ANALYSIS OF CONJECTURES AND PROOFS PRODUCED WHEN LEARNING TRIGONOMETRY

Jorge Fiallo
Dpto. de Matemáticas
Univ. Industrial de Santander
Santander (Colombia)

Angel Gutiérrez
Dpto. de Didáctica de la Matemática
Univ. de Valencia
Valencia (Spain)

Abstract. *Students usually learn mathematical proof based on contents of Euclidean geometry, calculus or numbers. Trigonometry is usually taught in a routine algorithmic way, but we show that also this topic can be used to teach students to prove conjectures. In this paper we describe a teaching experiment aimed to promote 10th grade students' ability to prove while meaningfully studying trigonometry with the help of a DGS. We present examples of the different types of proofs produced by the students, and show their progression during the teaching experiment.*

INTRODUCTION

The learning of proof is one of the most active research agendas in Mathematics Education. Mariotti (2006) suggests the existence of three research directions into this agenda: Analysis of the roles of proof in mathematics curricula, approaches to students' conceptions of proof, and teaching experiments to teach students to prove. Another very active research agenda is related to the use of new technologies, mainly computers, in teaching and learning mathematics. In particular, dynamic geometry software (DGS) has proved to be an excellent environment to learn geometry.

The integration of both research directions, teaching students to prove with the help of software, shows that computers are a powerful tool that can be successfully used to help students understand the need of mathematical proofs, and to explore, analyze and get data in order to state conjectures and to devise ways to prove them. Computer microworlds provide students with environments where the mathematical concepts are seen as objects that can be handled, transformed and observed.

Most research on teaching mathematical proof are based on Euclidean geometry, and some others on calculus or numbers, but very seldom students are asked to prove trigonometric properties. When studying trigonometry, quite often students just have to memorize a set of identities and to apply them to solve routine exercises. When students are asked to do proofs in trigonometry, proofs usually consist of algebraic transformations linking a side of an identity to the other side. On the contrary, to promote students' understanding of trigonometric concepts, they should be provided with tools and procedures that help them to analyze and relate concepts, to produce and prove conjectures, to meaningfully learn the relationships, concepts or properties.

In this paper we present results from a research aimed to get a better understanding of students' learning of proof processes by observing the ways students prove conjectures in trigonometry. The research was based on the design, experimentation and analysis of a teaching unit aimed to teach trigonometry to 10th grade students in a DGS environment and, at the same time, to induce students to prove the conjectures they get from their explorations.

RESEARCH FRAMEWORK

The analysis of students' answers focused on their abilities to prove the conjectures raised as answer to the activities. We understand *mathematical proof* in a wide sense, including formal proofs but also any attempt made by students to convince themselves, the teacher or other students of the truth of a mathematical statement or conjecture by means of explanations, verifications or justifications.

The analysis was based on the categories of proofs described in Marrades, Gutiérrez (2000) who, elaborating on the types of proofs identified by Bell (1976), Balacheff (1988), and Harel, Sowder (1998), defined an analytic frame-work to characterize students' answers to proof problems. Due to space limitation, we only include here short descriptions of the types of proofs integrating this framework:

A) *Empirical proofs*. The types of empirical proofs are:

* *Naive empirical proofs*, when a conjecture is proved by showing that it is true in examples selected without a specific criterion. Depending on how the examples are used, a naive empirical proof may be:
  - *Perceptual proof*, when it involves only visual or tactile perception of examples.
  - *Inductive proof*, when it also involves the use of mathematical elements or relationships found in the examples.
* *Crucial experiment proofs*, when a conjecture is proved by showing that it is true in a specific, carefully selected, example. A crucial experiment proof may be:
  - *Example-based proof*, when it only shows the existence of an example or the lack of counter-examples.
  - *Constructive proof*, when it focuses on the way of getting the example.
  - *Analytical proof*, when it is based on properties empirically observed in the example or in auxiliary elements.
  - *Intellectual proof*, when it is based on empirical observation of the example, but the justification mainly uses accepted abstract properties or relationships among elements of the example.
* *Generic example proofs*, when the proofs are based on a specific example, seen as a characteristic representative of its class.
  - The four above defined types of crucial experiment proofs are used to discriminate generic example proofs too.

B) *Deductive proofs*. The types of deductive proofs are:

* *Thought experiment proofs*, when a specific example is used to help organize the proof. Depending on how the example is used, a crucial experiment proof may be:

- *Transformative proof*, when it is based on mental operations producing a transformation of the initial problem into another equivalent one.
- *Structural proof*, when it consists of a sequence of logical deductions derived from the data of the problem and axioms, definitions or accepted theorems.

* *Formal deduction proofs,* when they do not have the help of specific examples.

- The two above defined types of thought experiment proofs are used to discriminate formal deduction proofs too.

METHODOLOGY

During the two weeks previous to the beginning of the classes, the first author met each teacher to make them aware of the research aims, teaching objectives and methodology, their expected role as teachers, etc. Our conception of proof in mathematics was specially emphasized to the teachers.

The teachers were the responsible for the teaching, and the researcher acted as a participant observer, taking field notes, observing students’ behaviour and collaborating with the teachers by answering some students’ questions and queries.

Data gathered during the teaching experiment to analyze students’ activity were students’ answers to a written diagnostic test to show their previous knowledge and proof abilities, groups’ answers written in the activity sheets, concept maps filled in by the groups at the end of several sets of activities, three written exams posed by the teachers during and at the end of the teaching experiment, and videotapes recording daily actions and dialogs of two groups from each school.

THE TEACHING EXPERIMENT

The sample.

The teaching experiment was carried out with 100 grade-10 students (aged 15-16) in three whole classroom mixed ability groups from three secondary schools at Santander (Colombia). The sample was selected on the base of availability of schools and teachers, who showed their interest to collaborate in this experiment. In Colombia, 10th grade is the first year of non-compulsory secondary school, and trigonometry is taught for the first time in this grade. The students from two schools were average students, and those from the third school were above average students. They had never been asked before to prove mathematical statements or conjectures.

The experiment.

The teaching experiment took place as part of the ordinary classes of mathematics, for a period of about 12 week. Each group had two 90 minute classes per week in a computer room using Cabri II+. In two schools the students worked in small groups (2 or 3 students per group) with a computer, and in the other school each student worked with a computer. Students could use Cabri freely to solve all the proof problems posed. They didn't have previous experience in using a DGS. The teaching unit included five activities:

1) Introduction of trigonometric ratios of right triangles.

2) Introduction of trigonometric ratios of angles in standard position.

3) Visualization and vector representations of trigonometric ratios.

4) The Pythagoras Theorem and related trigonometric identities.

5) The sine of the addition of two angles.

Each activity was integrated by a set of related sub-activities. Some examples of sub-activities are shown in next section; space limitation impedes us to include more details about their content.

The experimental teaching unit was designed in a guided discovery teaching style, with the first parts of activities 1 to 3 setting the ground knowledge on trigonometric ratios (both graphical, algebraic, and analytical) to be used in the second parts of those activities and in activities 4 and 5.

As a main objective of the teaching unit was the development of students' proving abilities, they were asked from the very beginning to analyze and prove any conjecture they made. A frequent way for the groups of students to solve an activity was to make a figure in Cabri, or open a file with a figure, to explore the figure looking for a conjecture, to debate this conjecture, to write the group's conclusions and arguments in the activity sheets, and finally to participate in a class discussion with the other groups and the teacher. As the students hadn't used Cabri before these classes, and there was a limited time to teach them to use Cabri, students were only asked to make easy figures, and they were provided with files containing already made figures in the other activities; then Cabri was mainly a tool to visualize and dynamically explore and analyze trigonometric definitions, properties and relationships.

EXAMPLES OF PROOFS OF TRIGONOMETRIC PROPERTIES

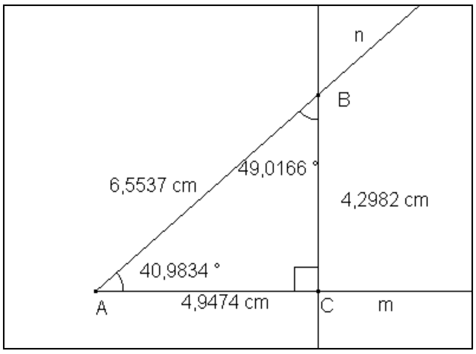
In this section we present examples of different types of proofs produced by the students. It is interesting to note that, even with a rather small sample of students, who had never been asked before to prove conjectures, we obtained a quite large variety of types of proofs, showing that trigonometry may be a rich context to make students engage in learning to prove.

Examples of empirical proofs.

Naive empiricism. In activity 1.3.1 students were asked to create in Cabri a right triangle ABC (Figure 1a), based on two rays *m* and *n* and a straight line perpendicular to *m*, and to get and prove a relationship between the acute angles A and B.

Students in group G1G filled in a table (Figure 1b) with measures for angles A and B taken from the screen and they raise a conjecture: *The measures of angle A and angle B add the same as angle C*. Then this dialog took place:

1 Researcher: Angle C, how much does it measure?
2 Students: 90º.
3 Researcher: Is it the same for any triangle? Is it always true?



- a -

- b -

1	61.9330	28.0670
2	59.2209	30.7791
3	31.4057	58.5943
4	30.4019	59.5981
5	22.2341	67.7659
6	11.4087	78.5913
7	12.3369	77.6631

Figure 1: a) Right triangle for activity 1.3.1. b) Table filled in by students.

4 Students: I think so.
7 Researcher: Then, how would you justify that the addition is really 90º?
8 Students: Because we have several measures here in the table, and if we add them it is ever 90º, any two we take.
9 Researcher: Is this enough to justify it [the conjecture]?
10 Students: Yes.

In this dialog the students showed an inductive naive empirical proof, since the proof is based on the data in the table, collected without any specific criterion.

Crucial experiment. In activity 2.3.1 students were asked to find and prove a relationship between sin(A) and sin(-A).

During the whole class discussion, student C10 explained to the class her answer. After having drawn on the board an acute angle A and angle -A (Figure 2), the student explained:

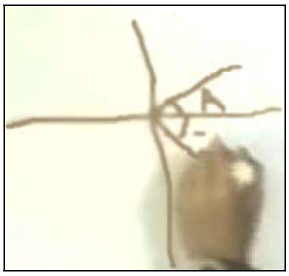


Figure 2: Students C10 drew angles A and -A on the board.

3	C10:	Then the values of this [pointing to ∠ A] are the same as the absolute values of this [pointing to ∠ -A].
4	C10:	Then here, let's say this point [marking a point on the terminal side of ∠ -A] and this point [marking a point on the terminal side of ∠ A] ...
5	C10:	Let's say this [point] is 3 and ... 2 [writing (3,2) next to the terminal side of ∠ A].
6	C10:	Then this [point] would be 3 and -2 [writing (3,-2) next to the terminal side of ∠ -A]. Three, minus two [whispering].
7	C10:	Then ... we can say that the values, as *x* is the same because *x*, as *x* is ... [pointing to the positive end of axis X] it is shared [by the points on terminal sides of ∠ A and ∠ -A], and *y* [pointing to values 2 and -2] is ... the absolute values are equal, so sine is supposed to be *y* over *r* [writing $\frac{y}{r}$].
16	C10:	Then we have that the radius is the same for both [angles, pointing to the terminal sides of ∠ A and ∠ -A].
18	C10:	And *y* is the absolute value is ... the absolute value is equal [pointing to values 2 and -2], but here ... [pointing to the sign of -2].
20	C10:	In A it would be $\frac{2}{r}$, and this would be $\frac{-2}{r}$ [writing them on the board].
21	C10:	Then, as *r* is equal, it would be ... I mean the divisions would be equal, only would be different the signs, so the results would be opposite additive or inverse additive.
22	Teacher:	Then, what is the relationship between sin(A) and sin(-A)?
23	C10:	We have that sin(A) = -sin(-A) [writing the identity on the board].

Student C10's proof was a combination of statements based on particular values of angle A (it was in the 1st quadrant) and coordinates, (2,3) and (-2,3), and attempts to generalize, by using symbols like $\frac{y}{r}$. The arguments she used were correct, but they were based on a specific example. Most likely, when the student was solving the activity with the computer, she dragged the terminal side of angle A and noted that, for any angle A in the circle, angles A and -A had the same abscise and inverse ordinates. Anyway, when explaining her conjecture to the class, she didn't feel the need to mention the angles A in other quadrants. Therefore, this is a case of

<u>intellectual crucial experiment</u> proof.

Generic example. In activity 2.3.2 students were asked to find relationships between the trigonometric ratios for angles A, A-90º and 90º-A, and to probe their conjectures "by using accepted mathematical properties".

Student F02 was working on a drawing (Figure 3; letters V and W added to make easier the references). Then the following dialog took place (during the dialog the student was pointing to the objects on the screen he was mentioning):

1 Researcher:		What did you find?
2 F02:		I changed a little the figure and added another ... another perpendicular line going through point S, which is ... which is the intersection of the circle and the side of [angle] 90º-A.
3 Researcher:		Ok.
4 F02:		When I did it, ... this ... this triangle, which is A, S, and axis X, appeared [triangle ASW]. I noted that it is equal to triangle A, P and axis X [triangle PAV]. It [line AP] was the first line a drew.

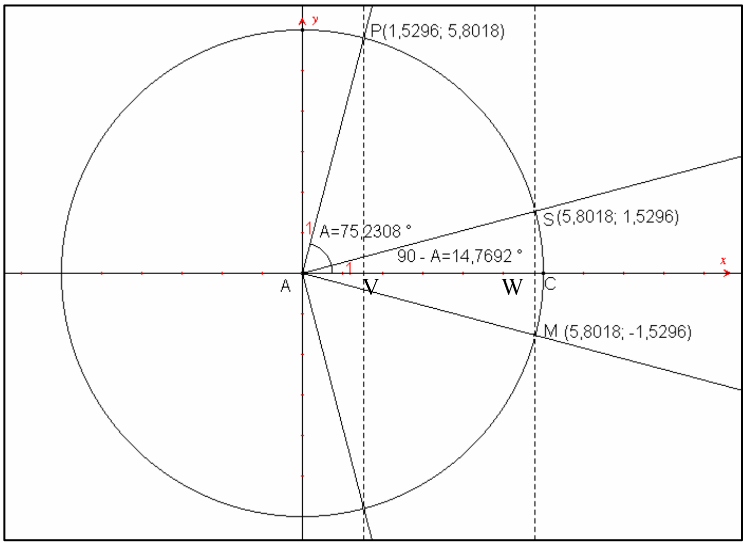


Figure 3: Drawing for activity 2.3.2.

5 Researcher:		Ok.
8 F02:		Then, the angle between S and C [∠ CAS] is equal to the angle between Y and P [∠ YAP], there isn't much to say about this [i.e., the equality is evident], but I can say that P and C [∠ PAC] is equal to S and Y [∠ SAY].
9 F02:		Why? Because with angle Y the angle between Y and C is 90º.
10 Researcher:		Ok.
11 F02:		Then, when I subtract this angle, angle A, this angle appears [∠ 90º-A].
12 F02:		As the triangles are equal, I can say that the sine of this one [∠ SAW], which is the opposite [side] over the hypotenuse, shall be equal to cosine ... to cosine of A, which is this distance here [AV] over the

hypotenuse. As the hypotenuse is the radius, it will ever be the same and, from the figure I have made, I get that ... these two distances [SC and AV] are equal, so the ... the ... the values of these two ... these two [trigonometric] ratios are equal, equal to sine of A and ... [stopped by the researcher]

13 Researcher: Equal in the signs too?

14 F02: Equal in the sign too [while moving the head up and down].

17 Researcher: Then, we can say that sine of A is equal to ...?

18 F02: To cosine of ... of ... 90-A.

19 F02: And also that cosine of A is equal to sine of 90-A.

Student F02 stated a conjecture starting with an example intended to represent all the angles in the circle; the student didn't check nor mention angles in other quadrants. The proof included several references to the way the example had been made. The student tried to produce abstract arguments, although they really referred to properties or elements of the drawing. The drawing in the screen showed the measures of angles and coordinates of points, but the student didn't use them explicitly. Then F02's proof is a case of <u>constructive generic example</u> proof.

Examples of deductive proofs.

Mental experiment. In a problem included in an exam, the students were asked to find a relationship between cot(360º-α) and cot(α), and to prove it.

Student C16 drew the diagram and then she wrote this proof:

cot(360º-α), cot α

360º - acute = in the 4th quadrant (θ)

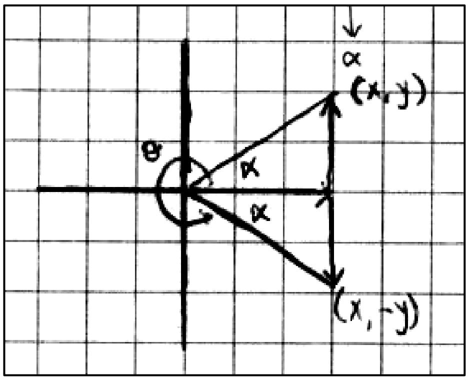
$\cot\theta = \dfrac{x}{-y}$ $\qquad$ $\cot\alpha = \dfrac{x}{y}$

$\cot\theta < 0$ $\qquad$ $\cot\alpha > 0$

$|\cot\theta| = |\cot\alpha|$

$-\cot\theta = \cot\alpha$

$\boxed{-\cot(360^{\circ}-\alpha) = \cot\alpha}$



The student first drew an acute angle α and the angle 360º-α in the Cartesian plane. Then she assigned coordinates (x,y) and (x,-y) to two points in the terminal sides of these angles. She knew that the triangles drawn in the diagram are congruent because this property had been studied in a previous activity. The student also used the label θ to name the angle 360º-α, and then she wrote algebraic transformations to deduce the correct relationship. The diagram drawn only played the role of an auxiliary abstract example, but the student organized the proof of the conjecture based o it. Therefore this is a case of a <u>structural mental experiment</u> proof.

Formal deductive. In activity 3.4.8 students were asked to find a relationship between sin(360º-α) and sin(α), and to prove it. Student C16 wrote this proof:

IV QUADRANT $\sin(360^{\circ}\text{-}\alpha) = \sin\theta$

$$\sin\theta \text{ and } \sin\alpha$$

$$-A \quad \text{and} \quad A$$

$$\boxed{-\sin\theta = \sin\alpha}$$

$$\sin\theta = \frac{y}{r} \qquad \sin\alpha = \frac{y}{r}$$

$$y < 0 \qquad y > 0$$

$$\sin\theta < 0 \qquad \sin\alpha > 0$$

$$-\sin\theta = \sin\alpha$$

First student C16 wrote the conjecture sin(360º-α) = sin α and, like in the previous example, she changed 360-α to θ. She noted that the terminal side of ∠ 360-α is the same of ∠ -α and reminded the identity, proved in activity 2, sin(A) = -sin(-A). Then she wrote the conjecture -sin θ = sin α and afterwards she proved it by using coordinates of points in the sides of the angles and algebraic expressions. The student assumed that α is an angle of reference in the 1st quadrant, so 360º-α (-A) is in the 4th quadrant, and the ordinates associated to these angles have the same absolute value but opposite signs. This is a decontextualized proof based on transforming the initial problem (find a relationship between angles 360º-α and α) into another one (find a relationship between angles A and -A), so it is a case of a transformative formal deductive proof.

SUMMARY OF RESULTS AND CONCLUSIONS

Table 1 synthesizes the types of proofs produced by the students in each activity. The cells with thick border represent the most frequent type of proof for each activity.

Act iv.	F	FE	PNE	INE	ECE	CCE	ACE	ICE	EGE	CGE	AGE	IGE	FD	TTE	STE	TFD	SFD
1																	
2																	
3																	
4																	
5																	

Codes: **F** = Failed. **FE** = Failed empirical. **PNE** = Perceptive naive empirical. **INE** = Inductive naive empirical. **ECE** = Example-based crucial experiment. **CCE** = Constructive crucial experiment. **ACE** = Analytical crucial experiment. **ICE** = Intellectual crucial experiment. **EGE** = Example-based generic example. **CGE** = Constructive generic example. **AGE** = Analytical generic example. **IGE** = Intellectual generic example. **FD** =

Failed deductive. **TTE** = Transformative thought experiment. **STE** = Structural thought experiment. **TFD** = Transformative formal deduction. **SFD** = Structural formal deduction.

Table 1: Types of proofs produced by the students.

We can note, along the teaching experiment, a change in the types of proofs produced by students, from empirical proofs (in activities 1 and 2) to deductive proofs (in activities 4 and 5). Failed proofs (when students didn't succeed in writing any proof, even a wrong one) were limited to activities 1 and 2. The students making more deductive proofs were those who showed, from the beginning of the experiment, a tendency to present their arguments deductively and also showed a better knowledge of the necessary previous mathematics contents.

The types of proofs produced by the students were also related to the content of the activities. We can observe that most proofs written to answer activity 1 were empirical, partly because students were asked to drag the figures on the screen and to draw conclusions out. The least frequent type of proofs was the generic example, mainly because the kind of activities and questions posed to students didn't promote this type of proofs. Activity 2 was the one with most variety of types of proofs; this may be a consequence of having included in this activity an assessment questionnaire with five proof problems. Activity 3 is the first one having many more deductive proofs than empirical ones; more specifically, the type of proof most frequently produced by the students was the transformative formal deduction; we believe that the help provided to students by the visualization of properties in the Cabri figures was a main reason for this result. In activities 4 and 5, only deductive proofs were produced to solve them, the most frequent types of proofs being the most abstract and formal ones; Anyway, the help of the Cabri figures to visualize conjectures and to suggest the students ways to prove them was decisive in their success. The answers to these activities showed also a clear progress of the students in one of the school towards deductive proofs based on general geometric and algebraic properties induced by the visual proofs suggested by the figures in Cabri.

Finally, without purpose of generalization, we can conclude from this research that trigonometry is a rich field whose teaching can be organized around discovering activities as a way to promote learning with understanding and the development of abilities of proving.

REFERENCES

Balacheff, N.: 1988, 'Aspects of proof in pupils' practice of school mathematics', in D. Pimm (ed.), *Mathematics, teachers and children*, Hodder & Stoughtonpp, London, pp. 216-235.

Bell, A.W.: 1976, 'A study of pupil's proof-explanations in mathematical situations', *Educational Studies in Mathematics* 7(1), 23-40.

Harel, G. and Sowder, L.: 1998, ‘Students' proof schemes: Results from exploratory studies’, in A.H. Schoenfeld, J. Kaput and E. Dubinsky (eds.), *Research in collegiate mathematics education*, American Mathematical Society, Providence, USA, Vol. III, pp. 234-283.

Mariotti, M.A.: 2006, ‘Proof and Proving in Mathematics Education’, in A. Gutiérrez and P. Boero (eds.), *Handbook of research on the psychology of mathematics education*, Sense Publishers, Rotterdam, The Netherlands, pp. 173-204.

Marrades, R. and Gutiérrez, A.: 2000, ‘Proofs produced by secondary school students learning geometry in a dynamic computer environment’, *Educational Studies in Mathematics* 44(1/2), 87-125.