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Visualization in school mathematics analyzed from two points of view

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In this book, terms like spatial reasoning, spatial thinking, or spatial visualization are used by the authors to refer, maybe with some subtle differences, to the use of elements, abilities or skills, vocabulary, and gestures having to do with characteristics of mathematical concepts which are perceived through the sight. In this text, I will use the term *visualization* to refer to all them.

The interest of researchers on the role of visualization in school mathematics did not begin in mathematics education, but in educational psychology. As I sowed in Gutiérrez (1996), several relevant psychologists, like Denis, Kosslyn, Krutetskii, Paivio, Shepard, Yakimanskaya, and others, made seminal works to characterize visualization that influenced the emergence of the mathematics education approach to visualization. However, mathematics educators' interest in focusing specifically on the teaching and learning of mathematics made them explore their own way, and open new approaches that became specific theoretical constructs, like those proposed by Bishop, K. Clements, Gutiérrez, Mitchelmore, Presmeg, or Wheatley along the 80s and 90s. Nowadays, the mathematics education research on visualization is not a part of the educational psychology research on this topic, but they continue having relevant links. The chapters in this book show some of those links.

The use of visualization and visual strategies in school mathematics is usually associated to the teaching and learning of geometry, as shown in several chapters of this book by mathematics educators and many other publications (Presmeg, 2006). However, visualization is useful also to understand and learn any other content area of school mathematics (arithmetic, algebra, functions, statistics, etc.), since all them may benefit from the use of some kind of visual representations like graphs, diagrams, schemes, dynamic representations of calculations, etc., as shown by several papers in Hitt (2002). Both in educational psychology and mathematics educations, there have been researchers interested in exploring the role of visualization in other areas of school mathematics, from kindergarten to future teachers and other undergraduate students. In this book, there is a wide interest for elementary arithmetic: Casey and Fell reflect on the relationship between visualization and different aspects of elementary arithmetic, namely the choice of strategies by young children when acquiring numeracy and early addition and subtraction, and the use of more or less complex strategies when making higher level arithmetic in the upper primary grades, like decomposition of numbers or the choice of calculations strategies of counting-all, counting-on, decomposition, and retrieval. These results agree with well-known results by mathematics educators, like Fuson, describing procedures used by children that combine visual images, motor actions with fingers and verbal recitations to count and calculate additions or subtractions. These strategies are eventually internalized as dynamic mental images (Presmeg, 1986b) and used by children to progress to the learning of more automatic and algorithmic calculation procedures.

Looking at middle primary school grades, Cipora, Schroeder, Soltanlou, and Nuerk's chapter summarizes publications exploring, by means of statistical correlation techniques, the kinds of relationships between space and arithmetic. They have found narrow correlations between visualization and the use of multi-digit numbers.

Jirout and Newcombe present results about the relationship between arithmetic proportionality and a specific kind of visualization, named *spatial scaling*, based on the ability to reason about contexts where some spatial relationships are identified and then applied to a different sized context. They mention, as a difficulty in teaching and learning numbers in primary school, that it is not always clear what numeracy or number means. However, if the teaching of numbers and arithmetic is approached from a phenomenological point of view (Freudenthal, 1983), the changes in the meaning of numeracy and number along the primary and secondary grades (from natural to complex numbers) may be seen as a continuous of increasingly complex mathematical objects created to solve increasingly complex real problems, each new kind of numbers including the previous ones. Numbers are different because they solve different problems.

An important aspect of learning numbers raised by Jirout and Newcombe is the need to make explicit the differential characteristics of the visual representations of each new set of numbers. Most part of their chapter is devoted to analyze spatial scaling and proportional relationships in the context of relative magnitudes. They compile results demonstrating that visual representations of numbers and operations are necessary for a good understanding of early arithmetic and a basis for later understanding of mathematics. Although the most usual context for spatial scaling is that of distances in maps and real world, they present other context where visualization plays a relevant role in making the contexts accessible to primary school children. This kind of results, conclusions, and proposals is also present in mathematics education research publications, like those synthesized in some handbooks for arithmetic in general (Verschaffel, Greer, and Torbeyns, 2006), rational number in particular (Lamon, 2007), and other areas of school primary school mathematics (Mulligan, Vergnaud, 2006).

A particularity of the educational psychology chapters in the book is that all them focus on young children, like most references they mention, but some of them not only deal with geometry, but also with other curricular topics and mathematical concepts. Casey and Fell, besides thinking about the context of elementary arithmetic, discuss the issue of the relationship between visualization and measurement sense. For instance, visualization is very helpful to develop de concept of array and apply it to calculate or estimate measurement of lengths, surfaces or volumes with the help of mental representations of the number line, and tiled surfaces or volumes. This is also related to the learning of fractions conceptualized as parts of the unit of measurement and the graphical representation of calculations with fractions.

Congdon, Vasilyeva, Mix, and Levine analyze the transition along primary grades from an intuitive perception to a metric understanding of space and the usefulness of visualization in this transition. They pay attention to the understanding of the unit of measurement because this concept is recognized as central in the process of acquisition of measurement. Congdon and colleagues review the well-known Piaget's results on this topic and relate them to the difficulties students show in the international assessment like TIMSS or PISA. A main reason for such failure is that teaching of measurement in schools tend to be algorithmic, based on memorizing formulas and applying them to calculate perimeters, areas or volumes of adequate figures, but teachers do not pay enough attention to the meaning of units of measurement and their manipulation. Congdon and colleagues' chapter also presents a detailed review of literature, from both educational psychology and mathematics education, related to teaching and understanding measurement. They show the evolution of the learning of length, area and volume, and angles along the primary grades and the role that visualization should play in such learning processes, by describing the different procedures and success of children using rote procedures or procedures where visual representations are part of a scaffolding for their learning.

These results are aligned with results from mathematics education, like David and Tomaz (2012), who showed that drawings and manipulatives helped students to gain an understanding of the concepts of area and area measurement deeper than their pairs receiving a more algorithmic teaching. Although the statistical comparison of pre and post-tests of experimental and control groups did not show significant differences, a qualitative analysis of students' procedures of solution showed clear differences.

It would have been interesting to know data from educational psychology research about higher educational levels, to see whether they support that visual images and visualization are not just accessory elements for mathematicians, teachers and students, but they play a relevant role, since images may help us understand a new concept or suggest a way to prove a new conjecture (Giaquinto, 2007).

Other question analyzed by educational psychologists and mathematics educators is the relationship between students' use of visualization and their achievement in mathematics. Casey and Fell discuss literature showing a relationship between development of visualization skills and arithmetic skills in early grades (K-2) and, as a consequence, a relationship between good visualization skills and mathematical achievement. Their conclusion is that there is evidence for a relationship between the use of abilities of visualization and the development of addition and subtraction skills in kindergarten and grade 1. This agrees with Young, Levine, and Mix, who conclude that teachers support to visual reasoning is an effective way to promote students' achievement.

In the same vein, the chapters by Lowrie and Logan, and Gutiérrez, Ramírez, Benedicto, Beltrán, and Jaime analyze the relationship, confirmed by many studies, between visualization and performance or mathematical talent; likewise, the chapter by Sinclair, Moss, Hawes, and Stephenson focus on children's drawings, as a vehicle to show their visual reasoning, and mathematical achievement.

In spite of the many data brought by the different chapters in this book in favor of such relationship, there is also literature concluding the opposite. Krutetskii (1976) described the components of the *structure of mathematical giftedness* and also mentioned some elements of mental mathematical activity that he considered are not obligatory components of the structure, such as the computational ability, the memory for symbols, numbers, and formulas, the ability for spatial concepts, and the ability to visualize abstract mathematical relationships and dependencies. Lean and Clements' (1981) analysis of literature concluded that there is not a clear support for the relationship between visualization and mathematical performance. Presmeg (1986a) stated that most talented students prefer non-visual procedures due to several factors like the nature of mathematics they study, economy of time, preferences of their teachers, etc. However, more recent authors, like Rivera (2011), Gruessing (2011), Ramírez (2012), and Paz-Baruch, Leikin, and Leikin (2016), offered conclusions relating expertise in the use of visualization abilities and mathematical talent.

As a closing synthesis, the chapters in this book show that educational psychology and mathematics education share an interest to analyze the role of visualization in teaching and learning mathematics. There is also agreement in some results and conclusions, but there are clear differences in specific research objectives; namely, educational psychology seems to be mostly interested by the elementary school level, while mathematics education explores also higher school, undergraduate, and graduate levels and even professional mathematicians' activity. For instance, Giaquinto (2007) and Alcock and Inglis (2010) analyze the role of visualization in highly formalized mathematics areas, like algebra or calculus, or the activity of writing formal proofs. They show that this kind of mathematical activity, purely textual and symbolic, is based on the application of axioms, definitions, theorems, etc., but visualization plays an important role to help giving sense to such manipulations of symbols and making them.

There are also differences in research methodologies since educational psychology prefers psychometric methods, showing panoramic pictures of broad questions, while mathematics education prefers qualitative methods, producing fine grained results answering specific questions. Those commonalities and differences are good basis for productive interactions and exchange of ideas between educational psychology and mathematics education.

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