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Geometry 13

Angel Gutiérrez

Core readings

The Core readings addressed in this chapter are:

Gutiérrez, A., 1996. Visualization in 3-dimensional geometry: in search of a framework. In L. Puig and A. Gutiérrez, eds, *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1. Valencia, Spain: PME, 3–19. Available at www.uv.es/angel.gutierrez/archivos1/textospdf/Gut96c.pdf [accessed on 18 May 2013].

Gutiérrez, A. and Jaime, A., 1998. On the assessment of the van Hiele levels of reasoning. *Focus on Learning Problems in Mathematics*, 20 (2/3), 27–46. Available at www.uv.es/angel.gutierrez/archivos1/textospdf/GutJai98.pdf [accessed on 16 May 2013].

Presmeg, N. C., 1986. Visualization in high school mathematics. *For the Learning of Mathematics*, 6 (3), 42–6.

Vinner, S., 1991. The role of definitions in the teaching and learning of mathematics. In D. Tall, ed., *Advanced mathematical thinking*. Dordrecht, Netherlands: Kluwer, 65–81.

Introduction

Traditionally, geometry has been a 'poor relation' in school mathematics curricula, and textbooks and teachers have tended to reduce the content taught to some basic definitions, properties and formulae. In recent years, research and teaching experience has shown that some difficulties encountered by students when learning other areas of mathematics could be overcome if students had deeper knowledge of geometry and geometric reasoning. As a consequence, teachers are becoming aware of the importance of geometry in school curricula, and researchers are working on providing teachers with knowledge and tools that could help them improve their practice. This makes research in geometry education an important field of research with many interesting directions. Gutiérrez and Boero (2006) and Battista (2007) provide overviews of the current state-of-the-art, and propose questions for further research.

In such a context, this chapter introduces readers to some essential elements of research in geometry education, and prepares the ground for them to undertake research in this field. The chapter is divided into three sections, devoted to three theoretical frameworks which are relevant for research on different aspects of geometry teaching and learning at any educational level.

Teachers should be aware of the visual abilities and skills that students use when drawing or seeing pictures, drawings or diagrams; this is particularly critical when learning three-dimensional geometry. Therefore, the first section of the chapter introduces readers to the main characteristics of a useful theoretical framework that identifies and analyses the elements of visualization used by students when solving geometric problems.

The second section describes the constructs defined by Shlomo Vinner to explain the processes of learning mathematical concepts in geometrically rich contexts. Vinner's model describes students' conceptions of mathematical concepts, and helps teachers and researchers to better understand students' learning processes, outcomes and errors, as well as to design effective teaching materials.

The third section presents the van Hiele model of geometrical reasoning. This influential model has proved to be very useful for describing and analysing students' mathematical reasoning when they are studying geometric content, and it is widely used in mathematics education. In fact, it is the framework most often used to organize the teaching of geometry, from national curricula (for instance, the NCTM Principles and Standards in the United States, and the Singapore National Curriculum) to the design of classroom activities. The van Hiele model is also very useful for providing teachers and researchers with accurate data for the assessment of students' geometrical reasoning.

Spatial visualization in geometry

Visual thinking is necessary in any area of mathematics, at all levels, and especially in geometric contexts, and it is very important for students to develop their visualization skills. Therefore, a relevant research question involves characterizing students' mental visual activity at different school grades. The development of dynamic geometry environments, and other software able to graphically represent mathematical concepts taught at any school level, has opened a new research field (see the chapter by Ruthven, this volume).

Visual thinking is also useful in many other disciplines (e.g. in medicine, in order to use Computed Axial Tomography and other three-dimensional-image techniques; in geography, for map reading; in chemistry, for modelling complex molecules; in architecture and engineering; etc.) and in everyday activities (to anticipate trajectories when driving;

to estimate object sizes, etc.). This wide range of applications has resulted in a lack of coordination or agreement among researchers from different specialities, so it is easy to find discrepancies in the use of terms or their meanings in the literature (see the core reading Gutiérrez 1996 for a deeper discussion on this issue). As an illustration, cognitive psychologists tend to define ‘mental image’ as a quasi-picture created in the mind from memory, whereas mathematics educators give a wider meaning to this same term, as we will see later.

Several approaches for analysing visualization in school mathematics can be found in the mathematics education literature. I shall present here an approach, from the core reading by Presmeg. This approach was first proposed in the 1980s, and is still useful to researchers and teachers. In any geometric activity, we can differentiate between external actions and internal, mental actions. External actions obtain information from outside, produce outcomes and communicate with others. By contrast, mental actions consist of processing external information to transform it into internal data, analysing internal data to generate new internal information, and converting internal information and results into external outcomes.

When the mental actions are based on the visualization of geometric objects, researchers have differentiated three components integrating those actions:

1. The *mental objects* handled. Presmeg (1986) called these objects *mental images*. She observed teachers and students, and identified several types of mental images: concrete pictorial images, pattern images, memory images of formulae, kinaesthetic images and dynamic images.
2. The *mental processes* that transform external or mental information into mental images, and vice versa. The process of creation of mental images occurs when students look at pictures in the textbook, on the blackboard, computer screen, and so on, when they read or hear a text and represent the information graphically in their mind, and also when students transform other mental images. Bishop (1983) called this process *visual processing* (VP) of information.

   After having created mental images, students may analyse them, retrieving information needed to solve the problem they are working on, and exteriorize the information by using appropriate language and graphical representations. Bishop (1983) called this process *interpretation of figural information* (IFI).
3. Students need to have learned some *visualization abilities* in order to perform the above-mentioned mental visual processes while solving a problem, in the same way as a person should have learned some manual abilities to use a screwdriver or a hammer in order to join two pieces of wood. Del Grande (1990) compiled a list of mental visualization abilities necessary to solve geometric problems. Abilities like figure-ground perception, perceptual constancy, mental rotation, perception of spatial positions or spatial relationships, visual discrimination, visual memory, and others are necessary to solve geometry problems, especially in three-dimensional geometry. See Del Grande (1990) for details.

From the point of view of research and teaching, the most important of the components mentioned above are the mental images. Presmeg (1986) introduced the different types of mental images she identified in her research experiments and included a description of each image, with characteristic examples of students using them to solve mathematics problems. The scope of Presmeg's article extends beyond geometry within school mathematics, and certain types of images are more often used in some areas of mathematics than in others.

In each of the core readings summarized in the previous paragraphs, the authors introduce one element that is relevant to understanding how individuals use visualization, but an integration of the three components was necessary. Gutiérrez (1996) presented the integration of visualization processes, images and abilities into a theoretical framework, with examples of students' outcomes to complete the description of the framework and facilitate its understanding. The diagram (Figure 13.1), adapted from Gutiérrez (1996), summarizes the elements of mathematical visualization used to solve a problem of geometry.

We can exemplify and apply this diagram practically. What comes to your mind after reading the word 'pyramid'? Most probably this external input has prompted you to create the mental image of a square-based pyramid lying on its (horizontal) base (IFI to a concrete pictorial image). Now, make your mental pyramid rotate to lie on one of its triangular faces. This is a different mental image, generated from the previous one (IFI + VP to a dynamic image). Finally, draw the pyramid you have in your mind (VP to an external representation). Different readers may have used different visualization abilities for the same process in this task, but most likely perceptual constancy, mental rotation, perception of spatial positions or visual memory have been used by some readers.

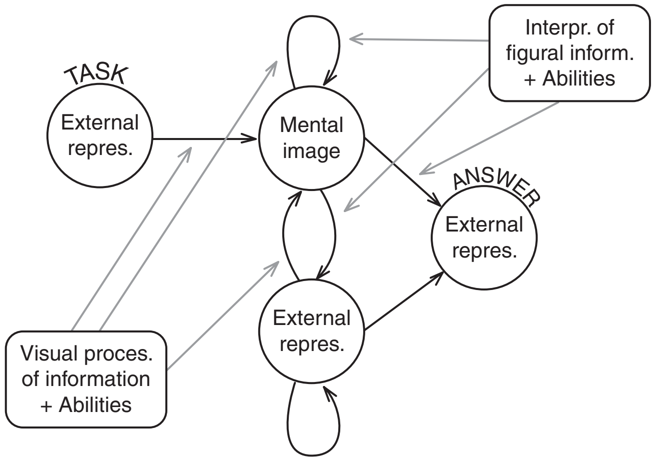

Figure 13.1 Elements of mathematical visualization used to solve a problem of geometry

The learning of basic geometric concepts

The experience of both researchers and teachers shows that most students feel more confident when they learn new mathematical concepts represented in specific examples than when students are just presented with a definition, often in their textbook (Watson and Mason 2005). An interesting research activity is to analyse students' learning and understanding of mathematical concepts when they are taught using examples and non-examples.

A student is asked for the definition of a regular polygon. He replies: *A regular polygon is a polygon having all its sides equal and all its angles equal* – that is, the standard definition. Now the student is given some geometric figures and he is asked to select the regular polygons. One of the figures that he selects is a rectangle. When the researcher asks the student why this rectangle is a regular polygon, he replies: *Because it has four equal angles*.

Another student is asked for the definition of a square. She replies*: A square is a quadrilateral having four equal sides and four right angles* – that is, again, the standard definition. Now the student is given some drawings of quadrilaterals and she is asked to mark the squares. One of the polygons she does not mark is represented in Figure 13.2. When the researcher asks the student why this quadrilateral is not a square, she replies: *Because it is not in the correct position*.

Both answers, which many teachers will recognize, appear similar because both students can repeat the definition: however, they differ in the following respect. The first student understands the meaning of having all the sides equal, and he is able to discriminate between polygons having, or not having, this property; he can also correctly manage the property of having all the angles equal. But he does not understand that both properties have to be true at the same time for a polygon to be regular.

The second student can identify squares only when they are in the standard position, that is, resting on a horizontal base. She does not understand the definition of square and, in fact, she does not use it to classify the quadrilaterals but uses the prototypical *image* of square that she has seen in textbooks and the blackboard as the target against which to match the drawings in the exercise.

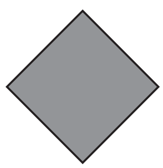

Figure 13.2 Is it a square?

A powerful framework that explains these two students' behaviours was proposed by Vinner (1991), as follows. The information that students receive in their mathematics classes and outside school is of two types:

- Graphical: this includes pictures, drawings, physical objects, and so on that students see in textbooks, blackboards, and elsewhere. It works like a collection of photos.
- Verbal: this includes definitions, theorems, formulas, and so on that students read in textbooks or hear from teachers or other persons. It works like a collection of newspaper cut-outs.

Neuropsychologists tell us that the human brain stores verbal and graphical information in different places. Vinner represented those places in the memory as two boxes: the graphical box is called *concept image* and the verbal box is called *concept definition* (Figure 13.3).

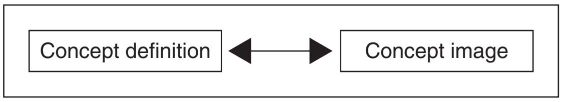

Figure 13.3 Concept image and concept definition as two connected boxes

Teachers should aim to enable students to *connect* the two boxes.

Typically, when students are introduced to a geometric concept, they populate their concept definition and concept image 'boxes' with the contents learned, but students are not always taught how to establish relationships between them. As a consequence, when students feel that the question formulated by the teacher asks for a definition, property, formula, and so on, they access their concept definition and, when students feel that the question asks them to identify or reproduce a shape they resort to their concept image.

In the example of identification of squares, the student did not establish a relationship between her definition and her concept image of a square. Furthermore, her concept image was limited to prototypical images – squares with a horizontal base – so she decided to reject the 'diamond' shape because it was not similar to any other held in her concept image of a square.

Vinner's model of acquisition of mathematical concepts offers a resolution to such learning difficulties. Vinner (1991) is an extended compilation of previous publications by this researcher and other colleagues in which he presents the different components of the model in detail and provides many examples. The chapter by Vinner also discusses different patterns of students' behaviours depending on their concept definitions, concept images and the relationships between them that students are able to manage.

Vinner suggests starting teaching a new concept with a carefully organized set of examples and non-examples, to help students learn the concept in the same way as everyday concepts are learned, that is, by comparing examples and non-examples, and identifying discriminating properties as follows:

- the comparison of examples and non-examples should highlight the properties of the examples which are not present in the non-examples, that is, the necessary properties of the concept;
- comparison of two different examples should indicate a property of one example which is not present in the other example, that is, a non-necessary property of the concept;
- the necessary properties identified should enable students to formulate a definition of the concept, and to generate links between this definition and their concept images.

Figure 13.4 presents the way a Spanish textbook introduces the concept of polyhedron in Grade 2 (student age 13–14) of the secondary school (Colera et al. 1997). This is a quite simple but effective illustration of the use of examples and non-examples in typical geometry teaching. An application of Vinner's model is the following procedure of designing a sequence of examples and non-examples to teach a certain geometric concept, based on the following steps (Hershkowitz 1990):

- Decide on the definition of the concept to be taught.
- Select the necessary properties of the concept that students should discover.
- Select the non-necessary properties that students often select erroneously in the identification of a shape as an example or a non-example. Non-necessary attributes such as shape, position, number of sides or faces, and so on are often accepted as necessary properties by students.
- For each necessary property, draw an example and a non-example differing in this property.
- For each non-necessary property, draw two examples differing in this property.

Figure 13.5 shows the result of applying these steps to the concept of a right prism.

Vinner's model may also be used as a research framework to evaluate teachers' and students' understanding. Hershkowitz et al. (1987) and Gutiérrez and Jaime (1999) present findings of research based on this model. Watson and Mason (2005) elaborated on Vinner's

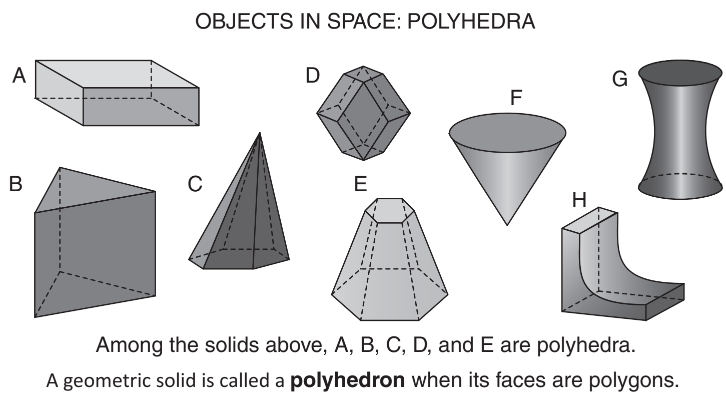

Figure 13.4 Introducing the concept of polyhedron (Colera et al. 1997)

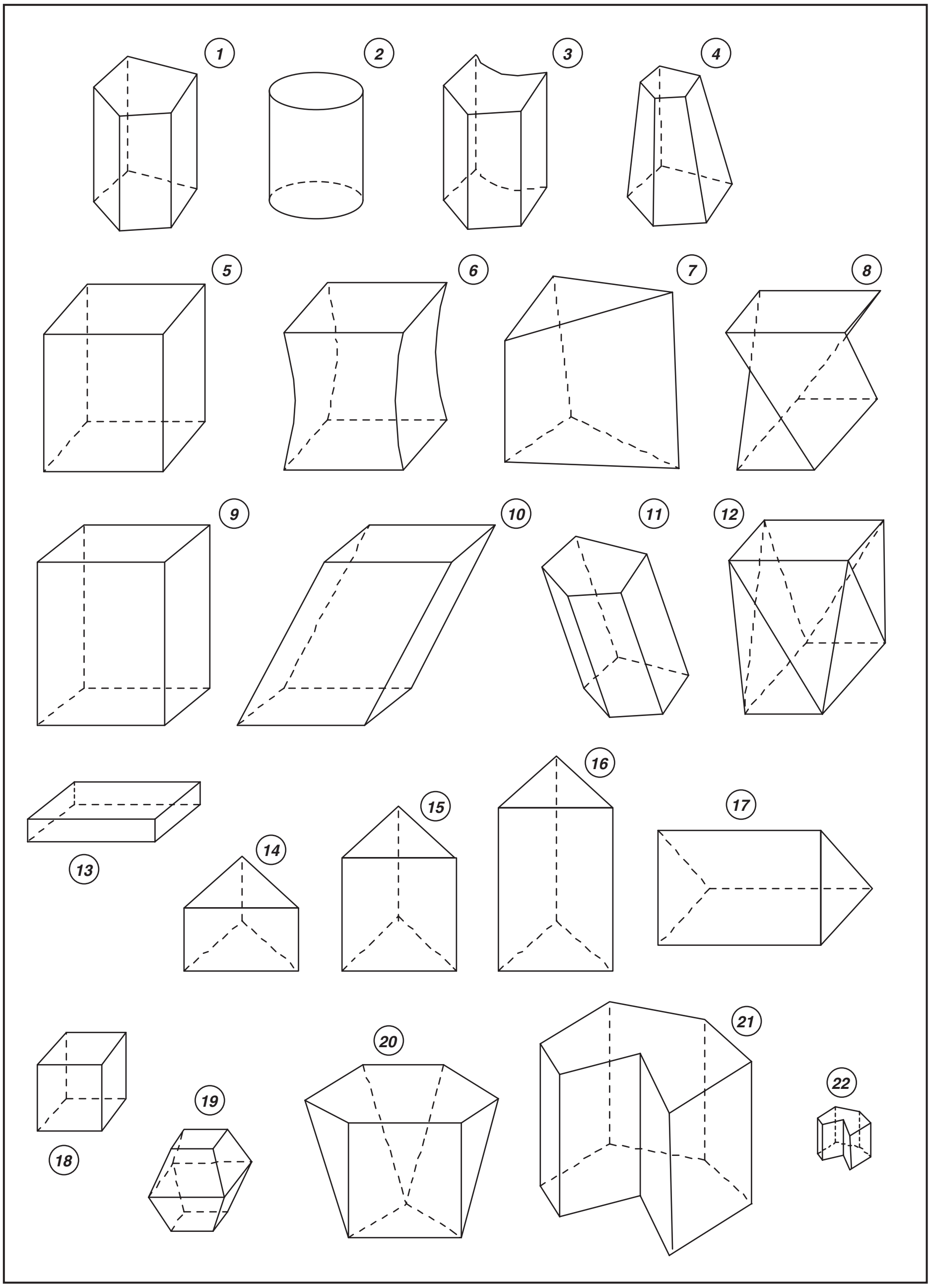

Figure 13.5 Introducing the concept 'right prism' following Hershkowitz (1990)

concept image to define students' *example spaces* as 'small pool[s] of ideas that simply appear in response to particular tasks in particular situations' from which examples produced by students arise. Vinner's model is still widely used by other researchers, for example, Gilboa et al. (2011), and several articles in the *Educational Studies in Mathematics* special issue 69 (2) (2008).

The van Hiele model of students' geometrical reasoning

People studying mathematics, from kindergarten children to professional mathematicians, exhibit different modes of mathematical reasoning. The *van Hiele model of geometrical reasoning* characterizes those modes (or *levels*) of reasoning in geometrical contexts. Furthermore, the van Hiele model provides guidelines for teachers to plan their lessons and to help their pupils develop their reasoning. The guidelines to teachers are known as *the phases of learning*. Due to space limitation, I will focus on the use of the levels as a framework to evaluate students' geometrical thinking. A more comprehensive study of the van Hiele model from a research perspective should begin with Clements and Battista (1992) and Battista (2007).

The main characteristics of the van Hiele levels follow. The core reading Gutiérrez and Jaime (1998) gives a more detailed account.

Level 1: Students recognize geometric concepts by their physical appearance, and in a global way, without explicitly distinguishing their mathematical components or properties.

Level 2: Students recognize the mathematical components and properties of geometric concepts. They are able to verify conjectures through empirical reasoning and generalization. Students can only manage basic logical relationships between mathematical properties.

Level 3: Students can manage any logical relationship. They are able to prove conjectures using informal deductive reasoning. Students can understand simple formal proofs, but they cannot construct them themselves.

Level 4: Students understand the need for rigorous reasoning and they can write formal deductive proofs. They understand the function of axioms, hypotheses, definitions, and so on.

Level 5: Students are able to manage different axiomatic systems, and they can analyse and compare properties of geometric objects in two axiomatic systems (for instance, Euclidean geometry and spherical geometry).

The first researchers using the van Hiele model considered – in keeping with the van Hieles' early writings – that a given student always performed at *the same* level; consequently, the assessment procedures tried to elucidate *which* level of reasoning students had attained (Burger and Shaughnessy 1986; Fuys et al. 1988; Usiskin 1982). However, these authors were unable to assign a level of reasoning to a significant number of subjects in their experiments due to contradictory results because some students 'failed' the questions focusing on one level and 'succeeded' in the questions focusing on a higher level, and other students gave answers that showed a mixture of levels of reasoning.

For instance, a Spanish student in Grade 3 of the secondary school (student age 14–15) was given a sheet with drawings of several quadrilaterals, and was asked to mark squares (C), rectangles (E) and rhombuses (O); after having marked the shapes (Figure 13.6), the student was asked to explain what he had paid attention to when making his classification of the quadrilaterals. Some of his answers were:

> For squares: [I paid attention] *to its equal and parallel sides, and the angle of 90°* (a level 2 answer).
>
> For rectangles: *its long shape with 4 parallel sides making 4 angles of 90°* (an answer mainly at level 2 but with aspects of level 1).
>
> For rhombuses: *its four sides, 2 slanted parallel sides and 2 right parallel sides* (a level 1 answer; the intended meaning of the terms 'slanted' and 'right' is unclear).
>
> Is shape 1 a square? *Yes. Because of its 4 parallel sides, and its width and short shape* (answer mainly at level 1, but partly at level 2).
>
> Is shape 5 a rectangle? *No, because its sides do not form 90° angles* (a level 2 answer).

The van Hiele levels have some core characteristics that ought to be taken into account when using them to assess students' geometrical reasoning, or to design teaching materials, as follows:

- The levels are *sequential*: levels are ordered, so that progression from one level is always to the next level.
- The levels are *local*: showing a level of reasoning in a certain topic of geometry does not necessarily imply showing the same level in a different topic. Geometrical reasoning is highly dependent on knowledge of mathematical content, so students and teachers may be reasoning at a high level in one geometrical topic but at a low level in another geometrical topic that they are just beginning to study. Several researchers have administered similar questionnaires based on different geometric topics to sample groups of students or teachers, and all of them have reported that the levels of reasoning in most participants depended on the topics (Clements and Battista 1992, p. 429).

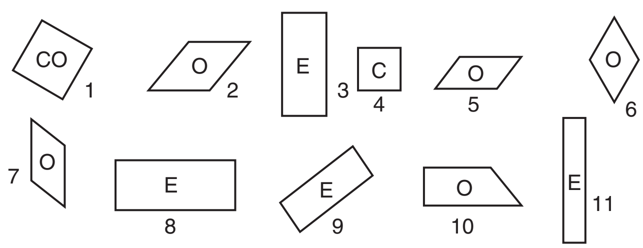

Figure 13.6 A student's classification of some quadrilateral drawings

For instance, Gutiérrez and Jaime (1988, cited in Clements and Battista 1992) administered two tests to a group of 232 pre-service primary school teachers, one test based on plane shapes and the other on solids. They found that only 10 per cent of the students showed the same level of reasoning in both tests; 80 per cent of the students showed levels 2 or 3 in plane geometry, and 9 per cent of the participants showed levels 2 or 3 in solid geometry.

- Each level has a characteristic *language*: students at different levels may give different meanings to the same term, for instance to 'proof'. For example, suppose that a secondary school teacher asks his pupils to deduce the formula for the sum of the interior angles of an *n*-sided polygon. Students calculate the sums of several polygons (triangle, quadrilateral, pentagon, . . .) and they induce the formula *Sum of angles* = *180(n–2).* To prove that their formula is correct, the students show that it works for some polygons (n = 3, 4, 5, . . .). The teacher rejects the students' argument, and asks them for a 'general' proof, but the students do not understand why a deductive proof is necessary when (they reason) the examples clearly prove that the formula works. The reason for this didactical obstacle is that students are reasoning at level 2, so 'to prove' the formula means, for them, to check it in specific cases, while the teacher expects a level 3 proof, so for him 'to prove' the formula means to make a deductive abstract argument. As Pierre van Hiele (1959/1984, p. 246) wrote, 'two people who reason at two different levels cannot understand each other.'

Clements and Battista (1992) give a more detailed description of the core characteristics of the van Hiele levels, and present an analytic review of the relevant research literature.

Gutiérrez and Jaime (1998) and Gutiérrez et al. (1991) presented a comprehensive methodology for the assessment of individuals' levels of geometrical reasoning, based on an original approach to the structure of the levels. Their approach has proved to be useful for researchers (Battista 2007, p. 848) and overcomes earlier difficulties in applying the van Hiele theory. Gutiérrez and Jaime (1998) described and exemplified a technique for the design of questionnaires for the assessment of van Hiele levels, using multilevel questions and multiprocess super-items (these ideas are explained later): this approach optimizes the questionnaire efficiency, in terms of number of items and administration time. The work by Gutiérrez et al. (1991) presented a new conception for the assessment of the levels of reasoning based on the *continuity* of the levels and the possibility to measure the transition between levels.

According to Gutiérrez and Jaime (1998, p. 29), mathematics is a complex activity, integrated by five mathematical processes – recognition and description, use of definitions, formulation of definitions, classification and proof (see the chapter by Stylianides, this volume). Thus mathematical reasoning is a multiple activity that, in practice, depends on the mathematical processes required to solve a problem (see the chapter by Verschaffel et al., this volume). The acquisition of a van Hiele level implies the mastery of all the five processes of reasoning associated with the mathematical processes, so that the determination of a person's level of geometrical reasoning has to take into consideration the acquisition of each process of reasoning. Table 13.1, adapted from Gutiérrez and Jaime (1998, p. 32), summarizes the characteristics of each process of reasoning as practised in each van Hiele level.

Table 13.1 Distinctive attributes of the processes of reasoning in the van Hiele levels

	Level 1	Level 2	Level 3	Level 4
Recognition and description	Physical attributes	Mathematical properties	–	–
Use of definitions	–	Only definitions with simple structure	Any definition	Accept several equivalent definitions
Formulation of definitions	List of physical properties	List of mathematical properties	Set of necessary and sufficient properties	Can prove the equivalence of definitions
Classification	Exclusive, based on physical attributes	Either exclusive or inclusive, based on mathematical attributes	Move among inclusive and exclusive when definitions are changed	–
Proof	–	Empirical verification in examples	Informal deductive proofs	Formal mathematical proofs

Most primary and secondary school mathematics curricula pay more attention to some mathematical processes than to others; for instance, in the 1970s and 1980s, the curricula in those countries adopting the 'new mathematics' approach (for instance, United States and Spain, among many others) restricted experimental tasks (levels 1–2) in favour of extensive use of deductive proofs of properties (levels 3–4), while in the 1990s, deductive proofs were removed from their curricula and more emphasis was placed on the empirical verification of properties (level 2). Consequently, it should not be surprising to find that students are mastering some processes of reasoning at a certain level while they are still using other processes of reasoning at a lower level.

When designing a reliable test to evaluate students' levels of reasoning, it is necessary to ensure that each van Hiele level and each process of reasoning are evaluated – that is, each cell in Table 13.1 is evaluated. However, the use of multilevel questions and multiprocess super-items can help researchers to design short but still reliable tests, as follows. On the one hand, students' reasoning is not indicated by the fact of correctly having solved some problems, but by the way they have solved them. Different students may solve the same problem using different levels of reasoning; for instance, a proof problem may be solved by empirically checking the conjecture in one example, or a few examples (level 2), or by formulating a deductive argument in an informal (level 3) or formal (level 4) way. Similarly, a description of a geometric object may be physical (level 1) or mathematical (level 2). In the same way, the solution of a *multilevel* problem item may require the use of several mathematical processes. So, it is not necessary to include a different problem in the test in order to assess each level and process. On the other hand, a set of related questions and/or problems is more efficient than a set of independent questions and/or problems since related questions make it easier to graduate different difficulties or complexities. Then, as suggested by Gutiérrez and Jaime (1998), we can use *super-items* – sets of related questions having a common core – to discriminate the use of different van Hiele levels depending on the questions in the super-item answered by each student. In the core reading, Gutiérrez and Jaime (1998) give examples of eight super-items. Finally, since the levels of reasoning expected vary at different grades, it is more efficient to design different tests for students in different school grades.

These techniques allow the development of tests that reliably assess students' degrees of acquisition of the van Hiele levels and can be administered in a reasonable amount of time, typically 1 hour in class. Gutiérrez and Jaime (1998) presented a longitudinal study where a set of eight super-items was used to design three related tests to evaluate a sample of Spanish students from Grade 6 (student age 11–12, primary school) to Grade 12 (student age 17–18, end of secondary school).

Conclusion

In this chapter, I have introduced three theoretical frameworks that form part of the essential underpinning for research in geometry education. This chapter has set out the main features of each framework, as a starting point for more detailed reading of the core readings and the references indicated below.

It is worth mentioning that the three theories introduced here are compatible and complementary to each other. The information about a student's behaviour (with regard to reasoning, learning or representation) gained from one of the theoretical frameworks can be expected to warrant and reinforce information about their behaviour with respect to the other frameworks. For instance, knowing the kind of visual images and abilities students use can give clues about the type of concept image they are able to create, and their level of reasoning.

The frameworks introduced here are among the most important elements of research in geometry education, but they are not the only ones. Other important topics include research on the teaching and learning of proof, on the use of dynamic geometry software in primary and secondary school classrooms, on the view of geometry classrooms as communities of practice, and research on the teaching and learning of specific geometry topics. These research areas are addressed in other chapters of this book, and also in edited handbooks such as Gutiérrez and Boero (2006) and Lester (2007).

Further reading

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