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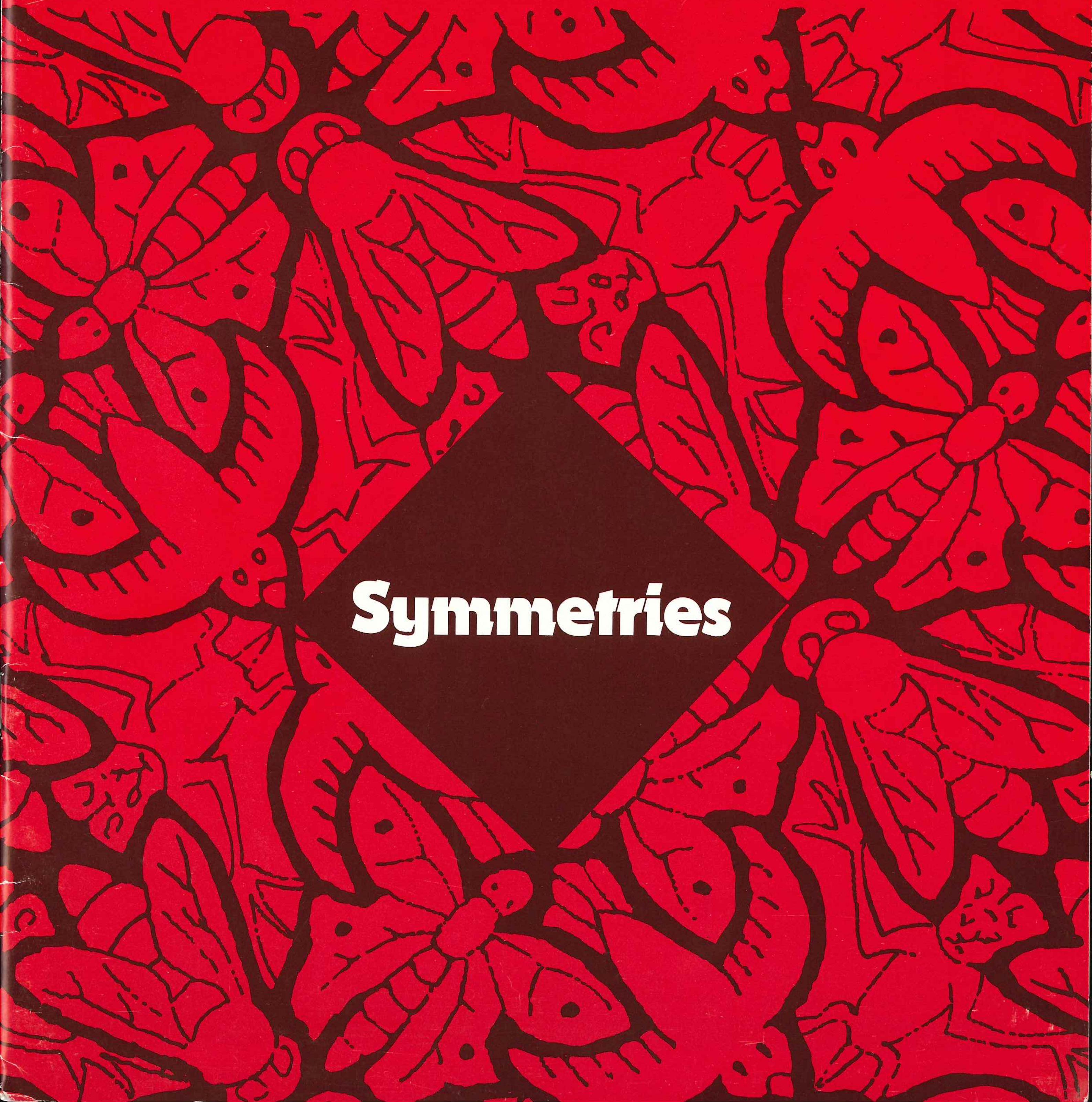

AN EXPERIENCE WITH M.C. ESCHER AND THE TESSELLATIONS

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Background to Experience

In this paper I would like to describe and commend an experience I have carried out during the last two academic years (1980-82) in my mathematics class of the Training University School of Teachers of Primary Level, after having studied the group of the isometries of the plane.

During a period of approximately six weeks, (three hours per week), we have studied in detail the isometries of the plane, both from the graphic viewpoint, by making operations in the cartesian plane, and from the algebraic viewpoint, by analyzing the compositions of rotations, symmetries and translations and by finding the subgroups contained in the group of the isometries of the plane.

For completing the theme and applying globally the whole of the previously acquired knowledge, I prepared working material, (acetate transparencies and slides for group work and photocopies for individual work), based on different patterns of tessellations plus some of the fascinating drawings of M. C. Escher which follow those models (without bearing in mind the changes of colour).

Every Mathematics teacher knows that the authentic understanding of a theme is not obtained by means of *pseudo-real problems* prepared in order to arrive at an agreeable solution by applying the method showed in the class, but with open problems which, sometimes, have a difficult solution or even no solution at all. The objectives of these activities are to accustom my pupils' minds both to move objects and to recognize the movements those objects have made. On the other hand, it is always agreeable for the pupils to forget for a while the blackboard and the theorems; we must agree that Escher is always successful and that this kind of design is ever present in actual life, in walls, pavements, clothes and in the most unexpected places and objects.

Tessellation Patterns

Now, I am going to explain in detail that experience which I consider highly positive and instructive both for my pupils and for me. In the first place I would like to advertise that I neither attempted to study the obtaining of patterns of tessellations nor to prove that there are exactly 17 different patterns.

After giving the pupils the personal working material, we begin a discussion about the nature of tessellations and how one builds them on pavements or walls in the houses, in order to arrive at some clear rules which explain the position that every tile must have in the network. As all the tiles are the same, these rules may be considered as isometries which carry a tile, from its initial place to any other

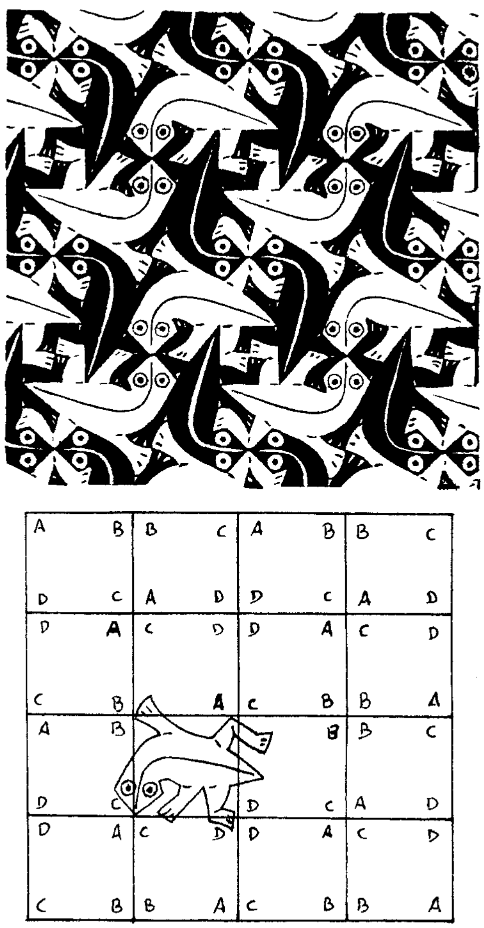
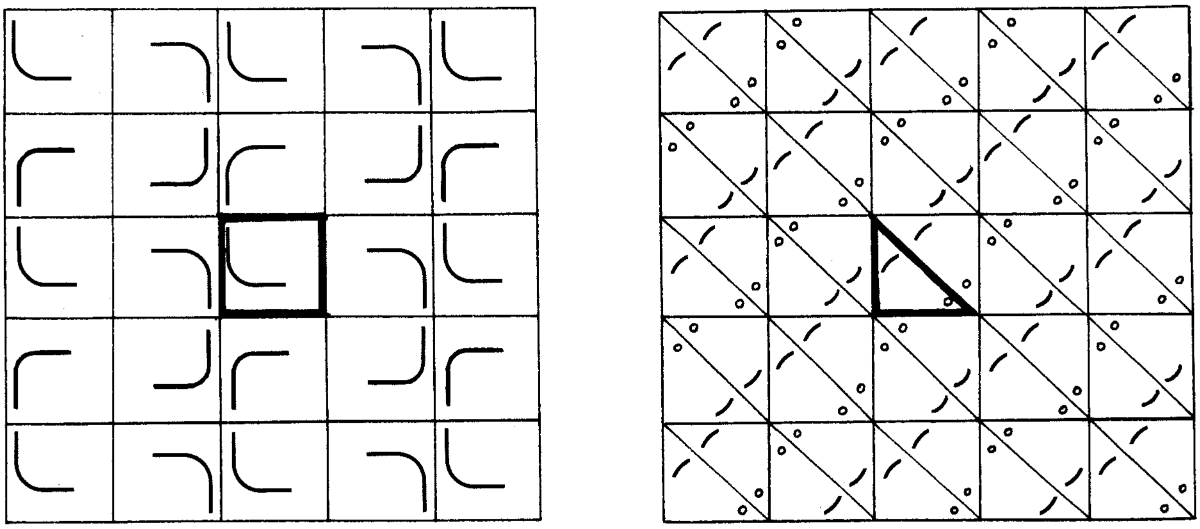
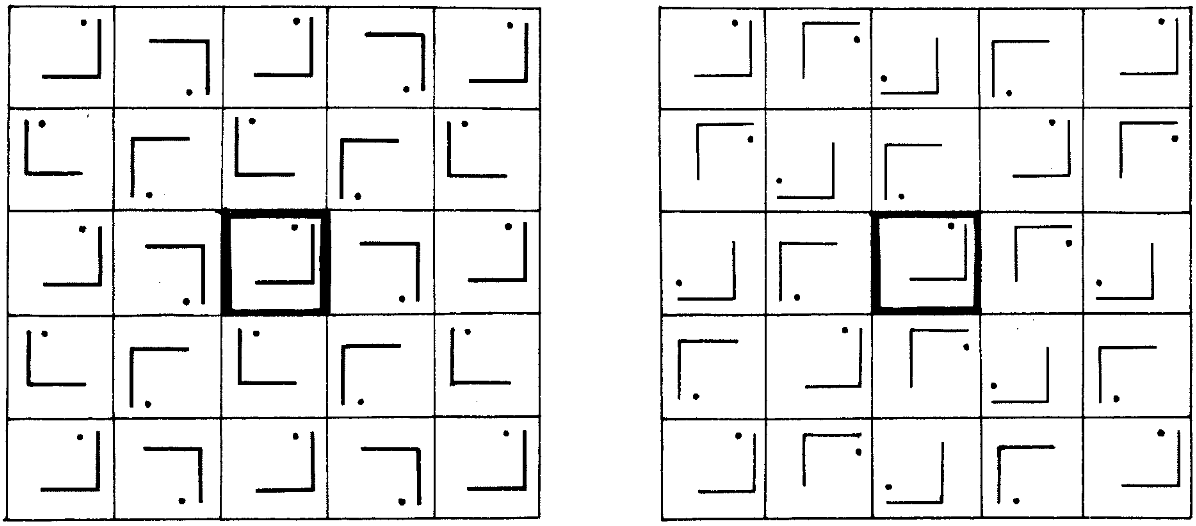


Fig. 1 These are an Escher's tessellation and its pattern.



Two patterns of pmg



Two patterns of pgg.

Fig. 2

place in the plane. Then, the problem is stated in the following terms:

"*Your working material is formed by some patterns of tessellations. The marked figure is the minimal motif (in other words, a tile) put on its initial place. The problem is to find which isometries allow that tile to move through the plane and, also, to select the smallest number of those movements which are necessary to cover the whole of the plane without leaving any free space.*"

In short, one has to find the generator movements of (the group of) the tessellation. Theoretically, there are 17 groups of movements which may cover all the cells in a plane pattern. However, some of those 17 groups may be applied to basically different figures, as parallelograms and triangles, giving results completely different (Fig. 2); it is even possible to apply the movements of a group in different ways to the same pattern (Fig. 2). You can see the difference of results in the figures. One can obtain up to 32 patterns.

As we had only quickly studied the glide reflections and we had little available time, we have only analyzed tessellations which don't include this movement. After a few days of individual work, we begin the study of the patterns of tessellations one by one, by means of transparencies. In the first place I present to them the patterns p1, pmm, p4 and p2 which are the easiest ones since they only have one isometry each (Fig. 3). In this way the pupils can easily overcome the first difficulty of recognizing each isometry. The difficulty is minimal in p1 (translations), greater in pmm (symmetries) and p4 (rotations around the corners) and it is greatest in p2 (rotations around the midpoint of the sides). After studying these four patterns pupils are already able to recognize each isometry and know how to find them: they notice the cells around the initial one and they compare their position with the initial position of the minimal motif. From this aspect, the students have no more difficulties in the next models of tessellations and they have caught one of my objectives (to recognize the movements).

But there is still a second difficulty to overcome, that is to find the least number of isometries which are necessary in order to cover the whole of the plane. The process consists of moving the minimal motif, from its initial place, so as to fill all the cells in the network. The complexity for the students starts when the motif detaches itself from its initial position, because it is more difficult to make the movement, specially if it is a rotation. Moreover, in this aspect, each tessellation gives place to different problems.

Although in the first patterns the pupils are lost and don't know what to look for nor how to do it, little by little they develop their technique. After some attempts, the students already know that for doing a symmetry they have to measure the distance from the cell to the axis (by using the more convenient unity, that is the cells of the pattern) (Fig. 4).

If there is a rotation, the pupils usually (specially when they *don't see* the movement) *build a radius* made by cells from the centre to the actual position of the motif; after they rotate the nearest cell to the centre and they reconstruct the radius over its new position (Fig. 5).

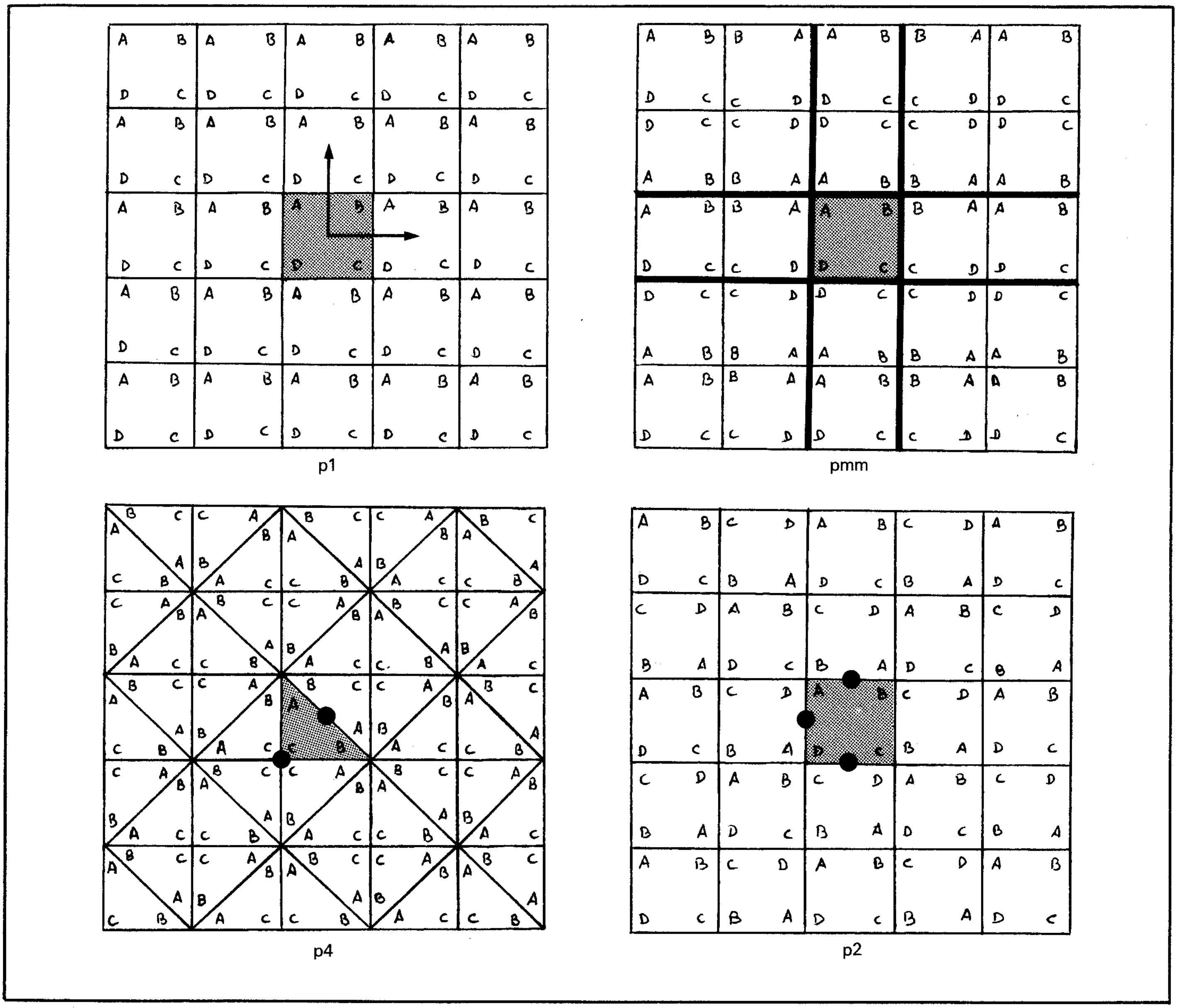


Fig. 3

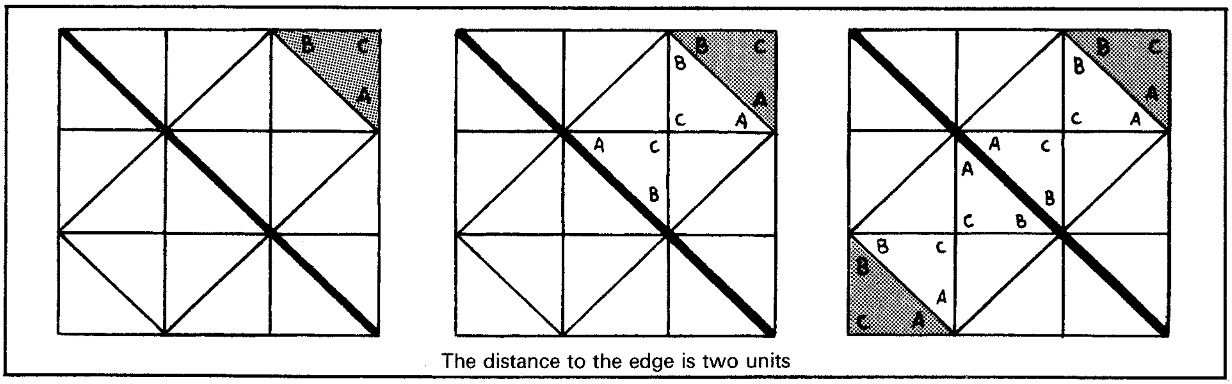


Fig. 4

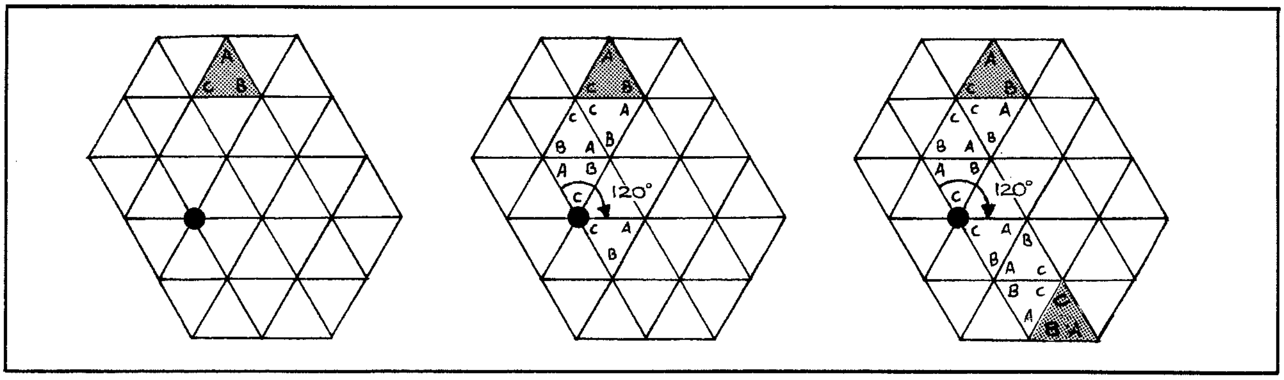


Fig. 5

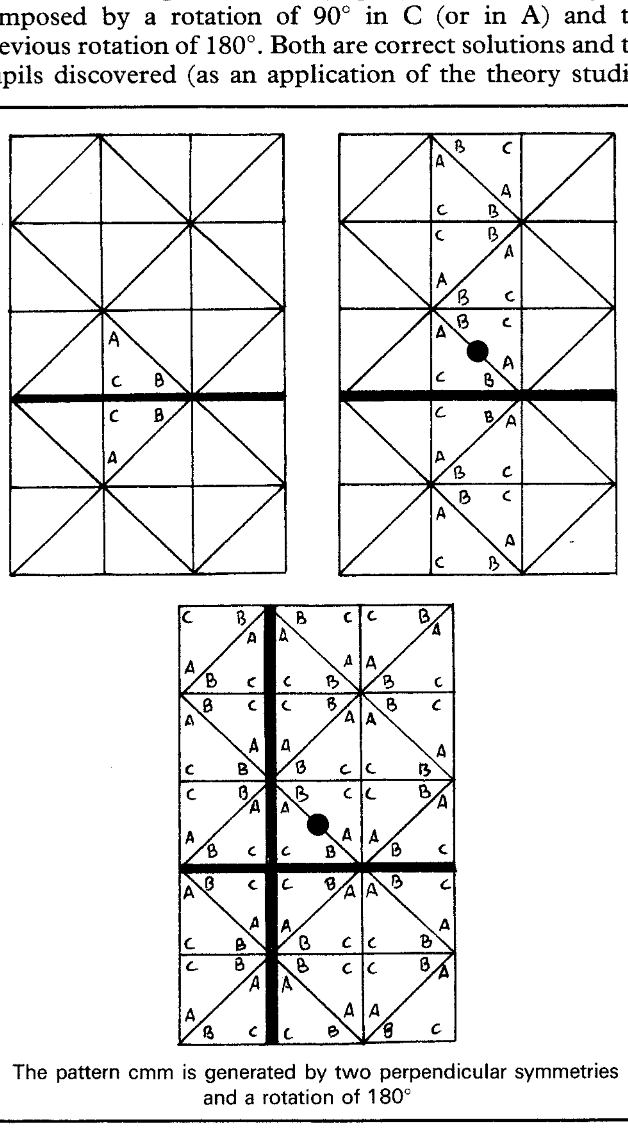
It is easy to observe how the students are acquiring security and quickness when they have repeated several times the same rotation, but how they lose these qualities if they face another rotation with a different angle (remember that the pupils are more than eighteen years old). Another of my objectives is this quickness in finding the position to which the minimal motif is displaced by an isometry.

In order to know how many isometries are necessary to generate the tessellation, a method is to begin by using only one isometry. When it is not possible to cover more cells, you add a new isometry and now, with both isometries independently, you move the minimal motif as much as possible; if still there are cells without cover, you add a third isometry and so on until all the cells in the tessellation are covered (Fig. 6). These activities played in some patterns (for example pm, p3, cmm, p31m, p3m1 besides others already cited) complete the first part of the experience.

Analysing the Patterns

I think this is the moment to comment on the non-uniqueness of the solution of a tessellation. The work in the classroom consists in the analysis of a solution proposed by some pupil, to see if it is correct and, if necessary, to modify it. Sometimes the proposed solutions are the standard generator system of that tessellation. In other cases, a pupil wishes to know if his solution, different to the one that we have studied, is correct. This situation gives occasion to deal with this interesting theme.

For example, in p4 with triangular motif, we studied the system composed by a rotation of 90° in B and a rotation of 180° in the midpoint of $\overline{AC}$ (Fig. 7) and also the system composed by a rotation of 90° in C (or in A) and the previous rotation of 180°. Both are correct solutions and the pupils discovered (as an application of the theory studied some days ago) that we obtain the rotation of 90° in B by composing the rotation in C (or in A) and the rotation of 180°, and also on the contrary. We stated this property by modifying the known property of generator systems of groups: "We always can substitute an isometry for the result of its composition with another isometry belonging to the system of generators." Then, in this example, we may obtain three different systems of generators for the pattern p4.



The pattern cmm is generated by two perpendicular symmetries and a rotation of 180°

Fig. 6

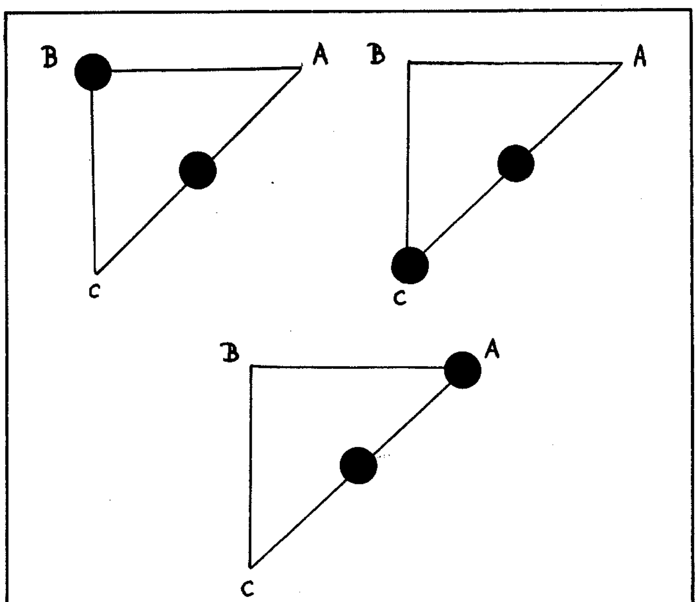


Fig. 7

Once that the different patterns of tessellations have been studied, we pass to the second part, which consists in analyzing some real tessellations. An inexhaustible source of covering of the plane is the samples of cloths or wallpapers; however, I cannot avoid the great attraction I feel for the M. C. Escher's periodic drawings, for which reason I have used transparencies and slides of some of them.

Immediately I observe the change that supposes for the pupils to pass from a network composed by squares, triangles or rhombs to a different kind of network. In these new networks, sometimes, one has to stop to interpret the drawing and look which animal or animals compose it (see, for example, in Fig. 8 the plate 41 of [2]).

Nevertheless, soon the pupils overcome this difficulty and it is compensated by the facility they have, after the practice of the first part, to find which movements are in the drawing. Whenever one takes as the minimal motif a half figure, it must have a symmetry completing it. If there are

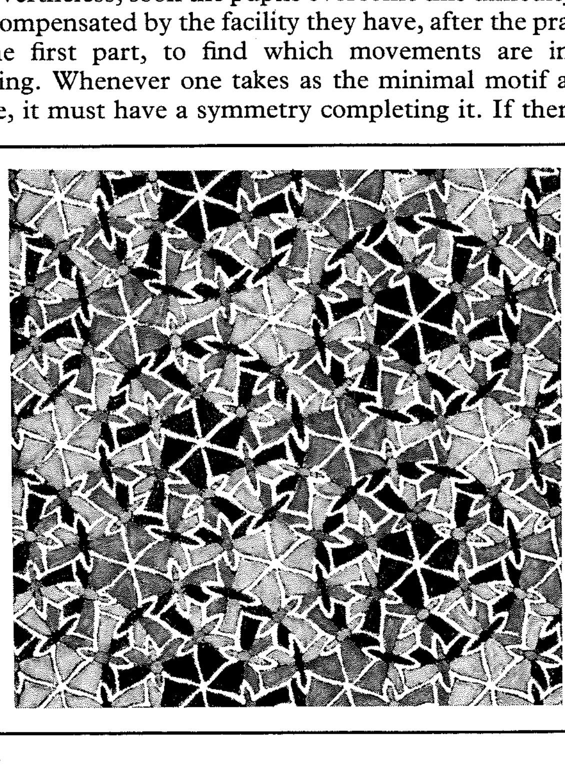


Fig. 8

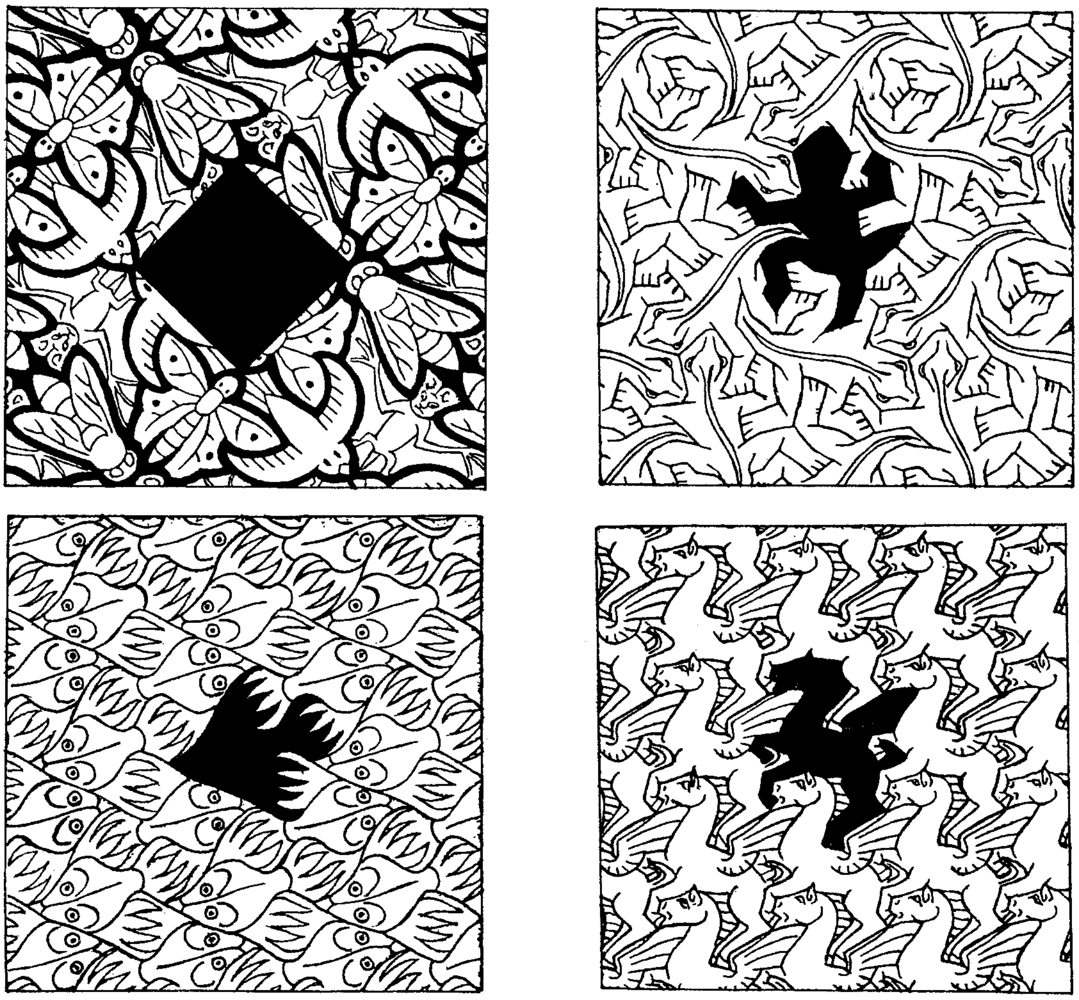


Fig. 9

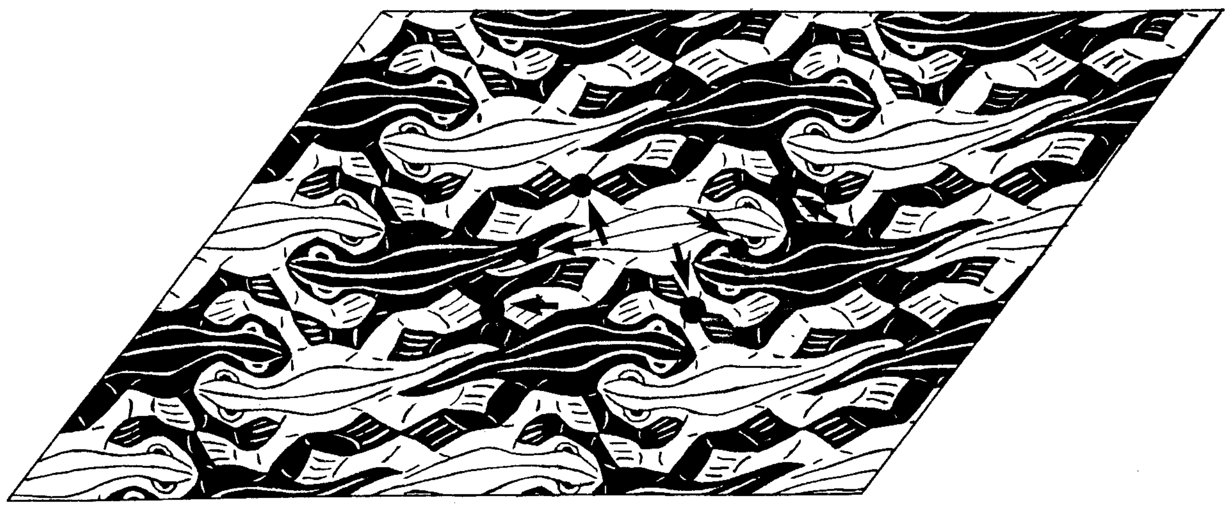


Fig. 10

figures with different senses of orientation, then there are rotations. If there are only two opposite senses, then there are rotations of 180° or glide reflections. When all the figures have the same position, the pattern is p1 (two translations). Fig. 9 shows some examples.

Sometimes it is difficult to find the centre of a rotation of 180°; my pupils joined by means of a segment (graphically or mentally) the two positions of some characteristic point (an eye, the mouth, the end of the tail, etc.) and they looked for the midpoint. Of course, they arrived at this technique after some erroneous trials, that is by the trial and error method.

There is a curious Escher's tessellation because it may lead to the mistake my pupils fell in. In plate 16 of [2] (Fig. 10) the minimal motif, obviously, is a lizard and there are 6 rotations of 180° (therefore it corresponds to the pattern p2) yet we cannot choose any three rotations, but we have to choose a rotation that maintains the colour and another one that reverses it. Did Escher realize the tessellation he had obtained?

In short, this second part doesn't state problems, excepting the initial ones I have pointed out, and it is more agreeable for the students than the first one because of the variety of drawings. With these experiences the pupils complete and make firm the concepts and they finish acquiring skill in the practical handling of the isometries of the plane.

References

1. Escher, M. C. and Locher, J. L. (1971) *The World of M. C. Escher*, Harry N. Abrams Publ., New York.
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