UNCERTAINTY AND 3D DYNAMIC GEOMETRY: A CATALYST FOR THE STEP FROM 2D TO 3D GEOMETRY

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Preservice teachers' training in 3-dimensional geometry can be strengthened with technological resources currently available in mathematics education. In this paper, we report how a task proposed to preservice teachers in a 3D geometry course promoted learning due to the intellectual need generated by uncertainty that encouraged argumentation. The analysis is part of a design-based research whose goal is to propose a model of 3D geometry tasks, to be solved by using dynamic geometry, that generate uncertainty. The results of the analysis of the interaction that took place while discussing the solutions to the task suggest that a suitable use of dynamic geometry, together with a carefully designed task, can generate uncertainty that provokes an intellectual need which students express in their arguments.

INTRODUCTION

Preservice teachers' difficulties to visualize in 3-dimensional geometry have been documented in various studies (Moore-Russo & Schroeder, 2007; Sgreccia, Amaya, & Massa, 2012). This is a problem due to the role visualization plays in argumentation and, in general, in student performance in 3-dimensional geometry. Researchers such as Prusak, Hershkowitz, & Schwarz (2013) present the need to design classroom tasks that encourage argumentation to favor students' construction of mathematical meaning. A strategy that we believe promotes argumentation is to pose problems which provoke students' uncertainty that induces the intellectual need to argue. In this paper we present a study of student reactions, when uncertainty was provoked by two representations of a *folded quadrilateral*, that is, a 3-dimensional quadrilateral whose vertices are not coplanar (Figure 1,2), obtained by redefining some of them. The task design was inspired by the use that Ferrara & Mammana (2014) make of the tool *Redefinition* in Cabri 3D. Our research objective in this paper is to analyze students' activity while solving a task designed to generate uncertainty, to identify ways the uncertainty emerged and how it influenced students' need to argue.

THEORETICAL FRAMEWORK

Students' motivation for learning arises from *uncertainty*, a concept that encompasses notions of conflict, doubt and perplexity resulting from social interaction in the classroom when solving a task that faces the students with a situation that is incompatible with their current knowledge or is not solvable with it (Zaslavsky, 2005). We agree with Stylianides & Stylianides (2009) that uncertainty acts as a mechanism that stimulates the emergence of the intellectual need to develop mathematical knowledge. *Intellectual need* is defined by Harel (2013) as the need to extend or reorganize knowledge to make it compatible with the situation that needs to be

2018. In Gómez, D. M. (Ed.), *Proceedings of the First PME Regional Conference: South America*, pp. 49-56. Rancagua, Chile: PME.

understood.

In the task that we analyze, for instance, uncertainty is provoked by the effect produced in the students when the representation of a *quadrilateral* is changed. Initially it was constructed with vertices *A, B, C* and *D* in plane α (determined by A, C y D in figure 1). Later, the Cabri 3D *Redefinition* tool was used on vertex *B* to move it out of plane α (Figure 1); afterwards, vertex *D* was also moved out of plane α, using the same tool (Figure 2).

Argumentation, to convince oneself or others (Harel & Sowder, 1998), is the manifestation of intellectual need provoked by uncertainty. In the situation that we analyze, uncertainty led to argumentation as the students examined the fulfillment or not of the definition of a folded quadrilateral, that had been constructed collectively in the class. Specifically, in the above situation described, students argued whether the last represented figure continued being a folded quadrilateral or not, a situation that forced them to visualize different planes in space.

METHOD

The task that gave way to the interaction that we analyze was planned, with other tasks, as part of a design-based research (Bakker & van Eerde, 2015), developed in a 3D geometry course of a pre-service secondary mathematics teachers' program, in a university in Bogotá, Colombia. The course consisted of 33 third semester students. They had studied two previous plane geometry courses, where a 2D DGS was frequently used. Cabri 3D was used for the first time in the experimental course. The students worked in groups of three.

Students were asked to *consider points A, B, C and D* in space and to study *the figure that is the union of the segments determined by these points, no two of which intersect in points different from their endpoints.* All the groups, except one, proposed as a solution a standard quadrilateral, with the four vertices in the plane (α). They did not imagine the situation that one group proposed, in which the four points were not coplanar. The teacher represented the situation suggested by the majority, with Cabri 3D, and then used *Redefinition* to move *B* out of plane α (Figure 1).The teacher encouraged the students to define the resulting geometric figure, unknown to them until then; they collectively defined the geometric object, and, due to the teacher's suggestion, labelled it *folded quadrilateral*: *A folded quadrilateral is a four-sided figure with four non-coplanar vertices, for which every three vertices are not collinear, and every vertex is the endpoint of exactly two segments.* Later, some students questioned what would happen if vertex *D* did not also belong to plane α. The teacher used the *Redefinition* tool again on *D* (Figure 2) and asked the students whether the resultant object was a folded quadrilateral or not.

The information used for the analysis in this paper corresponds to the discussion instigated by the teacher's question. It was obtained from two sources: the interaction between the teacher and the whole class, and the dialogue between one of the researchers who was in the classroom, and some students. Using the Complementary Accounts Strategy proposal suggested by Clarke (1997), in the next class the researcher showed certain moments of the teacher-student interaction extracted from the video of the classroom events and asked questions about them. This revision favored the exposure of ideas by the students; it permitted us to carefully track uncertainty moments, manifested in the students' facial expressions and in their argumentation about the situation, and to describe how these developed.

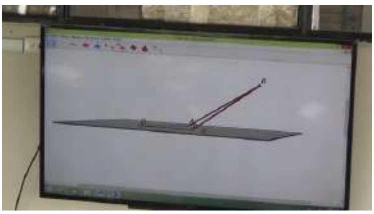

Figure 1. Solution with vertex *B* not belonging to plane α.

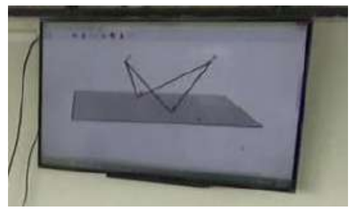

Figure 2. Folded quadrilateral with two vertices not in plane α.

Using the transcriptions of the interactions that took place, the analysis commences with the identification of indicators of uncertainty during the interaction with the teacher. Once these are found, the emergence of uncertainty is corroborated in the transcriptions of the dialogues with the researcher. Then, traces of intellectual necessity, expressed as argumentation, are looked for; the teacher's role and the effect of the use of Cabri 3D in the development of the task are identified. The analysis leads to the establishment of a route to articulate elements identified in the task design, and thus advance in the construction of an answer to the research question we have formulated.

ANALYSIS

In a preliminary discussion, before the implementation of the task, the research group considered that uncertainty could appear when the vertices of the quadrilateral are redefined to extract them from plane α (Figure 2); therefore, the class was questioned whether in each case a folded quadrilateral was represented. We had anticipated that if no student promotes further exploration, by extracting the second vertex, the teacher would do it. We expected that intellectual necessity would be expressed with arguments in favor of and against accepting it as such, a product of uncertainty generated by the situation.

The first redefinition, when point *B* is extracted from the plane, caused uncertainty; some students were surprised, expressed by the look on their faces, that such a figure actually satisfied the established properties. They could only imagine coplanar figures. During the teacher-guided production of the definition of a folded quadrilateral, uncertainty, as an expression of doubt, arose. This becomes evident, with Adriana's (the names are pseudonyms) objection to the proposed definition when she suggested that the possibility of redefining another vertex, as a point not in the plane, could modify the definition:

Teacher:	Then, we are going to list the properties and, from it, arises the definition. Yes? Then, four non-coplanar points [writes] (…) non-coplanar. Do I need four non-coplanar points?
Juan:	Every three not collinear.
Teacher:	Every three [writes] not collinear.
Adriana:	Teacher, if we redefine *D* and take it out of the plane (…)?

This possibility created the intellectual need for Adriana and Juan to question the established definition for a folded quadrilateral. They argued that the written list did not include the case of a representation in which two vertices were not in plane α. There are two options that can give place to different arguments: when the representation with two points out of the plane α is considered as an example of a folded quadrilateral and when it is not. The second was the case for Adriana and Juan, who felt that the definition lacked something: "We want [the definition to state that there is] exactly one point that is non-coplanar [in plane α]". This would prevent accepting, as a folded quadrilateral, a representation with two vertexes that are not points of plane α.

Once the teacher redefined vertex *D,* as a point that does not belong to α (Figure 2), she encouraged discussion by asking if the representation was an example of a geometric object, different from a folded quadrilateral. The intellectual need, which up to the moment was expressed with arguments against accepting the four-sided figure with two vertices not in α as a folded quadrilateral, is now expressed in favor by John.

Teacher:	Do I have another figure [different from a folded quadrilateral] that we may want to give another name to? (…) Doubly folded or something like that?
John:	But it is… It is the same [figure]!
Teacher:	It's the same? (…) Why?
John:	If we make a plane that contains points *B*, *A* and *C*, we are going to obtain the same thing.

The teacher illustrated John's idea with a representation of the plane that he mentions (Figure 3):

Teacher:	[…] You say that it should be the plane that contains *B*, *A* and *C*. And you say that one sees (a figure) like my folded quadrilateral (Figure 3). Does one see my same folded quadrilateral?
In chorus:	Yes!

John's intervention, which is expressed with confidence, together with the group's unified answer, transmitted the sensation of having reached consensus. Yet, the facial expression of various students reflected doubt. Since our interest was to obtain more information from those expressions of doubt, which we consider as indications of uncertainty, we decided to promote, during the next class, student interaction with the researcher, using the Complementary Accounts (Clarke, 1997) strategy. Presenting extracts of the previous class video, he asked questions about the students' comments and expressions.

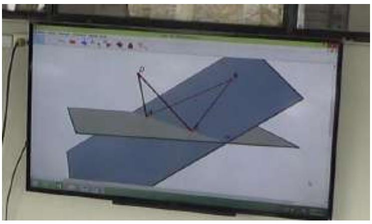

Figure 3. Quadrilateral with two vertices not in α

Figure 4. Lina illustrating her ideas

In their interaction with the researcher, Juan specified why he and Adriana thought that the definition should specify that exactly one vertex must not belong to the plane:

Juan: Well, I was thinking about the representation, but then […] we could see that something is missing [in the definition], because we could find a counterexample. Yes? That is, the representation (…) when we take another point out [from α], would stop being [a folded quadrilateral]. Yes?

Laura, who did not agree with the given definition of a folded quadrilateral, though she did not say so previously, and expresses this to the researcher:

Laura: Up to this moment of the class, we had three points [in α] and a [point] *B* that was not in the plane. The question was: if another vertex is taken out (…)? It seemed to me that it was not a folded quadrilateral.

Other students explained why they thought the written definition was correct:

Sergio: Nora said that it does not matter if we take [another] point [*D*] out [of the plane, because] *B*, *A* and C are going to determine a plane (…).

Santiago: Since there were two sides that intersect in a point, this already determines a plane.

Then Lina explains to the researcher how she had imagined the situation:

Lina: I imagined this visualization, but let's say in a drawing. It is that if we take first plane α and redefine *B* and [re]define *D*, for example, it would be something like this [she modelled with her hands: she placed a hand in horizontal position and the forefinger of another hand over the palm, but without touching it, indicating the position of a point not in plane α] (Figure 4). Two points *A* and *C* and another point *B*. I see this [plane] *B*, *A* and *C*, because [the points] are contained in a plane since they are not collinear; and we would have this plane (plane determined by the three points) and we would have the other one (α) [models this moving her finger from her palm in vertical position]. For me it continued being a folded quadrilateral.

Laura's next reflection was different because she questioned whether the use of the tool *Redefinition* to convert point *D* in a point that is not in α, could place it in the plane determined by points *A*, *B* and *C*. Then the figure would simply be a quadrilateral.

Laura: I believe that the issue is, when another point is redefined [*D*], is that moving it, the risk of moving it so that (…). In moving it, it could end up being in the same plane determined by [point *B*] that is not [in α] and [points] *A* and *C* that remained fixed in plane [α]. […] That *D* could be in the plane determined by *C*, *A* and *B* [a plane different from plane α]. Then it would not be a folded quadrilateral. Then I was saying, no. (…) But also, there are many possibilities that it might not end up [in that plane], in which case, yes it would be a folded quadrilateral.

Laura's explanation shows an interesting extension of visualization, because she is considering a plausible result from the use of the *Redefinition* tool of Cabri 3D.

Some students changed their previous idea about the resulting figure, because initially they did not recognize it had the configuration of a folded quadrilateral. They expressed this to the researcher:

Juan: Because I had not seen the other visualization of another plane. Because let's say a point, because anyway another point is going to remain outside. Then I said: something is needed.

Arturo: For me also [I was interpreting the same], only until the other plane was constructed.

Researcher: Only until the plane was constructed you knew (…)?

Arturo: (…) that it was folded.

Evidently, Juan's and Arturo's visualization of folded quadrilaterals broadened as a result of their observation of the representation of the plane determined by points *A*, *B* and *C*. Together with Santiago's argument: "two sides that intersect determine a plane", these became useful theoretical resources which the students could use to determine planes in 3D geometry.

Up to this point, the discussion illustrates two elements in the development of the task: the appearance of two divergent points of view regarding what is or not an acceptable representation of a folded quadrilateral, and the way the students try to resolve those differences. This development is what we now discuss in terms of the aspects incorporated in the task design: uncertainty and intellectual necessity.

DISCUSSION

The use of the Cabri 3D *Redefinition* tool allowed teacher and students to represent a figure, with two vertices outside plane α, that fulfilled the conditions stated in the definition of a folded quadrilateral. The modification of the initial representation of a folded quadrilateral, by taking another vertex out of plane α, generated doubt in the students, giving place to the desired uncertainty situation (is the new representation a folded quadrilateral?). This doubt was solved by means of arguments proposed by the students. The use of this tool, together with the construction of the plane determined by a new set of three non-collinear points, helped visualize the existence of planes in space, not represented in the initial configuration. In addition, *Redefinition* became a tool to solve 3D geometry problems. Having redefined two points, plane α became a distraction in the representation, and generated doubt in some students, which was solved when they noticed that plane α was not the only plane in space. The student's visual schema of space as a horizontal plane and some points not in that plane was modified.

The arguments exhibited by the students, especially in their interaction with the researchers, allowed us to identify that their concept image (Vinner, 1991) of a folded quadrilateral consisted of the specific position of three vertices lying in a horizontal plane, the other vertex above it. This motivated a class discussion to decide whether to accept or not other representations of the figure fitting the definition. Their arguments, as a result of the uncertainty provoked by the task, were a resource to analyze how to develop the intellectual need to clarify that only the properties given in the definition are needed to decide whether the representation is the defined figure. In addition, Laura's argument enriched our forecast of what can happen during the discussion of the solution of the task in future years.

As for the elements that must be articulated in the design and teacher management of a task, one is the transformation of familiar situations, without losing basic properties, to create doubt and to modify restricted images. Only if the students can visualize planes other than the one represented initially, will they be able to solve problems in 3D geometry. They must identify the geometric figures that are in each plane and the properties they have, so that their knowledge of plane geometry can become a resource to solve the problems.

CONCLUSIONS

The first relevant issue we observe is that uncertainty is produced and expressed when the students voice different points of view regarding what is or not a representation of a folded quadrilateral. Doubt probably arises due to the difficulty to visualize other planes in a configuration in which a plane is already represented. The second relevant issue is that uncertainty gives way to intellectual necessity, for the students resort to the axiomatic system conformed in the course, specifically how to determine planes, to be able to find a different plane than the one represented. The third issue is that the ideas the students expressed to the researcher should be heard in a normal class setting. Thus, only if a teacher is aware of the moments when uncertainty arises during a task development, and, precisely at that moment, induces students to communicate their ideas and promotes argumentation, will intellectual necessity be generated, and, as a consequence, meaning-making favored.

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