THEMATIC WORKING GROUP 7

GEOMETRICAL THINKING

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The Topic Group 7 of CERME-3 on "Geometrical Thinking" had more than 20 participants from countries all over Europe and from Mexico. During its sessions, the participants implicitly discussed 10 papers prepared for the Topic Group. No formal presentations of the 10 papers were made, but every author had a chance to present her/his main ideas. This report does not follow the order of the discussion in the group, but tries to account for the arguments brought forward and commented according to the structure given by the headings of the sections of the report.

1. PARADIGMS: "Natural" – "Intuition" –"Formalistic" – "Axiomatic"

The history of geometry is marked by two somehow contradictory trends: On the one hand, geometry is used as help for coping with situations in real and/or future life (including control of production and distribution of goods and services). On the other hand, geometry for more than two thousand years was the prototype of logical, mathematical thinking and writing after the publication of Euclid's "Elements" (for this way of looking onto geometry see for instance Heilbron 1998, p. 1ff). These contradictory perspectives are somehow mirrored in an approach widely discussed in the Topic Group, when Houdement and Kuzniak presented their approach to geometry. Following Gonseth (1945-55), they distinguish "Geometry I: natural geometry" (source of validation: sensitive; this geometry is intimately related to reality; experiment and deduction act on material objects), from "Geometry II: natural/axiomatic geometry" (hypothetical deductive laws are the source of validation; the set of axioms is as close as possible to intuition and may be incomplete; certainty is secured by demonstrations) and "Geometry III: formal axiomatic geometry" (axioms are no more based on the sensory reality, should be complete in the formal sense and independent of applications; consistency is the criterion of existence).

The discussions in the Topic Group showed that this more detailed description of geometry provides a method to classify geometrical thinking. It can also be helpful to interpret tasks eventually given to students and future teachers and can be used to classify the productions from the students, offering an orientation for the geometry teacher. In being closer to the learner's and teacher's world (at least before entrance to university) it also avoids the pitfalls of linear algebra / analytical geometry which often is so far off the geometry world of the student that it is no more taken as part of geometry (see also the paper by Gueudet-Chartier). Some participants even came up with the question if lower secondary teachers normally 'reach' an insight into geometry which includes the Geometry III level at all.

In addition to that, the Gonseth-classification throws some light on the problems discussed in section 3 (registers of representation) and 4 (instrumentation), because it complements the logico-mathematical classification (axiomatic theory – local ordering – problem solving) with relations to which instruments human beings use in geometry and how they use them.

2. DEVELOPMENTAL STAGES

It seems fair to say that the so-called 'Van Hiele levels' of geometrical reasoning are nowadays the most relevant theoretical framework to organise the teaching and learning of geometry. A summarised description of Van Hiele levels can be found in Kuzniak and Houdement's paper. The recent "Principles and standards for school mathematics" (NCTM, 2000) are a clear example of application of Van Hiele levels to curriculum design.

The description of the 'van Hiele levels' already gives hints to fundamental links between these levels and the model suggested by Houdement and Kuzniak (see section 1 above). Consequently and following the argumentation in the paper by Houdement and Kuzniak, these links were widely discussed in the Topic Group. On the other hand, there was some discussion if geometrical knowledge progresses through sequences of stages. Some of the papers in the group could be seen as contradictory to the traditional 'van Hiele levels'. This is especially true if the levels are linked to clearly identified and fixed ages or if individual persons are thought to necessarily follow the order of the respective levels. A position on these ideas on developmental and/or learning stages is obviously important when geometrical tasks or research projects are being constructed or discussed. If accepted, the levels can help to find and further develop appropriate tasks for developmental and research work – and they are obviously helpful for explorative activities to come across new, maybe even innovative ideas.

3. REGISTERS OF REPRESENTATION

A figure, in the sense of the most general object of geometry (either 2D or 3D), can be represented in many ways. We can distinguish three main groups of semiotic representation: material representation (in paper, cardboard, wood, plaster, etc.), a drawing (made either with pencils on a sheet of paper, or on a computer screen, with use of a geometric software, etc.), and a discursive representation (a description with words using a mixture of natural and formal languages). Each register bears its own internal functioning, with rules more or less explicit. Moreover, students have to move from one register to another, sometimes explicitly, sometimes implicitly, sometimes back and forth.

Questions about registers of semiotic representation and cognitive processes have been studied in depth by Duval (1995). He defines “semiotic representations as productions made by use of signs belonging to a system of representation which has its own constraints of meaning and functioning”. Semiotic representations are absolutely necessary to mathematical activity, because its objects cannot be directly perceived and must, therefore, be represented. Duval maintains that “semiotic representations are not only a means to externalise mental representations in order to communicate, but they are also essential for the cognitive activity of thinking”. He thus differentiates *semiosis*, comprehension or production of a mental representation, from *noesis*, conceptual comprehension of an object, while maintaining their inseparability. In cognitive activities linked to *semiosis*, he distinguishes three types of activity: the formation of a representation, which can be identified as belonging to a given register; the processing and transformation of a representation within the register where it was created; and finally *conversion*, i.e., the transformation of a semiotic representation from one register to another. He underlines the importance of the third activity by describing it as a necessary passage for coordinating registers attached to one same concept. Nevertheless, although the first two activities seem to be taken into account in mathematics teaching, the third is usually ignored. It acts as an underlying assumption that is never made explicit as long as the rules of operation within each register are acquired by the learner. Duval shows, however, that many students’ difficulties arise from their inability to carry out conversions between registers of semiotic representation. He also maintains that the possibility of conversion is one of the essential conditions of conceptualisation and, therefore, of *noesis*. “(Integrative) comprehension of a conceptual content rests on the coordination of at least two representation registers, and this coordination is revealed by the rapidity and spontaneity of the cognitive activity of conversion”.

In another text, Duval (1994) had worked more specifically on the study of the different ways of functioning of a figure in geometry. He distinguishes four types of apprehension of a figure. The most immediate is the *perceptive apprehension*, which immediately allows to recognise a form or an object according to ‘gestaltistic’ organisational laws and intra-figural indicators. The *discursive apprehension* is quite simple, a figure is seen according to a verbal description (let ABC be a triangle, with a right angle in A, …) which determines explicitly some of its properties. Indeed, no geometrical property is, in a strict sense, visible on a drawing (one can measure if two lines are parallels, but the uncertainty on the precision of the measurement is inevitable), it has to be either given in the hypothesis or coded, or proved by deductions. The *sequential apprehension* concerns the steps, and their order, according to which a figure must be constructed. It lies in the properties of the figure but also in the type of instruments used and the technical constraints they bear. The actions, often using intermediate constructions and external control, therefore modify the knowledge one has of the object represented. Finally, the *operative apprehension* is the most complicated. As Polya (1945) suggests, it is supposed to show the ‘idea’ of the solution of a problem. It lies on the possible modification of a figure (cutting it in different parts to rearrange like a jigsaw, changing its size, its form or its position).

The didactic difficulty is that these four modes of apprehension, and mostly the last three, play very different roles in the learning of geometry. Moreover they remain largely independent, and the possibilities of transfers are very poor. In spite of an intense training in the discursive and sequential modes of apprehension, there is no transfer on the competence in a heuristic exploitation of the figure, proper to the operational apprehension. Indeed, “there is a heuristic figural processing, which is independent of the construction activities, and which is independent of any reasoning activity, otherwise the use of the register of figures would only be a trick in order to present the process of a reasoning, in a form supposed to be more directly accessible. Yet, this figural processing appears to be non-obvious and very difficult to grasp for many students” (Duval 1994, 136).

Taking into account the variety of the registers of representation, and the heterogeneity of the reasoning processes they mobilise, induces the necessity of implementing different teaching activities specific for each and developing a certain mobility among them.

In our group about “geometrical thinking” the questions of registers have been discussed widely throughout most papers presented here.

The theoretical model offered by Houdement and Kuzniak specifies the use of figure and language in each of the three paradigms. It shows that a same figure or a same verbal description can be understood at different levels, which reveals the difficulties in the understanding of the various modes of representations of geometrical objects (see section 1).

Vighi’s, Acuna’s and Osorio’s contributions show that students have a certain rigidity in their figural representation, even at the perceptive level (a triangle is mostly seen as isosceles with a side horizontal). Moreover, Vighi shows that the word triangle refer to many objects of everyday life that can be an obstacle to the construction of the concept of triangle.

Lanciano’s paper deals with the construction of an instrument: The question here is to make the representation the students have of the instrument operative in its construction. She makes the hypotheses that the senses can assist the intelligence in a very productive way.

Kurina is interested in the heuristic power of the figure in geometrical thinking. His paper shows marvellous examples of the power of the operative apprehension of a figure in the solving of problems.

Cohen’s and Bako’s contributions refer to spatial representation (see section 5). One shows the difficulty of students in apprehending visually the interrelations of points, lines, and planes in space; the other deals with the question of plane representation of spatial objects. Moreover, Gueudet-Chartier is interested into the question of representation via geometrical objects (drawings mostly) of concepts of linear algebra. Using Fischbein's theory, she shows that students have great difficulty in using a figure as an intuitive basis of abstract concepts. She also points out the question of variability in the interpretation of a same drawing that can represent very different situations either in geometry or in linear algebra.

In Rolet's experiment the same task is offered in three different environments. In each, the registers of representation are different and indeed influence the action of the students in an essential way.

Finally, Perrin-Glorian's experiment shows the difficulties students have in restoring a drawing in the plane of which some parts have been erased. They have to overcome the perceptive level in order to attain the discursive level of representation of the figure necessary in order to accomplish the task.

4. INSTRUMENTATION:

Artefacts, Computers, and the ways they are used

There is broad consensus in the community of mathematics teachers and educators that learning geometry is much more effective if concepts, properties, relationships, etc. are presented to students materialised by means of instruments modelling their characteristics and properties. Furthermore, the use of didactic instruments is very convenient, if not necessary, in primary and lower secondary grades.

There is a huge pile of literature reporting the continuous efforts devoted by mathematics educators since long ago to explore the teaching and learning of geometry with the help of manipulatives, computers, and other tool. Even in the early 1970s, during the dominance of “modern mathematics”, there were mathematics educators working in this direction, like Z.P. Dienes. Dienes described, from the paradigm of “modern mathematics”, the process of learning abstract mathematical concepts and structures by children, summarised in *four principles* for the learning of mathematics (dynamic principle; perceptual variability principle; mathematical variability principle; and constructivity principle; see Dienes 1970), and *six stages* in the learning of mathematics (free play; playing by the rules; comparison; representation; symbolisation; and formalisation; see Dienes, 2000). According to Dienes, for children to learn mathematics, playing structured games and using manipulative tools are fundamental pieces in the two first principles and the three first stages.

The Van Hiele levels emphasise the importance of using instruments since, according to this model, children reasoning in 1st, 2nd or 3rd levels need physical objects, instruments or drawings to help them to solve tasks, to understand geometrical structures, and to organise their reasoning:

- Children in the first level (level 0 in Kuzniak and Houdement's notation) perceive geometrical objects as physical entities, so they can learn geometry only while interacting with real objects, drawings, computer figures, etc.

- Children in the second level are able to discover and generalise properties of geometrical concepts by induction from their observation of one or more examples. Therefore, the use of instruments and drawings is the base for their way of reasoning.

- Children in the third level begin to produce abstract deductive reasoning, but they still need concrete representations (physical, on a computer, or drawn) to help them to organise their deductive arguments.

An additional view of teaching and learning geometry from the cognitive side is represented by R. Duval. According to Duval (1998), learning geometry includes three types of cognitive processes, each of them having specific functions: Visualisation (space representation of 2D or 3D configurations), construction (of models representing geometrical structures), and reasoning (to organise descriptive or justificative discourses). To do processes of visualisation and construction, students have necessarily to manipulate instruments, to make drawings, or to interact with computers. Also to do some types of reasoning processes, namely those which are not totally abstract, students take advantage of instruments as help for and base of their reasoning.

Several papers presented to this Topic Group report on research paying attention to the role of instruments in the processes of teaching or learning geometry (Bakó, Lanciano, Rolet, and Larios). Most of them have used software in their experiments, although it has not been the main focus of the research but one of several instruments used.

Bakó experimented with several types of plane representations when teaching 3D geometry in secondary school, to determine if someone of them is more convenient than the others. Namely, she compared the usefulness of axonometric (i.e., parallel), central, and orthogonal projections to solve tasks consisting of finding as many different plane sections of a cube as possible, obtained with the help of either a transparent model of a cube or specific computer software. One of Bakó's conclusions is that software was more helpful in finding different sections, while manipulative models were more helpful in finding the way to obtain the sections.

Lanciano paid attention to the training of future teachers in the use and construction of manipulatives for their classes. In this experiment, she asked the future teachers to make several classical artefacts used to identify points in space. There have been other research projects, many of them carried out by Italian colleagues, where artefacts from past centuries were used to introduce or deepen the study of geometry by asking students to analyse the structure and ways of use of such artefacts.

Rolet's paper shows the necessary interdependence between the "working space" where students act and the instruments they use. In her experiments, primary school children were asked to build a cube, with the materials provided, in four different working spaces (the floor, large and small sheets of paper, and Cabri). Changing the size of the working space and the instruments students have to use is supposed to help students to move from the level of sensible space (physical geometrical experiments) to that of geometric space(abstract geometry).

5. SPATIAL GEOMETRY

Spatial geometry is always a difficult subject from nursery school until university, in every country. On the one side, 3-D objects are part of our everyday experience but, on the other side, they are usually represented in geometry by 2D drawings. Therefore our understanding of 3D geometry is interrelated with our understanding of 2D geometry. Nevertheless, in some cases, a conflict between our knowledge of plane geometry and our intuition of space may be the root of a deep misunderstanding.

In his work, Doan Huu (2001) showed that many students tend to ‘naturally’ transfer theorems from 2D geometry to 3D geometry. There is a strong tendency to use analogical thinking and students have real difficulties in confronting the theorem they know in 2D geometry with their intuition of space. If a student has not built a comprehension of a theorem in 2D independent to her figural representation of the statement, she will be in difficulty. Indeed, it is very difficult to transfer from 2D to 3D only on the basis of figural transposition. A more conceptual background is needed. In other words, understanding 3D geometry leads to reflect again about what we know in 2D geometry and some gaps, not yet visible, may appear to be crucial.

Cohen’s paper deals with the question of relations between points, lines and planes in space. She experimented a teaching process with concrete manipulative objects made of plastic and plasticine. She makes students see and act in 3D with real objects.

Bako is interested in the question of plane representation of 3D objects. In this sense, she claims that central projection should be used as an alternative to axonometric projection. Yet, she made an experiment with the use of computer software and with concrete objects and came to the conclusion that neither of the two types of representation (via axonometric and central projection) leads to better results. Indeed, students have very often great difficulty in ‘seeing in space’ and therefore in understanding the rules for reading a plane drawing representing a spatial object.

Lanciano’s experiment is interesting in the sense that she makes students work in the meso-space, their actions and their senses of space making them reflect on objects in 3D.

In Gueudet-Chartier’s work, not only 3D objects are involved but also multi-dimensional objects in linear algebra. Prototypical situations involving vectors may be represented by drawings in 2D or 3D. The drawings do not only represent the geometrical situation but are a paradigmatic model (in Fischbein’s sense) for a more general situation.

6. TEACH FUTURE TEACHERS

The field of teacher training is very broad, but this Working Group is interested in those aspects related to the teaching of geometry. Both group discussion and several papers included below have raised the difficulty for having a good teaching of geometry in primary school. Although all the papers which elaborate on this problem refer to France, it is present also in many other European countries, and all over the world. There is not a single reason for such situation: meagre contents of geometry both in official curricula and textbooks; poor organisation of contents, with emphasis on memoristic aspects (definitions, formulas, etc.); insufficient use of instruments and other didactic tools; consideration of geometry as a secondary (i.e., unimportant) part of school mathematics that can be skipped over if necessary; and, no less important, poor knowledge of geometry by many teachers.

As a consequence, most students entering pre-service teacher training programs have a very poor background in geometry, with many gaps and partial or erroneous knowledge. Unfortunately, when these students become in-service teachers, they reproduce their errors, misunderstandings, and negative habits, beliefs and behaviours, and transmit them to their pupils.

A serious problem in the teaching of geometry is the transmission of conceptual errors to the students (many times in an unconscious way). A clear instance is the use by teachers and textbooks of prototypical examples as the only way to teach geometrical concepts; R. Hershkowitz and S. Vinner have reported on pre- and in-service teachers' behaviour in several publications (Hershkowitz, 1989, 1990; Hershkowitz and Vinner, 1984; Vinner, 1991). Related to this issue, Cohen's paper presents a research aimed to identify and classify directions of planes and straight lines in space preferred by pre-service teachers when they have to solve problems of identification and drawing of perpendicular or parallel lines and planes.

A critical question with respect to the formation of future mathematics teachers (primary and secondary school) is to decide the extent of mathematical knowledge they should master and the quality of mathematical reasoning they should be able to produce. There is a rule almost universally accepted stating that teachers must know more mathematics and more deeply than what they have to teach. The point now is to agree how many more mathematics and how much more deeply. The Van Hiele levels provide us a good reference to answer the question on reasoning: Teachers should reason at least one level higher than students (although they have also to be aware that they should interact with their pupils on their level). Then, primary school teachers should be (at least) in the third level, and secondary school teachers should be (at least) in the fourth level.

Another theoretical framework complementing Van Hiele levels is provided in Kuzniak and Houdement's paper below. They define three paradigms characterising different forms of geometry, that is different amount and organisation of contents, and different ways of interaction, communication and reasoning. According to this model, primary school teachers should be, at least, in the second paradigm (Geometry II), and secondary school teachers should be in the third paradigm (Geometry III).

Finally, in relation to section 4 above, prospective primary or secondary mathematics teachers have to learn to use different didactic instruments useful in diverse mathematics areas, and they have to acquire the ability to adapt them to the objectives of their classes and to design new instruments (see Lanciano's paper below). Therefore, teacher-training courses should allow prospective teachers to learn the relationships among abstract geometrical concepts or properties and their concrete representations, and to discover geometric models in children's ordinary life environment.

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List of contributions

List of Thematic Groups