3 shows the new characteristics, to be added to those in Table 1 to make a more complete description of the levels of cognitive demand.

Procedures without connections
2.6 (representations). One or more representations may be used (arithmetical, geometrical, visual diagrams, manipulatives, etc.). When several representations are used, students use them independently, i.e., without establishing connections neither between them nor with the underlying concepts and ideas.
Procedures with connections
3.5 (explanations). Require explanations that focus on the underlying relationships by using specific examples.

Table 3: Characteristics added to the levels of *procedures without* and *with connections*

Having completed the definitions of the levels by merging Table 1 and Table 3, we were ready to refine the characteristics that induced wrong or multiple identifications of the cognitive demand in some problems, like the problem analysed above (number of diagonals of polygons).

Refining the characteristics of each level

A necessary feature of any set of disjoint categories is that their definitions have to make it clear the border between adjacent categories. As we showed above, this is not the case for the levels of cognitive demand. To refine the definitions of the levels, we made a systematic comparison of the characteristics in the same domain and decided to do some changes in their wording to make them more explicit and to clearly raise the particularities of each level.

The key difference between the levels of *procedures without connections* and *procedures with connections* is that, in the lower level, students do not need to be aware of the mathematical relationships implicit in the problem to solve it correctly but, in the higher level, students need to use consciously such relationships to solve correctly the problem. Table 4 shows the result of the comparison between those levels, where we have italicised the new characteristics (see Table 3) and the characteristics in Smith and Stein (1998) that we re-worded. Characteristic 3.4 was split because it included parts corresponding to two domains. The new wording of the characteristics of the levels has highlighted this key difference and now the border between those levels is clear.

If we repeat now the analysis of the problem in Figure 1, question 1a fits new characteristics 2.1, 2.3 and 2.4, because it focus students' attention to draw the diagonals from a vertex of each polygon and count them, so it can be easily solved without being aware of the relationship between the number of sides and diagonals from a vertex.

	Levels of cognitive demand	
Domains	**Procedures without connections**	**Procedures with connections**
Procedure of solution	2.1. Are algorithmic. *The procedure to be used* either is specifically called for or is evident *from the context. It is a simple procedure that students can follow without the need to connect to underlying concepts and ideas.*	3.2. *Are algorithmic*. They suggest explicitly or implicitly pathways to follow, that are general procedures *that students can follow only if they have established a close connection to underlying concepts and ideas*.

Objective	2.4. *Focus students' attention* on producing correct answers. *Students can solve them correctly without the need to understand underlying concepts and ideas.*	3.1. Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of *underlying* concepts and ideas.
Cognitive effort	2.2. Require limited *cognitive effort* for successful completion. Little ambiguity exists about what needs to be done and how to do it.	3.4a. *Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly.*
Implicit contents	2.3. *There may be implicit connection between the algorithms used and underlying concepts or ideas. However, students do not need to be aware of it to solve the problem correctly.*	3.4b. *Students need to engage with concepts and ideas that underlie the procedures to complete the problem successfully and that develop understanding.*
Explanations	2.5. *Require explanations* that focus solely on describing the procedure that was used.	*3.5. Require explanations that focus on the underlying relationships by using specific examples.*
Representations	*2.6. One or more representations may be used (arithmetical, geometrical, visual diagrams, manipulatives, etc.). When several representations are used, students use them independently, i.e., without establishing connections neither between them nor with the underlying concepts and ideas.*	3.3. Usually are represented in multiple ways, *(arithmetical, geometrical, visual diagrams, manipulatives, etc.). To solve correctly the problem, students have to establish connections between different representations by using underlying concepts and ideas, which help them develop meaning.*

Table 4: Comparison between the characteristics of problems in two levels of cognitive demand

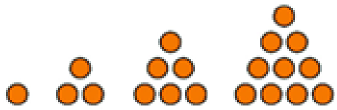
Question 1a also fits new characteristics 2.2 (since there is no ambiguity about how to solve it) and 2.6 (students will use geometrical and arithmetical representations but without needing to connect them), but it does not fit characteristic 2.5, since this question does not ask for an explanation (so, this analysis is also useful to uncover flaws in the statements of problems). On the other hand, question 1a does not fit new characteristics 3.1, 3.2, 3.3, 3.4a, 3.4b and 3.5. So, now it is clear that question 1a requires a cognitive demand in the level of algorithms without connections, which agrees with our experimental analysis of real students' answers.

Particularizing the new cognitive demand model to specific topics

As mentioned above, the Cognitive Demand model was generated after analysing problems that, in most cases, were related to school arithmetic or algebra. When we tried to use it to analyze problems in other areas of mathematics (plane geometry, geometric pattern problems and visualization), we found that the wording of quite characteristics of the levels were too generic and they did not help us to give meaning to the levels specific to those contexts. This forced us to re-word those characteristics of the levels to mention specific features of a given topic. We present here the particularization we have made to the context of geometric pattern problems.

Geometric pattern problems have proved to be a very fruitful way to introduce basic algebra to students (Amit & Neria, 2008; Rivera, 2013). A typical geometric pattern problem presents (Figure 2) a graphical representation of the first terms of a sequence of whole numbers, and asks students to calculate the value of certain terms of the sequence, to verbalize a general procedure to calculate the value of any given term, and to write an algebraic expression to calculate the value of any term.

You can see below a shape made with one dot, another shape made with three dots, and so on.



1. How many dots has the shape in the 4th position?
2. How many dots has the shape in the 6th position?
3. How many dots has the shape in the 20th position? How do you know it?
4. Is there some rule that could allow us calculate the number of dots of any given shape, for instance the one in the 100th position? Justify your answer.
5. Is there some rule that could allow us calculate the number of dots of the shape in the nth position? Justify your answer.

Figure 2: A typical statement of a geometric pattern problem

We are interested in analysing the relationships among the geometric patterns and the cognitive demand required by different kinds of students' answers. When we first used the definitions of the levels of cognitive demand (Tables 1 and 4) to classify answers to geometric pattern problems, we found that some characteristics were meaningless in this context, so we made a complete particularization of the characteristics of the levels to describe the answers to this specific type of problems.

Table 5 presents, as an example, the characteristics of the level of *procedures without connections* for the context of geometric pattern problems. It may be noted that most characteristics include reference to peculiar and unique aspects of those problems.

Procedures without connections (question 2)	
Procedure of solution	• Are algorithmic. The procedure consists in drawing a few terms by following the pattern of the terms in the statement, and counting the items. It can be followed without the need to connect to the arithmetic structure of the sequence.
Objective	• Focus students' attention on producing a correct answer, the number of items in an immediate or near term, but not on developing understanding of the structure of the sequence.
Cognitive effort	• Solving it correctly requires a limited cognitive effort. Little ambiguity exists about what has to be done and how to do it, because the statement clearly shows how to continue the sequence.
Implicit contents	• There is implicit connection between the underlying structure of the sequence and the procedure used. However, students do not need to be aware of it and they may answer the question by drawing terms and counting their items.
Explanations	• Require explanations that focus only on describing the procedure used. It is not necessary to identify the relationship between the answer and the term.
Representations	• A geometric representation is used to get the number of items and an arithme-

	tic one to write the result. Students use the representations without establishing connections neither between them nor with the structure of the sequence.

Table 5: Particularization of the Cognitive Demand model to the geometric pattern problems

This description of the levels of cognitive demand has proved to be very useful to analyze this kind of problems and students' answers to them.

Conclusions

We have presented a case of modification of a theoretical model to adapt it to the specific requirements of the analysis we had to do of our data. The Cognitive Demand model was a pertinent theoretical framework for our research project, with the potential to ground a deep analysis of our data, although the practice showed that the initial definition of this model, as formulated by its authors, did not fit well the requirements of our analysis. We have shown some difficulties that arose when we tried to apply the initial model. The way to overcome these difficulties was to analyze the theoretical model, to identify and understand the origin of and the reason for the difficulties, and to make adequate changes in the definition of the levels of cognitive demand to make it more accurate and useful. Finally, we had also to particularize the new definition of the levels to the specific context of geometric pattern problems. This general way of proceed may be applied, perhaps after an adequate adaptation, to modify other theoretical models not fitting adequately researchers' needs.

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