AN ANALYSIS OF THE STUDENTS' USE OF MENTAL IMAGES WHEN MAKING OR IMAGINING MOVEMENTS OF POLYHEDRA

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Abstract

We report a part of a research project focused on the analysis of the mental images, processes and abilities of visualization used by primary school students when they manipulate polyhedra. Students worked in a hands-on/paper/computer environment integrated by real solids, plane representations, and 3D software. The tasks presented here asked the students to rotate some polyhedra and to imagine their position after some rotations. The main part of the paper is devoted to describe and analyze the characteristics of environment and tasks.

The students exhibited ways of solving those tasks that reflect different levels of proficiency in the use of abilities and processes of visualization and mental representation of spatial relationships.

Introduction

During the last years we have carried out a research project* aimed to develop units for teaching space geometry in Primary Schools. As visualization and plane representations are two basic components of the learning of space geometry, a part of the research focused on the ways students build, manipulate, and draw 3-dimensional objects. We created a microworld, integrating real solids, plane representations, and 3D software, in which different types of problems with polyhedra were proposed to the students: Movement, comparison, plane representation, and building up of solids from plane representations. Because of the limited space, we don't report here the whole research project, but a group of tasks in which students were supposed to build mental images of polyhedra and to make mental rotations of them.

The aims of this report are a) to identify different kinds of students' behaviours arisen when they worked on tasks dealing with the movement of polyhedra, and b) to analyze how these behaviours correlate with the use, or lack of use, of different mental images, processes or abilities of spatial visualization. In the following pages we describe the part of the microworld relevant for the tasks described, present the procedures used by our pupils for solving the tasks, and analyze these procedures in terms of the processes or abilities of visualization used by the students.

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A theoretical framework

Spatial visualization has been studied from several approaches. The one most directly related to Mathematics Education and our research distinguishes three components of people's spatial visual activity: Images, processes, and abilities.

Presmeg (1986) defined a visual image as "a mental scheme depicting visual or spatial information". She identified several kinds of visual images; the most relevant ones for our research are:

- *Concrete images* are pictorial figurative images of physical objects. Our pupils dealt ever with this kind of images of the polyhedra they manipulated.

- *Kinaesthetic images* are images involving the physical movement of hands, head, etc. Our pupils very often moved their hands to represent a rotation before choosing the appropriate button on the computer screen, or while explaining to a researcher or to another child the movements they had just made.

- *Dynamic images* are images involving the mental movement of an object or some of its parts. These images were necessary for the students' work, since in most of the activities they had to rotate polyhedra or compare two different positions of the same solid.

Visual images are the units manipulated by people when they make a mental activity of spatial visualization. So when a person manipulates visual images, there is a flow of information between external objects and mental images or between different images. The main visualization processes controlling this flow were defined in Bishop (1983; 1989)*. Some of their components were relevant to our experiment:

- The process of *interpreting figural information* (IFI), that is, of reading, analyzing and understanding spatial representations (such as plane representations or mental images of polyhedra) in order to obtain some data from them.

- The *visual processing of images* (VP), that is, "the manipulation and transformation of visual representations and visual imagery".

We have also to consider different visualization abilities which may be used by students when solving 3-dimensional geometrical tasks. Del Grande (1987; 1990) made a summary of such abilities. The following ones were relevant for our study:

- *Figure-ground perception* is the ability to identify a specific figure by isolating it from a complex background. Our pupils used this ability when they had to identify faces of a transparent solid, since usually the edges created distracting shapes.

- *Perceptual constancy* is the ability to recognize that an object has invariant properties such as shape in spite of the variability of its position. We shall show below that one of our

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* Bishop (1983) defined abilities instead of processes, so he referred to the abilities to perform the corresponding processes.

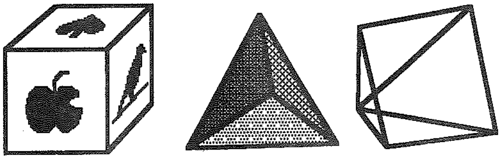
students had problems with respect to this ability.

- *Visual discrimination* is the ability to compare several objects, pictures and/or mental images, and to identify their similarities or differences. As the students were asked to move or compare polyhedra, the had to use this ability.

The environment

We devised a microworld based on different solids: Cubes, tetrahedra, octahedra, square pyramids, and rectangular prisms. These polyhedra were presented in several ways:

- The appearances of the polyhedra were (figure 1): a) Figurative opaque cubes, b) shaded opaque polyhedra and c) transparent (straw) polyhedra.
- There were 3 physical embodiments for the solids: Real solids (made of cardboard or straws), perspective 3-D representations on a computer, and perspective 3-D representations on paper (hard copies from the computer screen). The figures or shades on the faces were the same in the 3 cases (hard copies from the computer).



- a - - b - - c -

Figure 1.

We used computers Macintosh SE with 2 programs: 1) We created a Hypercard stack allowing to rotate a figurative cube 90° around the 3 axes crossing its faces (figure 2): When one of the 6 rotation arrows is marked, the computer automatically rotates the cube 90° in that direction.

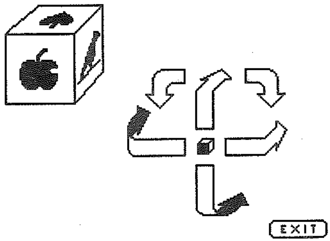


Figure 2.

2) The program Phoenix 3D allows to rotate freely solids around the axes X, Y, and Z (figure 3). After selecting one of the 6 direction arrows, the user has to determine the angle of rotation; for doing this, the software uses the metaphor of a hand turning in the chosen direction (in real time) a little pyramid that appears on the screen; when the user has decided the angle of rotation and releases the mouse button, the computer shows the new position of the solid.

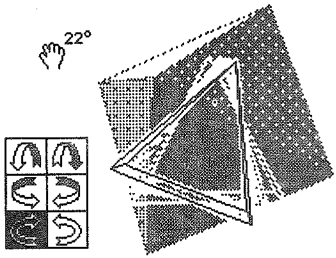


Figure 3.

The tasks

Several types of tasks, and several instances of each task, were proposed to the students, combining the different polyhedra, their appearances and embodiments, and the computer programs: Some times the students had to move a real solid and other times one on the computer; some times they had to compare a real solid to one on paper, other times to one on the computer, and other times solids on paper and the computer; etc. In this report we shall focus on two of those types of tasks: The first task asked the students to move a solid on the computer screen, from its current position into another position shown on paper. An example is this activity:

"Open the [Phoenix 3D] *file "Solid Tetrahedron". Move the tetrahedron on the screen to the first position you see below* [figure 4]. *Then, move it to the second one, and so on."*

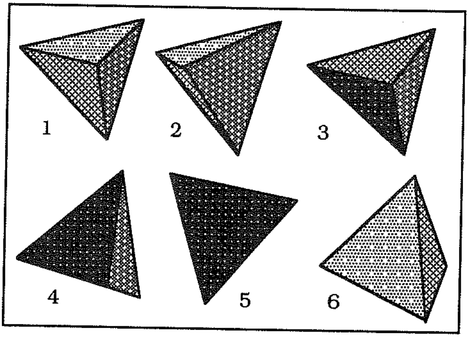


Figure 4.

For the second task, the students used a sheet of paper containing a picture of a figurative cube and several other views of the same cube with some blank faces. The students were asked to draw, if possible, the missing figures on the blank faces. An example is this activity:

"On the right there is a picture of a cube, and below there are some more pictures of the same cube [figure 8]. *Try to draw the figures on the blank faces of those cubes."*

Both tasks are related since the students should manipulate mental images of solids in both of them. The main difference between the tasks is due to the fact that the solid used can be rotated in the first task but no in the second one.

The subjects

Three 6th grade students (11-12 year olds) participated in the research. They had different ability levels: C was a high ability girl, E was an average ability boy, and M was a low-average ability girl. They had not received any previous instruction on techniques of visualization (mental rotations, plane representations, etc.). Their knowledge about 3-dimensional geometry previous to this experiment consisted on the learning of the main families of solids (prisms, pyramids, ...), their elements (faces, edges, ...) and basic properties (equalities, perpendicularity, ...). This knowledge was useful for our experiment, since the students only manipulated known polyhedra, so they did not suffered any difficulty for a lack of knowledge of some polyhedra's property.

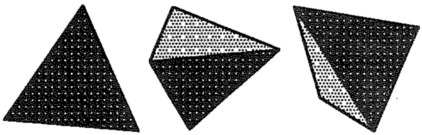
In the tasks not requiring computers, each student worked alone, but interactions among them were allowed. When the tasks required the use of computers, as only 2 computers were available, C and E worked together, and M worked alone. This grouping was decided because when M and another child worked together, as M was the less able, it resulted in M's inhibition, being the other student who solved most of the activities.

Analysis of answers

To analyze the diversity of the students' answers we have taken in consideration the different elements of visual thinking involved; such analysis should help us to understand how the students' capabilities of visualization develop. In this section we show and analyze some students' answers to the tasks described above. The students carried out the first type of tasks (movement of solids on the computer) by using two different strategies, reflected in the excerpts included below:

• M had to move the shaded tetrahedron from its current position on the screen (figure 5-1) to the position shown by the model, a real tetrahedron (figure 5-2). The following dialogue took place between M and a researcher (R):

- [First M moved the solid to position 5-3 by and]



1 2 3

Figure 5.

- M: *Ok.*
- R: *Is it in the same position?*
- M: *It is not the same face.*
- R: *This vertex here* [pointing to the model], *which one is here?* [pointing to the screen]
- M: *Ah! It is upside down.*
- [M made several rotations , but at the end the tetrahedron was nearly in the same position as before these turns]
- M: *Ok.*
- R: *Are they in the same position? It is the same as before. You have left it in the same position.* [M started turning the solid by] *Tell me. What are you looking for?*
- M: *I don't know.* [she continued rotating the same direction] *Look. It is wrong.*
- R: *Is that movement useful to you?*
- M: *No, because …*

- [R had to guide M, but she continued making complete rounds in a direction]

• In a similar activity, C and E had to move the shaded pyramid from its current position on the screen (figure 6-1) to the model's position (figure 6-2). E made the movements although, almost every time, C decided the movement to be made. This was the dialogue between C and E:

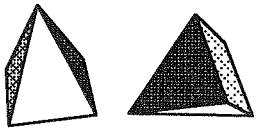


Figure 6.

- [C and E had moved the pyramid to the position in figure 7-1]

- C: *Now it has to be moved so that this* [pointing to the vertex in the screen] *goes there* [pointing to the vertex in the model].

- [E made the rotation , and the pyramid moved to position 7-2]

- C: *I see that. Now it has to be moved down* [E moves the pyramid by rotation , to position 7-3].

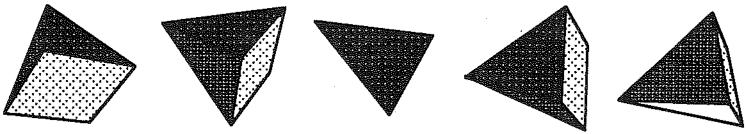


Figure 7.

- C: *Ok. Now, put it horizontal, horizontal. Do you know?* [C "drew" an horizontal line with her finger in front of the screen, and E moved the pyramid by ].

- C: *And now we have to see this face* [the base].

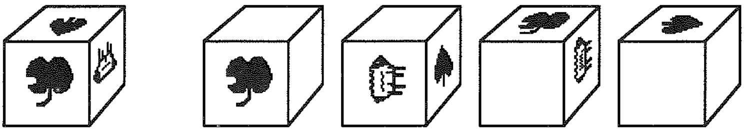
- [C directed E to rotate the pyramid by  and  to reach position 7-4. Then C realized that they had mistaken the model's front face for another with a similar shade. Then she continued directing E in the same way as for the previous excerpt until they put the pyramid in position 7-5]

While M usually made a search by guess, just rotating the solid until it happened to stop in a position resembling the model's one, C and E used efficiently the IFI and VP processes, pre-planning the search and deciding which movement (or sequence of 2 or 3 movements) should be done by comparing the positions of the movable solid and the model.

• When the same task had to be done with a figurative cube, all the 3 students used the same strategy: They moved the cube until the figure on the front face of the model appeared on the screen; then, they moved the cube to put that figure on the front face; finally, they turned the cube to put this figure in the correct position.

But having a correct strategy did not imply an accuracy in the movements: While C never needed to move the cubes more than a few steps, M some times made very long sequences of rotations and other times short ones. There is never a need of more than 5 rotations to move the cube to the position of the model, but often M made up to 10 rotations (in an occasion she needed 21 turns!). Such different behaviours mean different levels of proficiency in the use of visual abilities and processes: While C was able to transform correctly her mental visual movements into real rotations on the computer (i.e., to use properly the IFI process), many times M was unable to perform adequately such process.

• Finally, we refer to an E's answer to the second task described, in which it was evident that E had not completely acquired the ability of perceptual constancy. The students were provided with a sheet of paper containing a picture of a cube (figure 8-M), and other views of the same cube with some blank faces (figures 8-1 to 8-4). They were asked to draw, when possible, the missing figures on the blank faces.



M 1 2 3 4

Figure 8.

The students could not move the model, so they had to imagine the shapes and positions. In the cube 8-4, E drew an apple on the front face and a sheep on the right one. This is a part of the dialogue between E and a researcher:

- R: [watching E's drawings] *It is not clear to me. Where does the sheep look at?*
- E: *Downwards.*
- R: [pointing to cube 9-M] *Doesn't it look at the apple?*
- E: *Yes.*
- R: *But here* [cube 9-4] *no longer.*
- E: *Of course, because it moves.*
- R: *But do all faces move at one time, or only one face does?*
- E: *This* [face] *moves in this way, then that* [face] *moves in this way.*
- R: *And, does the sheep continue looking at the apple or no longer?*
- E: *No longer.*
- R: *Then, by turning the cube, can you make the sheep look at different figures?*

- E: *Of course.*

Conclusions

• The use of 3 different types of solids (figure 1) was relevant in this experiment, since it was evident that the figurative cubes were confidently manipulated by our pupils, while the transparent solids were the most difficult for them. So students of different ages or ability levels shall react in different ways to the various kinds of polyhedra.

• When selecting software for moving solids, it is important to know how each program asks for data. Some programs allow only rotations of a specific angle; others represent graphically the rotation's amplitude (no input for angles is needed); and others ask to key in the value of the angle to be rotated. The first type of software is the best for young children, but it may inhibit the development of analytical reasoning. The third type is the best for good visualizers, who are able to imagine complex movements and to calculate accurately the angles of rotation. The second type is good for students in an intermediate stage of development of their capabilities of visualization.

• Different types of activities have to be proposed to students to promote a complete development of their visual reasoning: Movements, comparisons, drawing of, and building from, plane representations are types of activities in which different kinds of mental images, abilities and processes have to be used by the students.

• Some questions have arisen from this exploratory research about the students' behaviours and mental activities: The use of analytical or visual ways of reasoning. The kinds of mental images (for instance, static or dynamic, global or local) generated by students of different ages or abilities. These questions should be investigated in future phases of the research.

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