reader reflections

(Continued from page 681)

Quadratic formula

Smith (1979) proposed to use a "stencil" of the parabola $y = ax^2 + bx + c$ to find the real roots of the equation $ax^2 + bx + c = 0$. He started from the definition of *parabola* as the set of points in the plane that are equidistant from a given point (h, k) and a given straight line $y = n$. He concluded that

$$x = h \pm \sqrt{n^2 - k^2},$$

which gives the roots in terms of h, k, and n. This result is different from the usual quadratic formula but can lead to it.

In class, students could begin with approximations of the roots from the parabolic stencil and then obtain a formula *from this parabola* that allows them to calculate the roots. Starting with the equation $x^2 + bx + c = 0$, we can sketch the curve $y = x^2 + bx + c$ shown in figure 1. Let (x_0, y_0) be its vertex. This point is the key for determining if the equation has real roots.

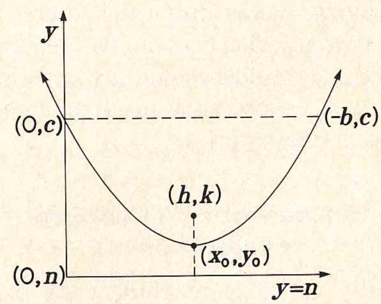

Fig. 1

Since the parabola cuts the y axis at $(0, c)$, by symmetry, there is another value x with the same ordinate c, namely, $(^-b, c)$. So the abscissa of the vertex is $x_0 = h = {}^-b/2$, and the ordinate is

$$(1) \qquad y_0 = \frac{b^2}{4} - \frac{b^2}{2} + c = c - \frac{b^2}{4}.$$

On the other hand, by the definition of parabola, $k - y_0 = y_0 - n$; therefore,

$$(2) \qquad y_0 = \frac{k + n}{2}.$$

So, from figure 1, the equations (1) or (2) have real roots if the curve cuts the x axis, that is, if

$$y_0 = c - \frac{b^2}{4} \leq 0, \text{ or } y_0 = \frac{k + n}{2} \leq 0.$$

(Continued on page 712)

reader reflections

(*Continued from page 694*)

The conditions for roots are

$b^2 - 4c > 0$ or $k < {}^-n$ (two real roots)
$b^2 - 4c = 0$ or $k = {}^-n$ (one real root)
$b^2 - 4c < 0$ or $k > {}^-n$ (no real root)

If the equation has real roots, they are $x = h \pm d$. See figure 2. Then, $k^2 + d^2 = 1^2 = n^2$; that is, $d = \sqrt{n^2 - k^2}$ and

(3) $$x = h \pm \sqrt{n^2 - k^2}.$$

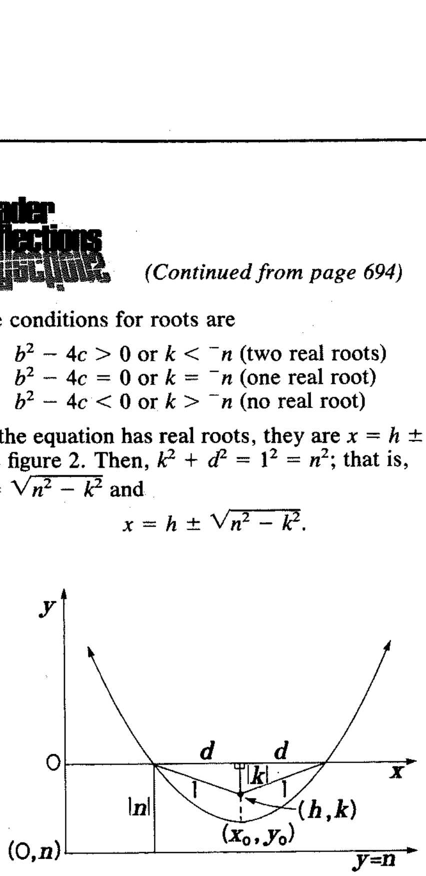

Fig. 2

The stencil of the parabolas $y = x^2 + bx + c$ is $y = x^2$ (fig. 3), in which $n = {}^-k$; therefore,

$$x^2 + (y - k)^2 = (y + k)^2$$
$$x^2 + y^2 - 2ky + k^2 = y^2 + 2ky + k^2$$
$$x^2 = 4ky$$
$$k = \frac{x^2}{4y}.$$

Since (1, 1) is on this parabola, we have $k = 1/4$ and $h = {}^-1/4$.

Then, in every parabola $y = x^2 + bx + c$, $k = y_0 + 1/4$, $n = y_0 - 1/4$, and so $k - n = 1/2$. By substituting these values in (3), we find that

$$x = \frac{{}^-b \pm \sqrt{b^2 - 4c}}{2}.$$

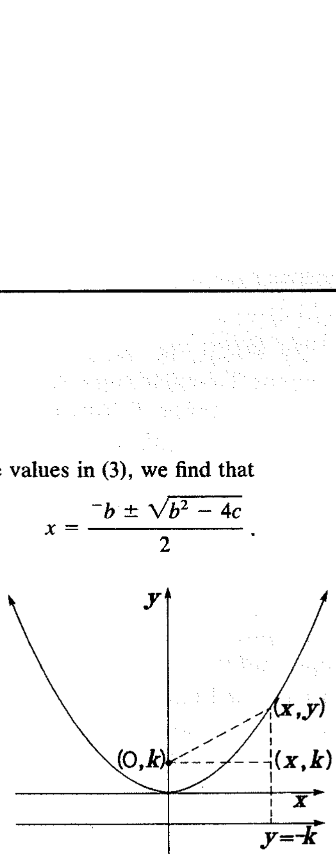

Fig. 3

If we have the general equation

$$x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = 0,$$

then

$$x = \frac{{}^-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Reference: Smith, Albertus. "A 'Stencil Method' for Solving Quadratics of the Type $ax^2 + bx + c = 0$ That Have Real Roots." *Mathematics Teacher* 72 (December 1979):661–67.

Angel Gutierrez Rodriguez
Escuela Universitaria del
Profesorado de E.G.B.
Alcalde Reig, 8
Valencia-13
SPAIN