# THE ASPECT OF POLYHEDRA AS A FACTOR INFLUENCING THE STUDENTS' ABILITY FOR ROTATING THEM

Angel Gutiérrez
*Departamento de Didáctica de la Matemática*
*Universidad de Valencia*

New generations of computers provide us with new tools that can be used for an effective teaching of Spatial Geometry. In this paper I report on the results of a research project that has analyzed the students' interaction with three pieces of software allowing them to move, in different ways, geometric solids on the computer screen. Usually, software provides us with several possibilities of presentation of the solids. Here is explored the hypothesis that the different aspects of solids provided by computers are appropriate for different levels of students' ability of spatial visualization.

## INTRODUCTION

Our ordinary life is plenty of interactions between plane and space, and most of them imply some kind of dynamic relation or information. Therefore, it is basic for children to acquire and develop abilities of visualization allowing them to manage 3D mental images, that is to create, transform, and analyze mental images representing 3D objects. This issue is particularly important for Spatial Geometry, since any kind of plane representation of spatial objects, as those usually present in textbooks, has some limitation and fails to transmit a part of the information (Parzysz, 1988). Therefore, in order to learn, from a textbook, what is a new kind of polyhedron, students have to create in their minds several images (several "pictures" of the solid, like those in Figure 1) and try to link them, that seldom happens. However, it would be much better if students could have an experience so rich as to allow them to create a dynamic image that could be moved in their minds in a way quite similar to a real solid being handled (like those in Figure 2).

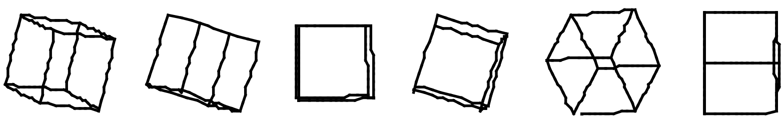


Figure 1. Independent images of a cube.

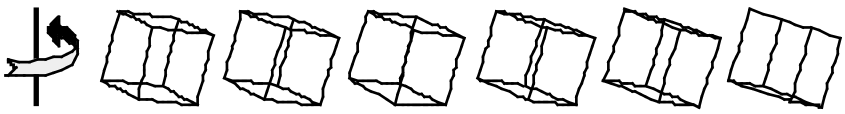


Figure 2. Linked images of a cube.

For the obvious reason of its similarity to reality, the most usual way of drawing polyhedra in textbooks is someone of the perspective or parallel representations. This is fine, but the best plane representation in the best textbook cannot provide at all a basic element of the instruction in Spatial Geometry, namely, Movement. Up to now, the dynamic approach to Spatial Geometry is being done by using real solids and by allowing children to handle them, but this approach, like that based only on textbooks, mainly generates static images in the students' mind. Furthermore, it is quite usual to find only some prototypical pictures of polyhedra, like those shown in Figure 3 that are some of the most frequent representations of a pyramid in textbooks and children's drawings.

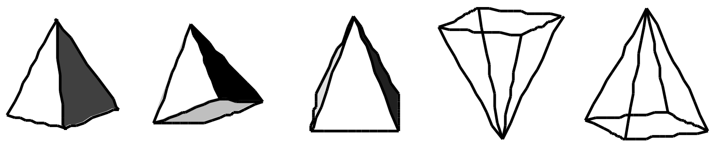


Figure 3. Usual images of a pyramid.

Now, however, cheaper and more powerful computers are giving to teachers some opportunities for creating new ways of teaching Geometry, and to students some tools for helping them to know more in depth the solid's structure and better develop their spatial imagination, so they would be able, for instance, to recognise those pictures in Figure 4 as representations of the same pyramid.

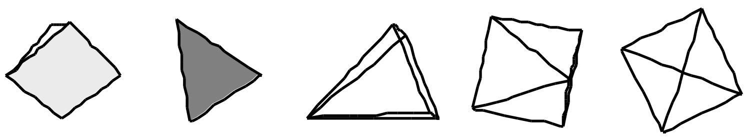


Figure 4. Infrequent images of a pyramid.

In this paper I will refer to a research, carried out at the University of Valencia during last academic year, having the aim of exploring how software allowing the movement of 3D figures on the computer screen can be used in the school and how children interact with it. This global objective of the research project has several sub-objectives. I will focus here on one of them: The question of which of the several representations of solids that can be provided by software are more appropriate for different children, depending on their age or their ability of spatial visualization.

## ORGANIZATION OF THE RESEARCH

### The conceptual base

The concept of spatial visualization has a wide range of meanings, depending on the field in which it is implemented. From the point of view of mathematics education, spatial visualization refers to a set of elements related to the generation and use of mental representations (mental images) of mathematical information. Of course, graphical images of geometric objects are the most evident example of mental images, but spatial visualization is an important tool for representing and understanding any other area of mathematics too (Dreyfus, 1991). This set of elements integrating spatial visualization can be divided into three main parts: *mental images, visualization processes,* and *visualization abilities.* Here are just listed the ones more relevant to the part of the research reported in this paper.

Mental images are the basic objects for spatial visualization and imagination. Students have to learn to construct, transform, and analyze them in order to acquire a good capability of spatial visualization. Some types of mental images described by Presmeg (1986) are:

Concrete images. They are pictorial figurative images of physical objects. The students in our experiments dealt ever with this kind of images of the polyhedra they manipulated. This is the simplest kind of mental images, and they are generated by every student, independently of the teaching method used by teachers. This is the kind of images we usually utilise for identifying polyhedra: When we have to identify a geometric solid that we are seeing, we first compare it with different pictures of solids we have in our mind, looking for similarities and differences.

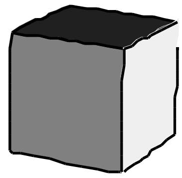


Figure 5.

Dynamic images. They are images involving the mental movement of an object or some of its parts. Therefore, these images allow us to observe a mental object from different points, changing the object's position. A characteristic of dynamic images is that they can present to us views of the solid that we really cannot see from our current physical position. This type of images was necessary for the students in our experiments since, in the activities we are reporting, they had to rotate mentally polyhedra in space or compare two different positions of the same solid.

Kynaesthetic images. They are images involving the physical movement of some part of the body (usually hands and arms, but also the head, etc.). This type of images was continuously used by our pupils since they moved very often their hands to represent a rotation before choosing the appropriate button on the computer screen, or while explaining to a researcher or another child the movements they had just made or were going to make.

Manipulation of mental images is not easy work. It requires of the use of some techniques and, of course, of some training. Students generate mental images of polyhedra or other spatial geometric objects because they have to use them to solve some problem. Then, there is a mental process of building an image (or recalling it from memory, depending on the cases), transforming it in some way, and obtaining some information intended to help in the resolution of the problem (Figure 6).

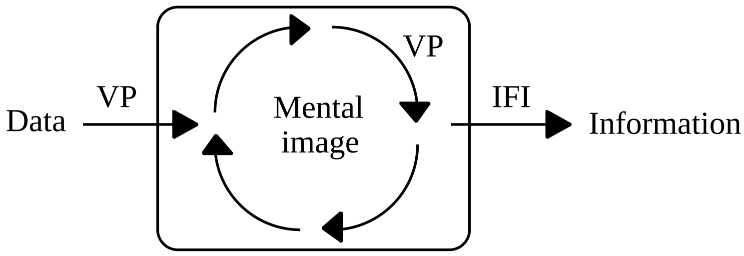



Figure 6.

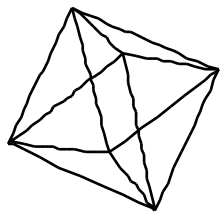
So, there are two inter-related processes involved in the handling of mental images and controlling the flow of information between external objects and mental images or between different images (Bishop, 1989):

The *visual processing* of information (VP) is the process of converting abstract and non-figurative data in visual images, and also that of converting a mental image into another one. The process of *interpreting figural information* (IFI) is the process of reading, analysing and understanding spatial representations, such as plane representations or mental images of polyhedra, in order to obtain some data from them. In some way, each process is opposite to the other one (Figure 6).

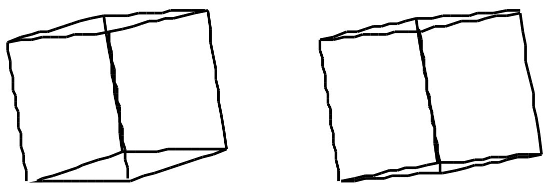
Finally, the third component in the activity of visualization are the abilities need to carry out the previous processes. The learning and improvement of these abilities is the key in the whole process of spatial visualisation. There are quite different types of abilities; some of them have a main physiological component, while others have a psychological nature. From those abilities compiled in Del Grande (1990), the following are relevant to this research:

<ins>Figure-ground perception</ins>. This is the ability to identify a specific figure by isolating it out of a complex background. It was used, for instance, when students have to identify a face of the octahedron in Figure 2, since usually the edges create distracting shapes.

<ins>Perceptual constancy</ins>. This is the ability to recognize that an object has constant properties such as shape, or adjacency of two faces in spite of the variability of its position. When children are dealing with opaque polyhedra, this ability allows them to use properties of the part of the solid they cannot see.



<ins>Figure 7</ins>.



- a - - b -

<ins>Figure 8</ins>. What are de differences between polyhedra -a- and -b- ?

<ins>Visual discrimination</ins>. This is the ability to compare several objects, pictures and/or mental images, and to identify their similarities or differences. It was used by our pupils in many activities that asked them to compare two representations of polyhedra or to move a polyhedron from its current position to another one shown by a picture (Figure 8).

The context.

As the main objective of this research was to provide the students with interaction between the computer and their usual way of learning Geometry (based on plane representations and, often, physical models), we designed a microworld in which the main element is the relationship between computer objects, pictures and real solids.

*Software.* We used three different pieces of software, each one of them having particular characteristics:

- One of the programs presents a cube on the screen, and it allows the students to rotate the cube 90° around any of the three axes passing through the midpoints of the faces (Figure 9). The faces of this cube are decorated with pictures, which help the students to maintain control of the movements and to relate two appearances of a given cube. We call it a "figurative cube".

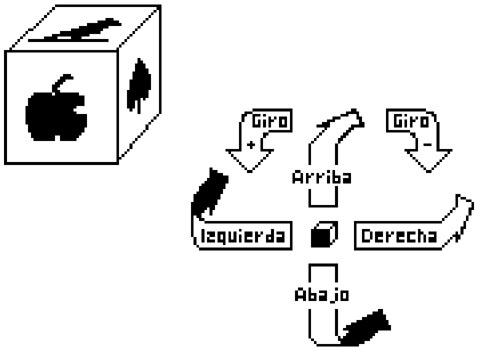



Figure 9.

- The second program presents several transparent polyhedra (cube, tetrahedron, octahedron, square pyramid, prism, and module of cubes; see Figure 10) in perspective view. The program allows the students to rotate the polyhedra freely around the standard X, Y, and Z axis. It presents on the screen six buttons, corresponding to the two directions of each rotation, and the program automatically rotates the solid in the appropriate direction while the student is pushing a button. Then, by using this program, the students can observe the continuous variation of the position of the polyhedra.

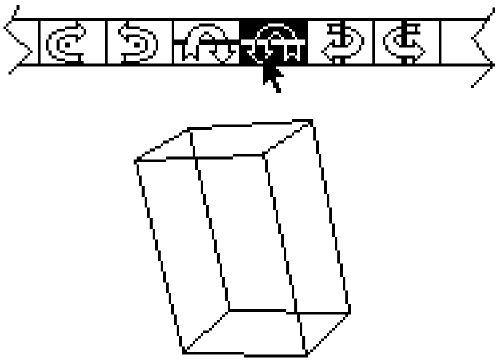


Figure 10.

- Finally, the third program presents on the screen the same solids as the second one, but with opaque faces, having each face a different regular shade (Figure 11). This program also makes the same rotations as the previous ones, but it does not provide continuous movement, but instantaneous change from the current position to the new one, determined by an angle given by the student.

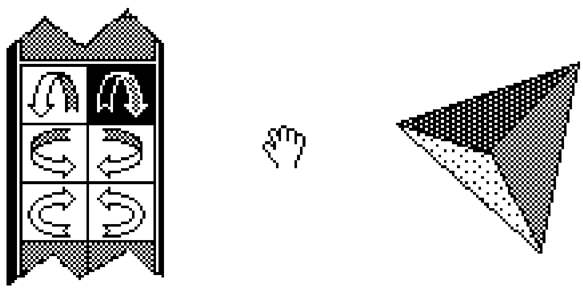


Figure 11.

*Pictures.* Students were provided with sheets of paper containing perspective representations of the polyhedra used in the experiment. In order to have the plane representations of the polyhedra as similar as possible to the solids on the computer, the pictures were hard copies from the computer screen. Figures 9, 10, and 11 represent both the shapes seen on the screen and the pictures printed on paper.

*Manipulatives.* The real solids were made of straws (for the transparent) or cardboard (for the opaque). In the case of the opaque ones, their faces were covered with the same shade as their equivalent on the computer. Furthermore, we took care of making similar the non-regular polyhedra (pyramid and prism) on the computer programs and the real one, that is, having proportional sides. In this way, we guaranteed the most similarity between the different representations of anyone of the polyhedra used in the experiment.

The students.

In this experiment participated groups of students in Grades 2, 4, 6, and 8 of a Primary School near the University. Three students in each grade (two in Grade 6) were selected so to have a range of ability levels in each group. The selection was made by their mathematics teachers on the basis of the students' level in mathematics, an they ranged (Table 1) from low average ability (↘) to high ability (↗).

Table 1
The pool of students.

| Grade | No. of students | Sex | | Ability | | | No. of sessions |
|---|---|---|---|---|---|---|---|
| | | M | F | ↘ | ≈ | ↗ | |
| **2nd** | 3 | 1 | 2 | 1 | 1 | 1 | 21 |
| **4th** | 3 | 0 | 3 | 1 | 1 | 1 | 23 |
| **6th** | 2 | 1 | 1 | 0 | 1 | 1 | 30 |
| **8th** | 3 | 0 | 3 | 0 | 2 | 1 | 22 |

Each group worked independently of the others. Two computers were available, so the 3 children in each group worked in 2 + 1. Each child had a booklet with the text and pictures of the activities and a set of physical solids.

The tasks.

As stated in the introduction, our objective in this part of the research was to observe the influence of the appearance of solids (figurative, opaque, transparent) on the students' behaviour. Then, we presented to the students a set of three tasks corresponding to the same activity: Given a polyhedron on the computer and a picture on paper of that polyhedron in a different position, students had to move the solid on the screen to match exactly the position show by the picture.

Each task was devoted to one of the possible appearances, and it presented six different pictures of the same polyhedron. Students were asked to move the solid on the screen to match Picture 1, then to match Picture 2, and so on.

This activity was done by each student with several polyhedra, but I will focus here only on the cube.

## RESULTS AND CONCLUSIONS

Students of different ages and ability levels have worked on the activities stated above, so we can obtain some conclusions with respect to the following questions:

1) Are there differences in the students' behaviour when moving the polyhedra depending on the kind of solid (figurative cube, transparent, opaque) used?

2) Is there some kind of solid more appropriate for each age or ability level?

Objective 1

The answer to the first question is affirmative, and it can be divided into several parts. It is very interesting to analyze the strategies used by students to solve the tasks. The predominant strategies for moving the cubes were different depending on the kind of solid, although they were the same for the different grades:

*Figurative cube.* In the case of the figurative cube, all the students looked for the figure in the front face of the cube, by rotating the cube until this figure appeared somewhere on the screen. Then they turned the cube so as to move this figure to the front face, and then they continued rotating the cube so to move the figure to the correct position. Figure 12 shows an example, where the cube was moved from the current position (left) to the new position (right), taking the sheep as the key figure.

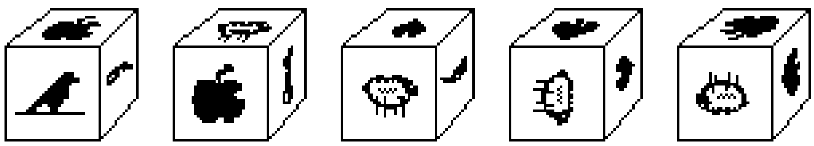


Figure 12. Strategy for moving a figurative cube.

The more able students (in Grades 6 and 8) improved this strategy by looking for any figure in the final position, then moving it to its face, and then rotating the cube so this figure goes to its correct position.

*Opaque cube.* In the case of the opaque cube, the usual strategy was to choose a relevant face (because of its colour or its shape) of the final position and to rotate the cube until it appeared on the screen. Then, students continued rotating the cube while looking for the other faces, and finally they tried to match the shapes and positions of the faces to the final position of the cube. An example is shown in Figure 13, where the black face played the role, because of its colour.

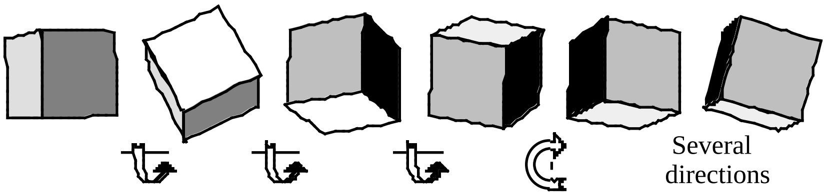



Figure 13. Strategy for moving an opaque cube.

This process was conducted by means of a continuous checking of colours, shapes, and positions between the cube on the screen and the model on the booklet.

*Transparent cube.* The usual strategy for moving the transparent cube was similar to that of the opaque one, except that students did not pay attention only to the faces, but to any relevant element of the cube (a face, a vertex, a couple of edges, etc.). Furthermore, as many times the relevant element was based on the relative position of two parts of the cube (for instance, two vertices one over the other), it was quite usual to choose a new relevant element from time to time.

However, having a strategy is not a guarantee for an efficient solution of the activity. The clearest example is that of children moving a figurative cube to a new position. When requested, they can explain their strategy for reaching the final position (the one I stated above), that is the strategy they are trying to follow, but, at the same time, they are unable to apply it in an efficient way, and they need to make up to 10 rotations, or more, to reach the final position.

For example, moving the cube in Figure 14 from position 1 to position 2 requires only three rotations (see Figure 12): Down ("Abajo", see Figure 9), Right ("Derecha"), and Right. But a 6th grader did the following sequence of nine rotations: Up ("Arriba"), Up, Up, Turn + ("Giro +"), Right, Up, Up, Up, and Left ("Izquierda").

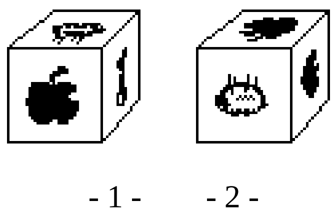



Figure 14. Move the cube from Position 1 to Position 2.

It was also interesting to analyze the students' reasons for selecting a particular rotation of a cube.

Many times, students chose rotations just by guess. However, it is necessary to distinguish two different cases: (1) When there is no useful information on the screen, the students have only the possibility of rotating the cube by guess, with the hope that the new position will bring some data; and (2) When there is useful information on the screen but the students are not able to identify and use it.

The last behaviour is a characteristic of less able students or more difficult activities. Table 2 shows the frequency of movements by guess made by students in different grades.

Table 2
Frequency of rotations by guess: High ( ■ ), intermediate ( ▒ ), and null ( □ ).

| | 2nd ↘ | 2nd ↗ | 4th ↘ | 4th ↗ | 6th ↘ | 6th ↗ | 8th ↘ | 8th ↗ |
|---|---|---|---|---|---|---|---|---|
| Figurative | ■ | □ | □ | □ | □ | □ | □ | □ |
| Opaque | ■ | ■ | □ | □ | □ | □ | □ | □ |
| Transparent | ■ | ■ | ■ | ▒ | ▒ | □ | □ | □ |

A third aspect producing differences in students' behaviour when moving the polyhedra is their ability for planning movements, that is, for deciding in advance a sequence of 2 or more turns to be made. This is an ability that is only acquired after a long experience with rotations of solids and with the use of mental images and spatial visualization. Less able or experienced children can only decide which rotation to do after they have compared the cube on the screen and the model in the booklet.

Table 3 shows that figurative cubes are much more easier to control by children than opaque and transparent cubes. This difference does not came only from children's ability, but also from the limited size of rotations of the figurative cube (only 90°) and the freedom of movements of the other kinds of cubes.

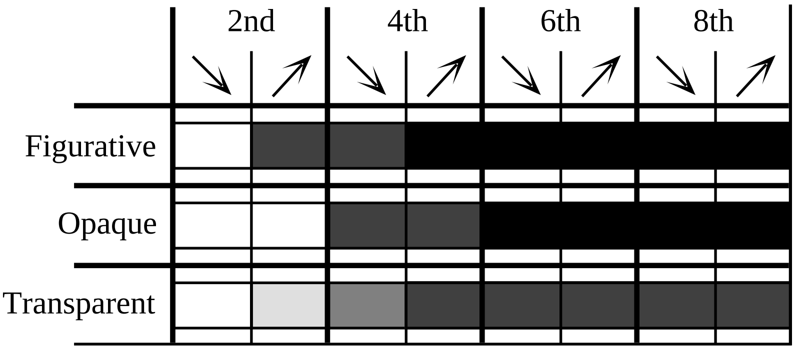
Table 3
Ability for planing several rotations in advance: High ( ■ ), intermediate ( ▒ ), and null ( □ ).

| | 2nd ↘ | 2nd ↗ | 4th ↘ | 4th ↗ | 6th ↘ | 6th ↗ | 8th ↘ | 8th ↗ |
|---|---|---|---|---|---|---|---|---|
| Figurative | □ | □ | □ | ■ | ■ | ■ | ■ | ■ |
| Opaque | □ | □ | □ | □ | □ | □ | ■ | ■ |
| Transparent | □ | □ | □ | □ | □ | □ | ▒ | ▒ |

Objective 2

In relation to the second question stated at the beginning of this section, the research I am reporting provides us with a clear picture, summarised in Table 4.

Table 4
Students' ability for using each kind of solid, from high ( ■ ) to null ( □ ).




*Figurative cube.* The figurative cube is the easiest to be rotated, for the reason stated above. Nevertheless, the less able 2nd graders could not rotate it efficiently, since they only used rotations by guess.

*Transparent cube.* The transparent cube is the most difficult to be rotated. Even the 8th graders were unable to move it efficiently. The main reason for this difficulty is that a transparent polyhedron shows its whole structure, so students have to learn to identify the real vertices, edges, and faces, and they have also to learn to distinguish the front and back parts of the solid.

## REFERENCES

Bishop, A. J. (1989). Review of research on visualization in mathematics education. *Focus on Learning Problems in Mathematics, 11.1,* 7-16.

Del Grande, J. (1990). Spatial sense. *Arithmetic Teacher, 37.6,* 14-20.

Dreyfus, T. (1991): On the status of visual reasoning in mathematics and mathematics education, in *Proceedings of the 15th PME Conference, 1,* 33-48.

Parzysz, B. (1988). "Knowing" vs. "seeing". Problems of the plane representation of space geometry figures. *Educational Studies in Mathematics, 19,* 79-92.

Presmeg, N. C. (1986). Visualization in high school mathematics. *For the Learning of Mathematics, 6.3,* 42-46.