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Similarity, Homothety and Thales theorem together for an effective teaching

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Abstract: In this paper we show partial results from a research aimed to analyse the ways of reasoning and evolution of some students when they carry out homework for a teaching unit related to similarity, homothety and Thales theorem. The analysis is based on Lemonidis (1991) and the Van Hiele model of reasoning (Gutiérrez and Jaime, 1998). The sample for this study was a group of 9th grade students (14-15 years old) in a school from Floridablanca (Santander, Colombia). A preliminary analysis from the data collected shows interesting ways of solving certain tasks in which the participants used a rich language and showed a variety of ways of reasoning.

Résumé: Dans cet article, nous montrons des résultats partiels d'une recherche visant à analyser les modes de raisonnement et de l'évolution de certains étudiants quand ils effectuent des devoirs pour une unité d'enseignement liés à la similitude, homothétie et théorème de Thalès. L'analyse est basée sur Lemonidis (1991) et le modèle de raisonnement de Van Hiele (Gutiérrez et Jaime, 1998). L'échantillon de cette étude était un groupe d'élèves de 9e année (14-15 ans) dans une école de Floridablanca (Santander, la Colombie). Une analyse préliminaire des données recueilliés montre façons intéressantes de résoudre certaines tâches dans lesquelles les participants ont utilisé un langage riche et ont montré une variété de modes de raisonnement.

Introduction

Several studies conclude that teaching of similarity is very precarious and, thus, it generates weak learning in students, which does not contribute to the effective development of students' geometric thinking (Gualdrón, 2006).
We present here some results of a teaching experiment based on an original teaching unit that integrates similarity, homothety, and the Thales Theorem. The teaching unit was designed to improve the ways of geometric reasoning of a group of students who participated in the study. The main difference between this instructional proposal and others is that the activities we have designed present in a connected way the concepts and properties of similarity, homothety, and the Thales Theorem. This helps student to better understand the topic and to learn tools useful to solve problems and to improve their geometrical reasoning.

Theoretical Framework

To design a teaching unit for similarity, homothety and the Thales theorem, and then to analyse students' performance in the unit, we have taken into account the analysis of teaching homothety by Lemonidis (1991) and the Van Hiele model of reasoning (Gutiérrez and Jaime, 1998; Gualdrón, 2007).
Lemonidis (1991) characterized three different approaches to similarity to be considered when teaching this topic:

a) *Intrafigural relationship*: when the correspondence between elements of a figure and elements of a similar figure is highlighted, but without considering the idea of transforming a figure in the other one. Within this approximation, we may distinguish:
   - When the figures are part of a Thales configuration, in which the homothety components are considered, with adequate reasons.

  - When the figures appear apart one from the other.

b) *Geometrical transformation seen as a tool*: the geometrical transformation is perceived as an application from the set of points in the plane into itself. This approach to similarity is useful to solve problems, for instance in trigonometry and calculus.

c) *Geometrical transformation seen as a mathematical object*: the geometrical transformation is characterized by an algebraic approach in which the objective is to find the transformation resulting from the composition (product) of two or more transformations.

The Van Hiele model of geometric reasoning is proving to be an excellent theoretical reference to organize and assess teaching and learning of geometry. However, similarity is a geometry topic in which research about the application of the Van Hiele model is very limited (Gualdrón, 2006). In this study we used the descriptors of level identified by Gualdrón (2006) and we have extended them to the specific contents of the teaching unit.
The activities in the teaching unit are mainly focused to students in Van Hiele level 2, although some students could be reasoning at level 1 or at level 3. Then, we include below the main descriptors of levels 1 to 3 for the topic of similarity, based on Gualdrón (2006):

- Descriptors of *Van Hiele level 1* for similarity:
  - Students recognize similar shapes based on their appearance that is based only on visual characteristics.
  - Students describe and compare shapes with terms like bigger, smaller, longer, etc.
  - Some students can use mathematical characteristics of similarity, but they do not do it in a consistent way. For instance, they may measure some corresponding angles and note that they are congruent.
  - Students identify the similarity of shapes in Thales configurations, but their arguments are visual.
  - Students can build or draw shapes being similar to a give one, but they do it visually, without taking into consideration mathematical properties like measure of angles or length of sides.
- Descriptors of *Van Hiele level 2* for similarity:
  - Students recognize similar shapes based on mathematical characteristics like measure of angles or length of sides.
  - Students may identify and generalize properties of similar shapes, like proportionality of corresponding sides, parallelism of sides when they are in a Thales configuration, etc.
  - Students assume that relative positions of similar shapes are irrelevant. They also assume that congruence is a particular case of similarity.
  - Students can build or draw shapes being similar to a give one taking into consideration mathematical properties like measure of angles, length of sides, or ratio, and making arithmetic calculations.
  - Students can use Thales configurations to prove that two shapes are similar.
  - Students may discover or induce some properties of similarity, for instance the cases of similarity of triangles.
  - Students can use the definition of similar shapes.
- Descriptors of *Van Hiele level 3* for similarity:
  - Students identify empirically the necessary and sufficient conditions for two triangles to be similar, and prove their truth deductively.
  - Students understand and use relationships among properties of similarity. For instance, they relate the characteristics of homothety to the different Thales configurations to prove that such configurations are cases of homotheties.

- Students identify necessary and sufficient properties of specific similar shapes. For instance, they acknowledge that having equal pairs of angles is necessary and sufficient for two triangles to be similar, but it is not sufficient for other polygons.
- Students can write informal deductive proofs of properties involving similarity of plane shapes.

Methodology

The sample for this study was a group of 27 grade 10 students (14-15 years old) in a secondary school in Floridablanca (Santander, Colombia). These students had studied in different grades the Thales theorem and homothety, but they had not studied the general concept of similarity before. Then, we planned the teaching unit to integrate the contents of similarity, homothety and Thales theorem, aiming to create on students a network of knowledge.

The teaching unit was designed taking into account the phases and levels of the Van Hiele model (Gutiérrez and Jaime, 1998) and the results of Lemonidis (1991). The activities were posed to students one after the other in the order determined by the consideration of Van Hiele phases. Each activity was presented to students in a sheet of paper, where students should justify the steps to the answer (numerically, graphically or verbally). Students could use ruler, square, compass, and other auxiliary drawing elements.

The teaching experiment took place in the ordinary classroom during the scheduled mathematics classes; the physical characteristics of the classroom allowed working in groups of three students. The teacher and a researcher (the first author) participated in all the experimental teaching sessions. The teacher was the responsible for the organization of the classes. The researcher was a participant observer, observing the students activity and, at the same time, cooperating with the teacher in tutoring and orientation to students during the classes.

The experiment was developed in nine sessions of 100 minutes, during 8 weeks. The students were organized in groups of three students, who discussed the possible solutions and then, each student wrote her own answer and conclusions.

The data collected consists on students' worksheets, video tapes of the classes showing the work made by the groups of students, interviews to some students, and researcher's field notes. The videotapes were done from. Some interviews were conducted in order to ask for clarifications and extra justifications, to ask students to explain what they have done, or to pose them other analogue tasks.

Analysis and Discussion

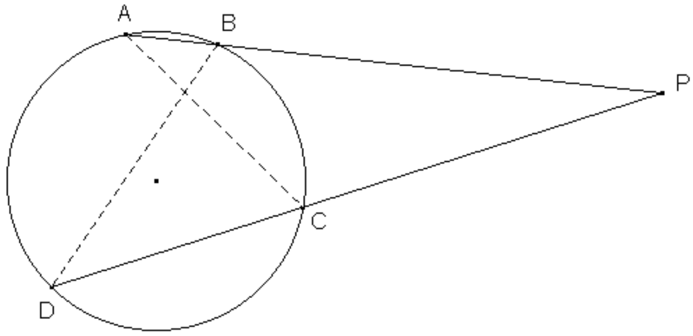
We present here the most relevant aspects of the data we have collected, with some representative examples of students' ways of reasoning and comments about them. We have focused on one activity (activity 24), shown in Figure 1. Each part of this activity asked students to prove that two triangles are similar.

The first part of the activity shows a intrafigural relationship among triangles APC and DPB, since the statement focus students' attention to corresponding sides of the triangles.

The second part of the activity presents the geometrical transformation of homothety as a tool to solve the problem, since identifying an homothety relating both triangles is the key to solve it.

Activity 24.

(1) Segments AP and DP belong to lines that intersect in point P and, at the same time, cut the circumference as shown in the diagram below. Justify that triangles APC and DPB are similar.



(2) Draw any triangle and, for each vertex, draw a line containing the vertex and parallel to the opposite side. In this way, you get a bigger triangle. Justify that this triangle is similar to the first one.

Figure 1. Activity 24.

One of our students, named Carlos (pseudonymous), used the homothety to prove the similarity of triangles in part 1 of the activity 24. In his answer, he first justified that there is a homothety, and then he used this homothety to justify the similarity of the triangles:

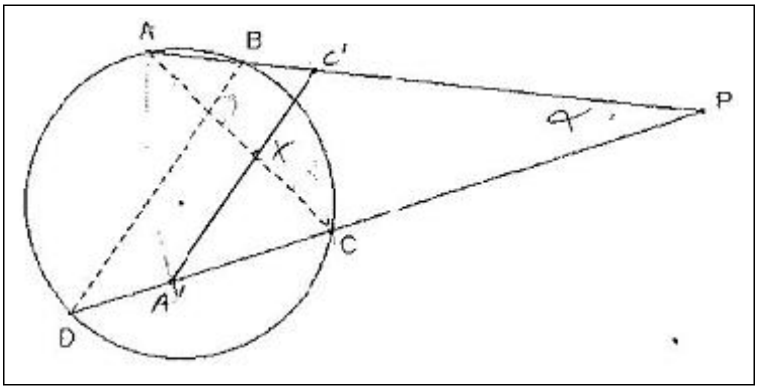


Figure 2. Carlos' answer to activity 24(1).

Carlos: *Translating* [copying] *distance PC over line PA and distance PA over line PD, we have triangle PA'C', which is similar to triangle DPB, since this process is just like tumbling the figure* [PAC]. *With this, it is possible to establish a homothety with centre P, since the correspondent vertices and P are collinear, PC'||PA and PA'||PD. And to prove that A'C'||BD, I used that, as <BAC and <BDC comprise the same arc, they are congruent, and as I have line PD, with two different lines cutting it forming the same angle, so the two lines are parallel and as these lines are BD and C'A', we have proved the last condition necessary to have a homothety.*

The previous answer shows a consistent use of Van Hiele level 3 reasoning, since the student did not use any specific measurement, but organized different properties of triangles and of the angles among parallel lines cut by another line to deductively prove that triangles APC and DPB are similar. The style of the proof is far from the formal proofs typical of level 4.

In his answer to activity 24(2), Carlos also used the homothety to justify that triangles ABC and A'B'C' are similar:

Carlos: *As the sides of the big triangle* [A'B'C'] *are parallel to the sides of the small one* [ABC] *(the correspondents), and knowing that, if the correspondent sides are parallel then they can be aligned to make an homothety and confirm the similarity, then we have a complete proof* *of the similarity* [of A'B'C' and ABC].

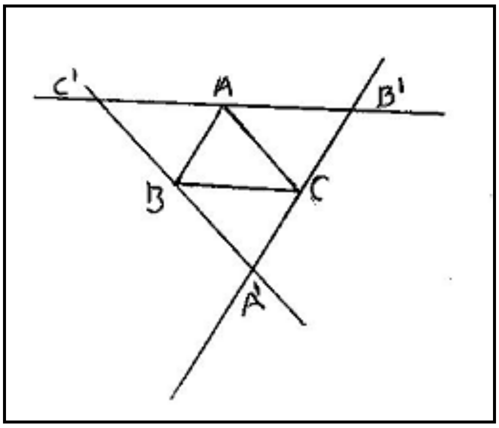


Figure 3. Carlos' answer to activity 24(2).

Carlos' answer to the second part of the activity is consistent with his previous answer, since he also produced a deductive proof. However, Carlos still has not fully acquired the ability of proof typical of Van Hiele level 3 since, in this answer, he did not feel the need to make explicit mention of Thales theorem nor to provide details to identify the focus of the homothety. One of the objectives of the teaching unit was to provide students with opportunities to practice and improve their proving abilities.

The results of the analysis seem to indicate that this way of teaching similarity (linking it to homothety and Thales theorem) was a factor highly positive for the acquisition by students of more and better abilities of reasoning.

Conclusions

Traditionally, the teaching of similarity is limited to the presentation of conditions for two geometrical figures to be similar, and then to the presentation of some graphic examples. Diverse studies (for example, Gualdrón, 2006) have shown that this way of teaching the subject limits extremely the geometric reasoning of students.

In this study we have analysed the arguments that a group of students stated to justify similarity among polygons, and we have found that students had more and better reasoning tools respect to results presented by previous studies (Gualdrón, 2006) where direct connections between similarity, homothety and the Thales Theorem were not established.

The results of this study contribute to the literature about effective ways of improving geometric reasoning, specifically in tasks related to similarity, by showing successful ways of connection among geometric subjects that, in most of cases, are taught in an isolated way.

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