CHILDREN'S ABILITY FOR USING DIFFERENT PLANE REPRESENTATIONS OF SPACE FIGURES*

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An important, but often neglected, aspect of the teaching and learning of Space Geometry is the use of plane representations of geometric solids (by 'use' I mean both draw and read). In this paper I present four ways of plane representation (by *layers, orthogonal, coded orthogonal*, and *isometric*) that can be used at Schools: I describe the representations, I analyze their particularities, and I make some suggestions to teachers about the way and time for teaching each one of them. This is based on the results of a research experiment carried out with primary school children in Grades 2, 4, 6, and 8.

INTRODUCTION

If we were thinking on an ideal school in the future, we could imagine classrooms with plenty of computers, many CDs lying on a shelf, and a big closet containing lots of apparatuses and manipulatives.

When the children in this classroom are starting the study of a new kind of geometric solid, the pyramid for instance, each pupil would be given by the teacher a pyramid and a CD for the computer. A few seconds later students shall have instant access to all the pyramids and other spatial objects they need for the lesson. First, the teacher asks them to spend some minutes handling their real pyramid, moving several pyramids on their computer's screen and transforming them. Afterwards, with guidance by the teacher, the children would talk in group about their findings, make verbal descriptions of their pyramids, and put together the properties they have discovered.

This may be a school of the 21st. Century, but today the textbooks are still the main source of information for school children, and likely it shall continue in this way for many years, at least in most countries in the world. In many schools, when children start to learn about pyramids (or cubes, prisms, cylinders, . . .), they only have available the description written in the book, accompanied by some colourful figures and by the teacher's explanation. In many other schools, the previous work is complemented with physical models of pyramids, used by the teacher to show what a pyramid is, and by pupils to discover properties and elements of pyramids. So we have to pay attention to the way textbooks transmit to children information about Space Geometry, and tray to solve the limitations of this way of communication.

Our ordinary life provides plenty of interactions between plane and space, and most of them imply the dissemination of some kind of spatial information by means of plane data (drawings, schemas, pictures, figures, etc.). And textbooks are still plane!! This issue is

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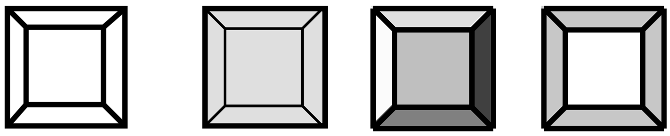
particularly important for Space Geometry, since any kind of plane representation of spatial objects has some loss of information. Therefore, a person reading a plane representation of a solid has to recover as much of the lost information as possible. This phenomenon is called "restitution of meaning" by Parzysz (1988), and he points out the importance of having "a *connivance* between the author of the representation and its reader, it being possible only because of a common geometrical culture. This connivance is concerned in the first place with the nature of the objects represented (e.g., point, straight line, triangle, circle, plane, pyramid, cylinder, …)". And this connivance is concerned in the second place with the type of representation used (both author and reader have to agree in using the same type of plane representation). For instance, the drawing in Figure 1-a can be interpreted in some different ways: We can think on the outer square as the front face of a cube (Figure 1-b) seen from very near, on the inner square as the front face of a truncated pyramid (Figure 1-c), on a plane frame (Figure 1-d), etc.



- a - - b - - c - - d -

Figure 1.

Therefore, it is basic for children to acquire and develop abilities allowing them to manage different 2D representations of 3D objects, that is, abilities allowing them to create, move, transform, and analyze mental images of 3D objects generated from the information brought by a plane drawing. In this paper I present some kinds of plane representations of 3D geometrical objects, the ones most frequently used in the classes of Space Geometry. Previous research has shown that, from an educational point of view, each of these representations is quite different from the others since students understand earlier and more easily some of them, while they have strong difficulty to grasp others. I analyze here the particularities of each plane representation and I make some suggestions to teachers about the way and time for using them.

THE TYPES OF PLANE REPRESENTATIONS

Plane representations of 3D objects are used in many areas of human activity. We usually think first of engineering and architecture, but there are many other professional areas where they are used, like medicine, geography, statistics, etc. (see Gaulin, Puchalska, 1987), and, of course, the teaching of Geometry. When teaching plane representations in Space Geometry, the activities proposed to pupils are usually based on modules made of small cubes that can be fixed one to another (Multilink or Centicube, for instance). This particular kind of geometric solids provides a nice context for all the representations, since it is easy for children to build and modify them, and provide teachers with a big variety of shapes and grades of difficulty, appropriate for students of any age or ability level.

Usually, each activity has its own needs, and some particular types of plane representations. The following are the most used in the context of teaching Mathematics:

Perspective (Figure 2). This is the kind of drawing naturally made by children. Therefore, its use and the process of learning and improvement has some special characteristics. Mitchelmore (1980) provided us with a very accurate description of the stages in the development of children's ability for perspective drawing. From then, experiments in different countries have supported his findings. I will not talk about perspective drawings because of its great difference from the other types of plane representation from the instructional point of view.

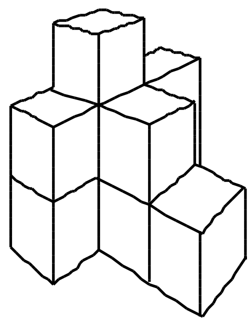
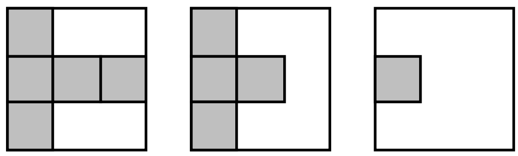


Figure 2. Perspective.

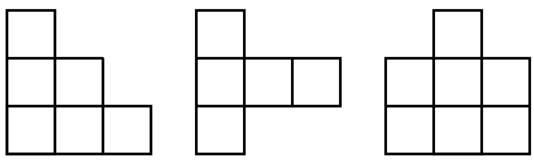
Layers (Figure 3). A representation by layers is made of several horizontal sections of the solid at some particular heights, in order to give an idea of the variations in shape and/or size from bottom (first layer) to top (last layer).



1st layer 2nd layer 3rd layer

Figure 3. Layers.

Orthogonal or *Side Views* (Figure 4). This kind of representation is very usual in technical drawing. The object is supposed to be into a cube, and projected orthogonally on the six faces of the cube. Each projection is one of the side views. In Geometry, only thee side views are usually provided: Front, top, and left or right sides, since each pair of opposite views (front/back, top/bottom, right/left) are symmetric.



Front Top Right

Figure 4. Orthogonal (side) views.

Coded Orthogonal or *Coded Side Views* (Figure 5). As orthogonal views are projections, information about some characteristics of the solid, as depth or slanted faces, are lost. To avoid this problem, some times additional graphical or textual information is added to the side views. For instance, plans of houses have some arrows added to show the directions of the roofs. When a module of cubes is represented, the relevant information that is worth to add to a side view is the number of cubes in each row perpendicular to that side. For this reason, we have named them “numeric views”.

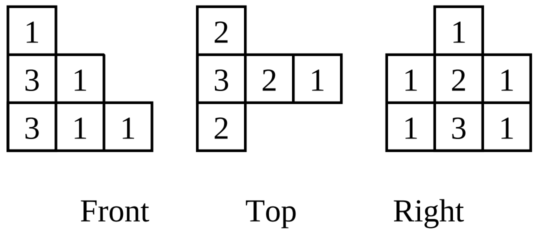


Figure 5. Coded orthogonal (numeric views).

Isometric (Figure 6). This is a type of parallel projection in which the three cartesian axes form angles of 120°. Isometric drawings are usually made on a net of equilateral triangles (isometric net), with the convention that the vertices of the solids have to match the points of the net. In particular, the isometric representation of a cube is a regular hexagon divided into three equal rhombuses, the visible faces of the cube.

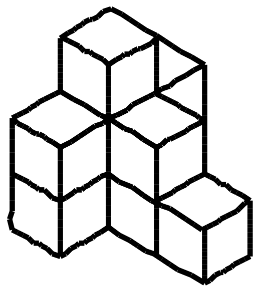


Figure 6. Isometric.

AN EXPERIMENT FOR TEACHING PLANE REPRESENTATIONS

In this paper I will report on a research experiment aimed to analyze how children use the above mentioned plane representations of modules. We were interested in observing the cognitive processes of learning to use each above mentioned plane representation, the difficulties, and the differences among the representations.

The tasks

Learning to use a type of plane representation is a two-way job: Students have to learn to draw plane representations of given solids, and also to build solids from given plane representations. Then, we designed activities in both directions:

- Given different plane representations of modules, the students had to build the corresponding physical modules.
- Given several modules, the students had to draw the different types of their plane representations. Usually we provided students with real modules, although some times we provided them with perspective views of modules on the computer; in this case, the modules could be freely rotated, so students were able to watch the modules from any position they liked.

We proposed to our pupils a third kind of activity, in which they had to relate two different types of representations, without building the module. For instance, given the isometric view of a module, students were asked to draw the representation of the module by layers. However, I do not refer here to these activities.

The students

In this experiment participated groups of students in Grades 2, 4, 6, and 8 of a Primary School near the University. Three students in each grade (two in grade 6) were selected so to have a range of ability levels in each group. The selection was made by their mathematics teachers on the basis of the students’ level in mathematics, an they ranged from low average ability (↘) to high ability (↗).

Each group of students from a grade worked independently of the groups from other grades. The children in each group worked alone on each activity, but when they finished, the should compare their results and talk about their ways of solution and the differences in their answers, if any.

Table 1
The pool of students.

	No. of	**Sex**		**Ability**			**No. of**
Grade	**students**	**M**	**F**	↘	=	↗	**Sessions**
2nd	3	1	2	1	1	1	21
4th	3	0	3	1	1	1	23
6th	2	1	1	0	1	1	30
8th	3	0	3	0	2	1	22

The children had no relevant previous instruction in this kind of activity. Those in higher grades had studied Space Geometry in their ordinary classes, and they knew names, characteristics, and basic properties of the usual polyhedron and solids of revolution. Someone of them knew how to draw a cube (the usual technique of drawing two congruent squares and linking their vertices).

Students were provided with all the necessary tools: Sheets of blank and isometric paper, pencil and rubber, Multilink cubes, and a booklet with the text and pictures of the different activities. We took all the children’s drawings, and the sessions were video-recorded for later analysis.

RESULTS AND CONCLUSIONS

Students of different ages and ability levels have worked on the activities stated above, so we can obtain some conclusions with respect to the following questions:

1) Are there differences in the students’ behaviour when dealing with the diverse types of plane representations? Is there some typical conceptual error or misconception?

2) Are there differences in the difficulty of the tasks asking students to draw a certain representation or to build a module from this type of representation?

3) Is it possible to order the types of representations according to their difficulty for students in each age or ability level?

Different students’ behaviours

Each kind of representation has its own difficulty for the students. Representations by layers are the easiest, but the students in lower grades had difficulties to understand the necessity of a relationship between the layers. We used the metaphor of a building, being each layer the plan of a floor, and each square a room. However, the young children are not able to coordinate the representations of the different layers, since they do not understand the meaning and utility of the framework that goes around each set of squares. Figure 7 shows an example of this behaviour; each layer is correct, but the framework is wrong since it does not shows the relationship between the three layers.

Figure 7.

In order to overcome this difficulty, we provided students with sets of blank layers, we asked them to draw several layer there, just crossing the appropriate squares, and we explained to the students with examples the necessity of the framework. They did correctly these cases (Figure 8), but when were asked again to draw sets of layers on blank paper, they did the same mistake as before (Figure 9).

These students had the same kind of difficulty when building modules from representations by layers. They were able to build correctly each layer, although the youngest made mistakes because they counted wrongly the number of cubes in a row. But they were unable to coordinate correctly the three layers, since they did not realize how to determine the place of the second layer on the first one, and the third layer on the second one.

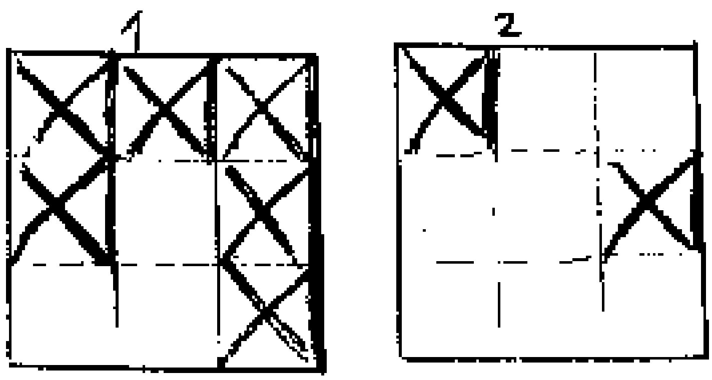


Figure 8.

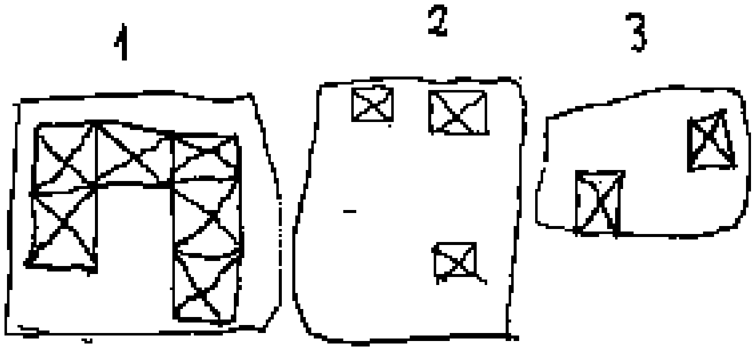
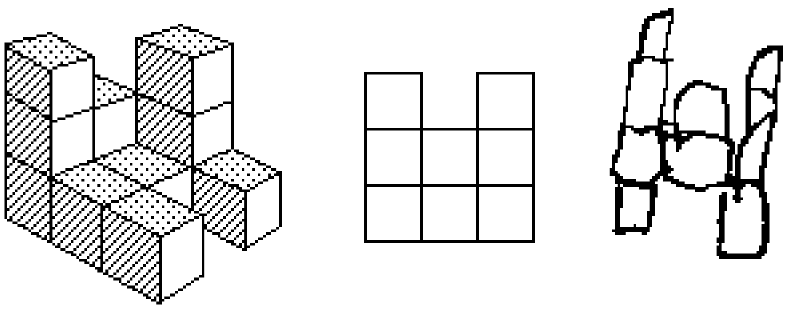


Figure 9.

Side and numeric views are an easy kind of plane representation when students have to draw it, but it is the most difficult for building the solids from, since it is necessary to coordinate the three views (or two views and the number of cubes in each row), and this can only be made by visualizing parts of the module to be built or by learning some sophisticated technique. This lack of coordination often resulted in students building a module symmetric to the real one.

The drawing of side or numeric views was for the students, in particular for those in 6th and 8th grades, a task as easy as that of drawing layer. But this type of representation highlighted another kind of difficulty for younger students: They were not able to isolate the faces in a plane from the faces they were seeing in other planes, that, therefore, had not be drawn. In other words, students in second and fourth grades showed that they had not acquired the ability of figure-ground perception (Gutiérrez, 1994). As can be recognized in Figure 10, these students tended to draw something in between perspective and orthogonal projection.



- a - - b - - c -

Figure 10. The module (a), its right view (b), and the student's drawing (c).

When building modules from side views, it was usual for students to select one of the views and to build a module (usually plane, that is, with all the cubes in the same layer) corresponding to the shape of that view. Afterwards, they selected another view and changed the positions of some cubes or added new ones trying to obtain a module fitting both the first and second views. Finally, the students checked their module against the third view and, if necessary, they tried to modify the solid, by adding or moving cubes, so it fitted the three views.

This is the theoretical plan most of the students followed, but having a good plan is not a guarantee for success, that is to build an appropriate module. Students in second grade were only able to build modules matching the first side they selected. They tried to match the second side by adding cubes in just a direction and a layer, so they were unable to match simultaneously the first and second sides. Students in higher grades were able to coordinate two sides, but they only can build a right module in the easier cases.

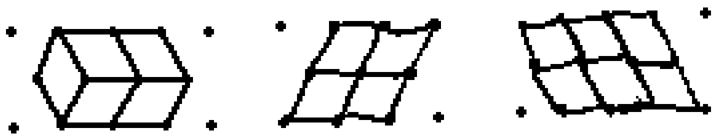
When building solids from numeric views, the procedure was quite similar, and also the kind of difficulties, although now students matched easily the number of cubes in each row, but they failed quite often in matching the module to the shapes of both views.

When students had to deal with isometric representations, their difficulties were in the opposite way to those in side or numeric representations: The isometric representation was the most difficult to be drawn, while building modules from their isometric representations was much easier than from side views, specially for students in Grades 4 and 6. For these activities, students were provided with sheets of isometric paper, pencil and eraser.

The difficulty of drawing isometric representations comes from the necessity of coordination between the different planes of the faces. From a practical point of view, this is translated into the necessity of coordination of the directions of perpendicular segments.

The activity of drawing an isometric view was very difficult for all the students when they tried it for the first time, so we trained them in this particular technique of drawing by asking them, first, to copy from paper to paper some very easy isometric representations (a cube alone or a module with just two cubes), and later to draw the representation of the same real modules. Even in such a simple tasks, the 2nd and 4th graders were quite unsuccessful. Only the more able 4th graders, and the 6th and 8th graders mastered reasonably the drawing of this representation.

Figure 11 shows two attempts of a student who was asked to copy the drawing in Figure 11-a.



- a - - b - - c -

Figure 11.

In the first attempt (fig. 11-b), the student began the drawing from the bottom right face of the module, and in the second attempt (fig. 11-c) the student began from the top right face. In both cases, the student's inability for modifying the direction of the segments was surprising for us.

Difficulty of each representation

As a global result, we observed that, for a given kind of plane representation, to draw and to build have different degrees of difficulty. The only exception is the representation by layers, since the same students who made mistakes in drawing layers also made mistakes when building from layers.

This is summarised in Tables 2 and 3, that present the degree of difficulty of each kind of plane representation for the students who participated in this research experiment.

Table 2
Students’ ability for drawing each kind of representation, from high (■) to null (❑).

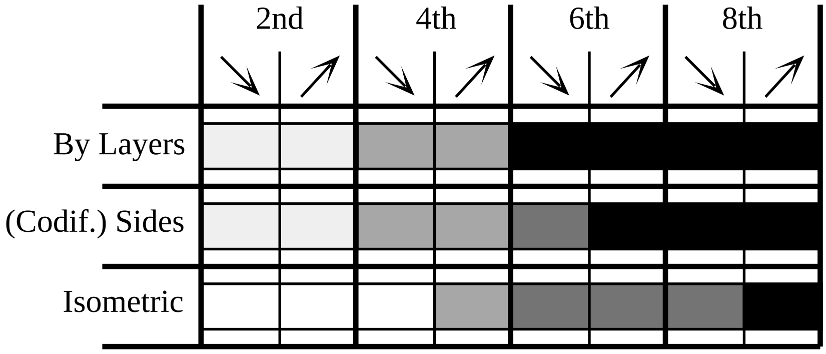
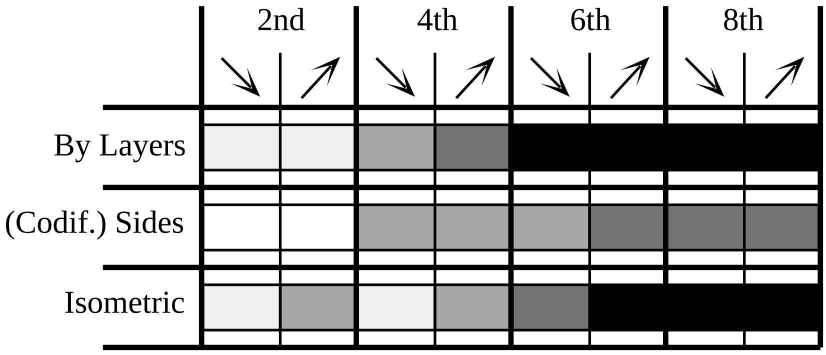


Table 3
Students’ ability for building from each kind of representation, from high (■) to null (❑).



From Tables 2 and 3, the following conclusions were made.

- There are important differences in the difficulty between building solids and drawing their plane representations, with the the representation by layers being the only exception. However, we cannot conclude that drawing is easier than building, nor vice versa, since drawing side views is easier than building from side views, but drawing isometric projections is more difficult than building from an isometric representation.

- Building from (coded) side views is the most difficult activity, since even the 8ht graders had difficulties with this task. On the other side, layers is the easiest kind of representation.
- Sixth grade is the first time where teachers have a reasonable probability of success, since second and fourth grades failed in mastering each kind of representation.

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