ANGEL GUTIÉRREZ and ADELA JAIME

PRESERVICE PRIMARY TEACHERS' UNDERSTANDING OF THE CONCEPT OF ALTITUDE OF A TRIANGLE

ABSTRACT. The way teachers understand mathematics strongly influences their teaching and what their pupils learn. Using Vinner's model of acquisition of mathematical concepts with its distinction of concept image and concept definition as a framework, we analyze concept images, difficulties, and errors related to the concept of altitude of a triangle exhibited by 190 preservice primary teachers in a written test. We describe the influence of two variables on the preservice teachers' performances: (a) the presence of a formal definition and (b) previous classroom activities that dealt with the concept of altitude. We categorize and analyze some common errors and identify the concept images that may lead to those errors. Finally, we present some implications of our results for teacher education.

One of the main foci of mathematics educators with respect to the geometry taught in primary and secondary school (in Spain, Grades 1–6 and 7–12, respectively) is to understand how geometrical concepts are learned by students at different educational levels. Strongly related to this focus is the interest of mathematics educators in understanding the learning processes of preservice and inservice teachers with respect to the geometric concepts and properties they are supposed to teach. In research on teachers' understanding of basic geometrical concepts, we find different approaches according to the context, the teaching methodology, the organization of the mathematical knowledge, and the kind of activities. This diversity is a consequence of the complexity of the problem, even for those concepts considered basic. The literature on preservice teacher education is quite extensive. In particular, several handbooks of research include overviews of this literature from varied viewpoints: Brown, Cooney, and Jones (1990) and Shulman (1990) discuss different research perspectives, or paradigms, to approach preservice and inservice teachers' training and beliefs. Clark and Peterson (1990) review results on teachers' thinking processes. Fennema and Franke (1992) review research on the relationship among teachers' knowledge, their teaching strategies, and their pupils' learning. Lanier and Little (1990) and Brown and Borko (1992) discuss issues related to teacher training processes.

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In this paper we present a study aimed at providing information about preservice primary teachers' conceptions of altitude of a triangle. Related to the objective of this paper, Brown, Cooney, and Jones (1990) suggested that often preservice teachers do not have the knowledge and understanding of mathematical content necessary to apply the methodological innovations proposed by new curricula. In addition, the limited mathematics knowledge of preservice teachers is an obstacle for their training on didactical and pedagogical knowledge (Brown & Borko, 1992). The results in Mason and Schell (1988) corroborate these findings for preservice primary teachers with respect to geometrical concepts.

Within the USA and Portuguese contexts, respectively, Mayberry (1983) and Matos (1985) studied the van Hiele levels of reasoning of preservice primary teachers on several geometric concepts. In Mayberry's study, based on clinical interviews, 48% of the American teachers showed reasoning at Level 2 or lower, 16.5% at Level 3, and 19% at Level 4. In Matos' study, based on a written multiple-choice test, 59% of the Portuguese teachers showed reasoning at Level 2 or lower, 18% at Level 3, and 4.5% at Level 4 or 5. In both studies, more than half of the preservice teachers were unable to understand deductive arguments, even informal ones, or to analyze definitions in order to classify families of polygons. Both researchers found a positive correlation between the van Hiele level assigned to teachers and their mathematical backgrounds in secondary school. In our experiments with Spanish preservice primary teachers (Gutiérrez & Jaime, 1987; Gutiérrez, Jaime, & Fortuny, 1991), we have obtained results that are similar to those in Mayberry's (1983) and Matos' (1985) studies. It seems that weak mathematical knowledge and reasoning skills of preservice primary teachers are independent of countries, educational systems, and techniques of assessment.

Linchevsky, Vinner, and Karsenty (1992) and Vinner, Linchevski, and Karsenty (1993) have examined in detail the conceptions preservice teachers have of mathematical definitions. Preservice high school teachers were asked to complete a questionnaire in which they were asked (a) to define an equilateral triangle and a rectangle, (b) to evaluate the correctness of several definitions of these concepts written by high school students, and (c) to reflect on a dialogue among a teacher and a student on the minimality of mathematical definitions. After the preservice teachers had completed the questionnaire, the researchers held a group discussion on the minimality of definitions, that is, the sufficiency of properties included in the definition, and on their arbitrariness, that is, the existence of *canonical* definitions and alternative equivalent definitions. The analysis of the answers showed that 65% of the preservice teachers believed that a definition could include non-necessary conditions, and only 33% of them understood the minimality principle. The students in the first category matched van Hiele Levels 2 or lower, and those in the second category matched Levels 3 or higher.

Based on their research, S. Vinner and R. Hershkowitz developed a model to explain the cognitive processes individuals use when they face new mathematical concepts, in particular, geometrical concepts. One of the core components of this model is the difference between a mathematical concept, the concept image created in a student's mind, and the concept definition verbalized by the student (Vinner, 1991; Vinner & Hershkowitz, 1980, 1983). According to this model, a good learning process results in the two elements, concept image and concept definition, being merged which then allows students to correctly discriminate examples of the concept. Unfortunately, however, for many students there is no link between the two elements, and students use one or the other according to the given task. In the next section, we present the Vinner model, which we used as a framework for our research. We then describe a study in which we analyzed preservice primary teachers' knowledge and understanding of the concept of altitude of a triangle and the influence of different variables on their understanding of the concept. In conclusion, we identify implications from our research for teacher education.

ON THE FORMATION OF GEOMETRICAL CONCEPTS

The van Hiele model of levels of geometric understanding is recognized as one of the most comprehensive models with respect to the learning of geometry. The Vinner model (Vinner, 1991; Vinner & Hershkowitz, 1980, 1983) is another useful framework to guide teachers and researchers in their activity of understanding students' mental processes. According to the latter model, when we listen to or read the name of a known concept or when we solve some task, our memory is stimulated and something is evoked. However, what is evoked is rarely only the formal definition of the concept, but rather a set of visual representations, images, properties, or experiences. This set of elements that can be recalled constitutes the *concept image*. For geometrical concepts, a student's concept image may include various figures the student remembers as examples of the concept, and the set of properties the student considers belonging to the concept. A student's concept image is viable when it allows the student to discriminate, without error, any example of the concept and when the associated properties are all necessary properties of the concept. On the other hand, as a result of the teaching methods, students may memorize a definition, which they may repeat when they are asked for it or if they are asked to identify an example. This verbal definition, which can be memorized and repeated by a student, is called the student's *concept definition*.

The properties included in a concept image are not always mathematical properties; irrelevant physical properties may also be considered, in particular by students who reason at the first van Hiele level (Burger & Shaughnessy, 1986; Clements & Battista, 1992; Fuys, Geddes, & Tischler, 1988; Gutiérrez et al., 1991). For instance, for many primary school children, the concept image of a triangle consists of a set of specific triangles in standard position and several properties of these figures, such as a triangle having a vertical (right) angle or slanted sides of equal length (Hershkowitz, 1989). The presence of irrelevant properties in students' concept images has also been reported by Wilson (1986) who studied the influence of irrelevant features of figures on the usefulness of counterexamples.

The concept definition expressed by a student may or may not coincide with the definition of the corresponding mathematical concept. On the other hand, the concept definition is not necessarily operationally linked to a student's concept image when the student is solving a task. For instance, when asked to define rectangles, many students include the condition that not all the sides are of the same length, although these students identify squares as rectangles when they are presented with figures. In contrast, other students state the definition of a rectangle as a parallelogram with right angles, but they do not accept squares as rectangles because all sides are of the same length (Wilson, 1990). Both behaviors evidence the differences that exist for students between concept image and concept definition.

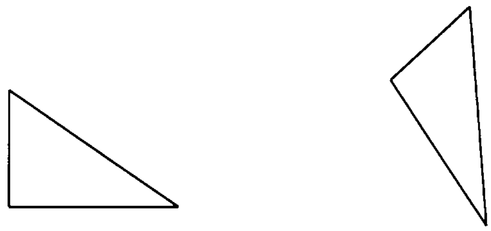
The Vinner model emphasizes that a student's experiences and the examples of a concept encountered, either in school or in other contexts, play an important role in the formation of a concept image. Very often students are given only a few examples of a geometrical concept, all of which have a common specific visual characteristic; these examples then become prototypes (Hershkowitz, 1989), and they are the only references available when the student is judging new cases. Therefore, a way to improve the quality of concept images is to try to detect the failings of students' concept images, to offer them a wider variety of examples, and to take into account especially those examples directly related to these errors. Unfortunately, many teachers suffer the same errors as their students, and so they cannot help them. An objective of this research is to provide specific directions on how to analyze teachers' concept images of a specific geometrical concept, the altitude of a triangle, as a first step in the improvement of their knowledge of this concept.

The Vinner model has been applied in several studies. For example, Vinner (1983, 1991) observed students who solved tasks based on concepts from calculus, such as function, tangent, continuity, and limit of a sequence. Vinner and Hershkowitz (Hershkowitz & Vinner, 1982, 1983; Vinner & Hershkowitz, 1980, 1983) studied primary and secondary school students' geometrical concepts. In some studies (Hershkowitz, 1989; Hershkowitz & Vinner, 1984), students as well as preservice and inservice teachers were included in the samples, and all solved the same tasks. Two types of tasks were used: (a) given sets of figures, participants were asked to identify examples of concepts such as angle, isosceles triangle, right triangle, diagonal, or quadrilateral, and (b) participants constructed examples of concepts, e.g., altitude of a triangle. In some experiments, a part of the sample was provided with a test that included the definitions of the concepts, whereas the other part of the sample received the same test without the definitions.

The main conclusions relevant to our study are:

- When teachers, preservice teachers, and students were asked to solve the same tasks, the same misconceptions were found in the three samples; the preservice and inservice teachers' concept images were only slightly better than those of the students (Hershkowitz, 1989; Hershkowitz & Vinner, 1984).
- The concept images of many teachers, prospective teachers, and students are based on a few prototypical figures and are independent of their concept definitions. For instance, a very common concept image of a diagonal of a quadrilateral includes only internal diagonals; consequently, only one diagonal is commonly drawn in concave quadrilaterals (Hershkowitz, 1989; Hershkowitz & Vinner, 1984; Vinner & Hershkowitz, 1980).
- With respect to identification tasks, the presence of the definition of a concept has almost no influence on the responses. However, in a task of construction of altitudes of triangles, the answers of subjects who were provided with the definition were significantly better (Hershkowitz & Vinner, 1982, 1983, 1984; Vinner & Hershkowitz, 1983).

Wilson (1988, 1990), in her studies of students in Grade 6 and above, got results similar to those of Vinner and Hershkowitz. Wilson asked students to draw examples of given geometric concepts, define the concepts, select examples among several figures, and answer questions about the concepts. The concepts selected were triangle, rectangle, and square. Wilson reported that students based their answers on inflexible prototypical instances. She also observed the presence of many



(a) (b)

Figure 1. Typical (a) and atypical (b) textbook examples of a right triangle.

inconsistencies among students' answers to the different parts of the test. Follow-up interviews showed that some students were aware of such inconsistencies, but they did not feel uncomfortable with them. Although Wilson did not use the Vinner model as the conceptual framework for her studies, her results can be interpreted in terms of that model. In particular, students used their concept images in some tasks and their concept definitions in other tasks. They were aware of the contradictions among images and definitions, but they were comfortable because they saw concept definitions and concept images as two independent tools.

In light of the Vinner model, three types of behaviors can be identified, according to the quality of the associated concept images (Hershkowitz, 1990):

- Individuals with the poorest concept images, developed from a few prototypical examples and visual properties, base their judgments on the visual aspect of these prototypes, compare them to the figures with which they have to work, and reject as examples those figures that do not coincide with the prototypes of their concept image. For instance, the concept images associated with a right triangle often include only those triangles with a horizontal and a vertical side (Figure 1a). Consequently, many students, preservice, and inservice teachers have great difficulty identifying right triangles without horizontal and vertical sides (Figure 1b) (Hershkowitz, 1989).
- Individuals with somewhat richer concept images make judgments based on a few prototypical examples plus some mathematical properties of those examples. They try to apply these properties to the figures with which they have to work, and they reject those figures that do not fit these properties. For instance, for many students, the concept image of a triangle includes the property that the triangle is an acute triangle and, as a consequence, has only internal altitudes; therefore, these students tend to say that there is no altitude to side a of the triangle in Figure 2, or they draw an internal segment (Hershkowitz, 1989).

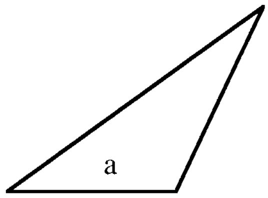


Figure 2. An obtuse triangle-frequently not part of a student's concept image.

- The third type of behavior described by the Vinner model corresponds to those individuals who have complete concept images, that is, images that include a wide variety of examples and all the important properties of these examples. The specific examples now play a complementary role to provide ideas or to verify conjectures which are later corroborated or formalized by the use of the mathematical properties. These students are able to make correct judgments based on the use and analysis of the critical properties of the concepts.

Finally, Vinner and Hershkowitz also showed that many teachers-at different educational levels-wrongly believe that their pupils' reasoning is based predominantly on the formal verbal definitions of the concepts, and that their concept images play a secondary role, subordinated, both in its formation and its use, to the concept definition. Probably these teachers take for granted as well that the concept definition used by their pupils is the mathematically correct one the students have been taught. However, the student activities are based, in most cases, only on their concept images because for many students the concept definition is inactive, except when they are explicitly asked to repeat it, or does not exist; they forgot or never learned the definition that they were taught by their teacher (Vinner, 1991).

PRESERVICE TEACHERS' CONCEPT IMAGES OF ALTITUDE OF A TRIANGLE

Based on Vinner and Hershkowitz's experiments on the concept of altitude of a triangle, we address and extend their conclusions. We present the results of a test designed to analyze preservice primary teachers' understanding of the concept of altitude of a triangle. We identify their reasoning processes and the influence of some variables, such as students' previous knowledge, the presence of a formal definition in the test, or the influence of learning activities that dealt with altitudes of triangles as part of the content of a course on mathematics education. The conclusions of this part refer to the identification of the preservice teachers' concept images and the way they make use of their concept images and the mathematical definition of altitude of a triangle in the resolution of specific tasks, tasks similar to those they will find in the textbooks when they begin their professional life.

Method

Instrument. The instrument for the study was a test made by rearranging the items used in Hershkowitz and Vinner (1982) and Vinner and Hershkowitz (1983). Originally, the test had been designed for students in Grades 6 to 8 (ages 11 to 14). One of the researchers' aims had been to investigate whether or not the presence of the formal definition of altitude influenced the students' answers. Vinner and Hershkowitz (1983) found that the presence of a definition in a construction task significantly improved results. In addition, irrelevant attributes of the triangles, such as position or shape, induced a significant number of wrong answers both when the definition was provided and when it was not provided (Hershkowitz, 1989). Incorrect responses were classified according to the critical attributes of the concept of altitude used by the students, but the reasons for the errors were not analyzed.

Figure 3 shows one version of the test used in our study. The alternate version differed from the one depicted only by the omission of the definition of the altitude. The definition of altitude used in the test was the one included in Spanish textbooks and taught by primary teachers. There were no time limits to complete the test.

Participants and procedure. The test was administered to four groups of students in the Primary Teacher Training School of the University of Valencia (190 students). Like all Spanish university students, these students had studied mathematics from Grade 1 to 10. Most students in the sample had also studied mathematics, as an optional subject, in Grades 11 and/or 12 of secondary school, although we do not have information about the exact amount. Euclidean geometry, however, is usually studied only in primary and lower secondary school (Grades 1 to 10). About half of the students in Group A (A2) were administered the version of the test that contained the definition of the altitude of a triangle. The other students in Group A (A1) took the alternate version of the test. All students in Groups B and C first took the test without the definition of the altitude. Immediately after completion, they took the test again, and this time the test did contain the definition. Students in Group D were administered the test twice without the definition, once before and once after they had worked on activities involving altitudes of triangles. Table I summarizes the information about the distribution of the students in the different groups.

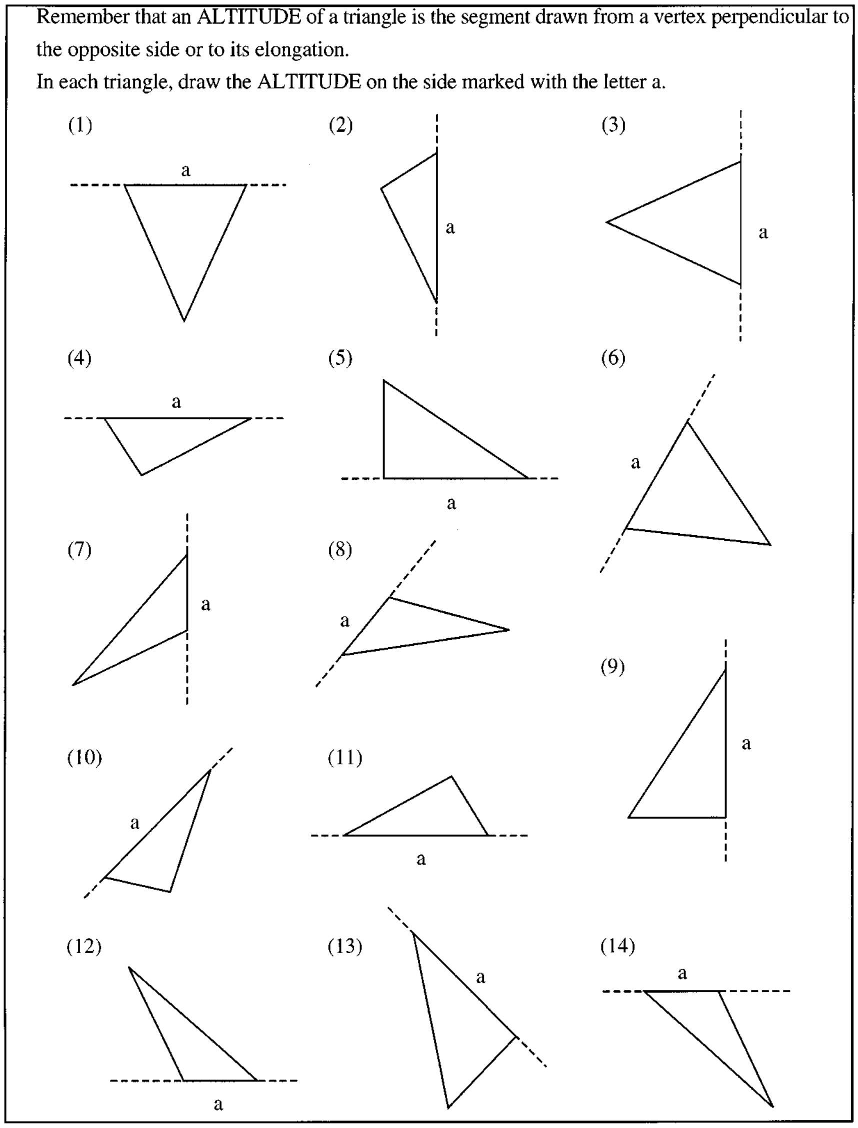


Figure 3. Test items including the definition of the altitude of a triangle.

Coding. To code the answers, we considered correct answers to be those that satisfied the mathematical requirements of the question. We allowed, however, a small error in the perpendicularity of the altitudes because the students were working without drawing tools. We coded as wrong answers those in which the segments did not meet the mathematical requirements

TABLE I

Distribution of the Preservice Teachers Participating in the Study

		Number of students taking the test		
Group	College year	without / with definition	pre-test / post-test	Total
A1	3rd	28 / 0	—	28
A2	3rd	0 / 31	—	31
B	1st	33 / 33	—	33
C	3rd	34 / 34	—	34
D	2nd	64 / 0	64* / 49	64
	Total	159 / 98	64 / 49	190

*15 students answered only the pre-test.

of the question or those in which the segment was clearly not perpendicular to the base. We analyze the various kinds of errors later in the section.

Influence of the Presence of the Definition of Altitude

Because the concept of *altitude of a triangle* is considered important by primary school teachers, we can assume that all students in the sample had studied the concept at some point during their primary school years. On the other hand, because Euclidean Geometry is not taught in upper secondary school, we can also assume that students had not formally dealt with the concept for several years. The administration of the test without definition of the concept of altitude allowed us to obtain information about what the students remembered several years after encountering the concept, and thus gain insights into the nature of their present concept images. If the definition was included in the test, it possibly served as a reminder for the students. That is, the definition might have activated the students' concept image, although this concept image might not have matched the given definition. In other cases, the presence of the definition might have allowed students to analyze the definition and thus enabled them to answer an item. Therefore, the version of the test with the definition allowed us to gain information about the change in the students' concept image fostered by the definition.

The results of the tests for Groups A1 and A2 (28 and 31 students, respectively), B (33 students), and C (34 students) are summarized in Figure 4 and Table II. The design used for Groups B and C proved to be richer than the one for Groups A1 and A2, because the twofold administration of the test, first without and then with the definition, provided us

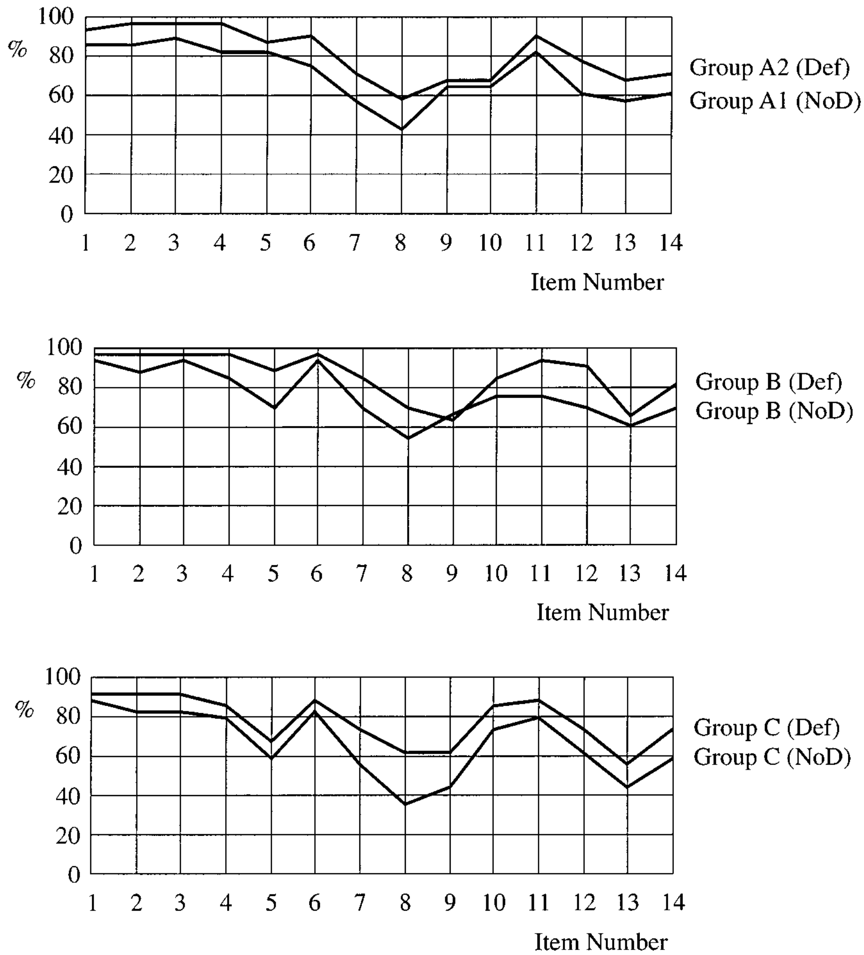


Figure 4. Percentages of correct responses in Groups A1, A2, B, and C.

with information about the changes in the ways students responded to the items before and after they had read the definition.

Correct responses. Overall, there were more correct responses in the test containing the definition of altitude than in the test without the definition, except for Item 9 in Group B. The definition seemed to provide the students with information that helped them improve their understanding of the concept of altitude. Although Groups B and C were administered the same test twice, we believe that the influence of the practice effect was minimal because students had unlimited time to check all responses and correct those they thought wrong before they returned the first test to the teacher.

TABLE II

Responses (%) to the Test Without Definition (NoD) and With Definition (Def) of Altitude

		Correct responses (%)				Incorrect responses (%)				No response (%)			
Item	Cond.	A1	A2	B	C	A1	A2	B	C	A1	A2	B	C
1	NoD	85.7		93.9	88.2	14.3		6.1	11.8				
	Def		93.5	97.0	91.2		6.5	3.0	8.8				
2	NoD	85.7		87.9	82.4	14.3		12.2	17.6				
	Def		96.8	97.0	91.2		3.2	3.0	8.8				
3	NoD	89.3		93.9	82.4	10.7		6.1	14.7				2.9
	Def		96.8	97.0	91.2		3.3	3.0	8.8				
4	NoD	82.1		84.8	79.4	17.9		15.1	20.6				
	Def		96.8	97.0	85.3		3.3	3.0	14.7				
5	NoD	82.1		69.7	58.8	17.9		24.2	35.3			6.1	5.9
	Def		87.1	78.8	67.7		12.9	15.1	26.5			6.1	5.9
6	NoD	75.0		93.9	82.4	25.0		6.1	17.6				
	Def		90.3	97.0	88.2		9.7	3.0	11.8				
7	NoD	57.1		69.7	55.9	42.8		27.3	38.2			3.0	5.9
	Def		71.0	84.8	73.5		29.1	15.1	26.5				
8	NoD	42.9		54.5	35.3	57.2		45.4	61.8				2.9
	Def		58.1	69.7	61.8		42.0	30.3	35.3				2.9
9	NoD	64.3		66.7	44.1	35.7		27.2	47.1			6.1	8.8
	Def		67.7	63.6	61.8		25.8	30.3	35.3		6.5	6.1	2.9
10	NoD	64.3		75.8	73.5	32.1		21.2	11.8	3.6		3.0	14.7
	Def		67.7	84.8	85.3		25.8	15.1	11.8		6.5		2.9
11	NoD	82.1		75.8	79.4	17.9		21.2	17.6			3.0	2.9
	Def		90.3	93.9	88.2		6.5	6.0	8.8		3.2		2.9
12	NoD	60.7		69.7	61.8	35.7		27.3	38.2	3.6		3.0	
	Def		77.4	90.9	73.5		22.6	9.1	23.6				2.9
13	NoD	57.1		60.6	44.1	39.3		36.4	52.9	3.6		3.0	2.9
	Def		67.7	66.7	55.9		29.1	30.3	41.2		3.2	3.0	2.9
14	NoD	60.7		69.7	58.8	39.3		27.3	41.2			3.0	
	Def		71.0	81.8	73.5		25.8	18.2	26.5		3.2		

Difficulty of items. About the same items seemed to be easy and difficult, respectively, for all groups. Based on the percentage of correct answers in each group, the easiest items (3, 1, 2, 6, 4, and 11, ordered from easy to less easy) are, except one, the triangles with internal altitudes. The most difficult items (8, 13, 9, 7, 14, 5, and 12, ordered from more to less difficult) are the triangles with non-internal altitudes, that is, obtuse or right triangles.

Influence of position. Items 10 and 11 are the same triangles, but rotated. The graphs show a clear difference between the percentages of correct answers for these two items from all groups of students. This can be interpreted as a consequence of the different rotation of the figure, the easier item (11) appearing in the prototypical position with a vertical altitude. The same effect can be seen for Items 7, 8, and 12, where the easiest item (12) is a triangle in the prototypical position, whereas the base of the triangle in Item 8, the most difficult one, is slanted.

Items with no responses. The number of items with no responses was low in each group, even when the test did not provide the definition. Most of the blank answers corresponded to Items 5, 9, 10, and 13. Three of those four items involved right triangles. The fourth, Item 13, was a triangle with an internal altitude, but the altitude had to be drawn in an unusual position. From the low number of items without a response, we can conclude that all the students had at least some concept image of altitude which they were able to recall.

Influence of Instruction on Students' Concept Images

Group D was administered the test without the definition of altitude both times, but before and after the students did some work with altitudes of triangles as part of the second year course of their teacher education program. The aim was to observe the influence of instruction that included attention to the altitude of a triangle.

The intervention. As a part of a course on didactic of mathematics, Group D worked one hour weekly during the year on problem-solving tasks based on Cabri (Version 1.7 for PC). The aims of this course were twofold: (a) to improve the students' knowledge and understanding of the mathematics taught in primary schools, and (b) to teach elements of didactic of mathematics students would need for their professional activity. During the instructional time, students solved geometric construction problems, made and proved or disproved conjectures, and designed simulations of

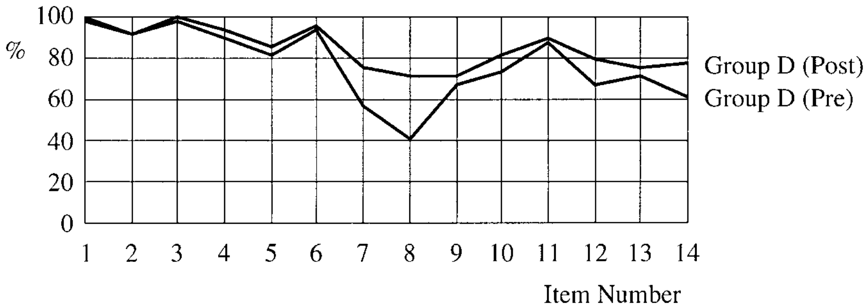


Figure 5. Percentages of correct responses in Group D.

geometrical properties studied in primary school. All tasks were related to basic geometrical concepts, such as triangles, quadrilaterals, parallelism, or perpendicularity. Computers were operated by groups of two or three students. For each task, students first completed the task on their own. In the whole-group discussion that followed the group activities, students discussed various ways of solving the problem, as well as their findings, conjectures, and difficulties. Finally, there was a phase of institutionalization by the teacher (Douady & Perrin-Glorian, 1989).

For three consecutive sessions, students worked on tasks involving triangles. In particular, students studied the construction of special line segments, such as altitude and median, and points (e.g., orthocenter, barycenter). They observed, conjectured, and verified the properties of segments and points. One of the properties studied was the position, internal or external, of the segments and centers as relative to various kinds of triangles. Although the concept of altitude of a triangle was used, its definition or characteristic properties were not explicitly stated by the teacher. Students drew altitudes by using the perpendicular line command in the Cabri menu.

Pre- and post-test. The pre-test was administered one week before students began the three classes devoted to triangles, and the post-test was administered two weeks after these classes. Figure 5 and Table III summarize the responses obtained from the 49 students who participated in both the pre-test and the post-test. Figure 5 shows the improvement in the results of the post-test in relation to the pre-test. It may be appreciated that the greatest differences between the correct answers on the pre- and post-test happen in Items 7, 8, 12, and 14, which are among the most difficult items. These data suggest that, in the courses for preservice teachers, instruction and directed activities influence students' concept images, as would be expected.

TABLE III

Responses (%) to Pre- and Post-Test in Group D (n = 49)

	Correct responses		Incorrect responses		No response	
Item	Pre-test	Post-test	Pre-test	Post-test	Pre-test	Post-test
1	98.0	100	2.0	0		
2	91.8	91.8	8.2	8.2		
3	98.0	100	2.0	0		
4	89.8	93.9	8.2	6.1	2.0	
5	81.6	85.7	16.3	14.3	2.0	
6	93.9	95.9	6.1	4.1		
7	57.1	75.5	42.9	20.4		4.1
8	40.8	71.4	57.1	26.5	2.0	2.0
9	67.3	71.4	30.6	26.5	2.0	2.0
10	73.5	81.6	26.5	18.4		
11	87.8	89.8	12.2	10.2		
12	67.3	79.6	32.7	20.4		
13	71.4	75.5	28.6	22.4		2.0
14	61.2	77.6	38.8	22.4		

Error Analysis

In this section we analyze the answers of the 159 students from groups A1, B, C, and D (pre-test) who answered the test *without* the definition of altitude (refer to Table I). Our aim is to present a catalogue of responses characteristic of different kinds of partial or poor concept images. The student drawings were coded as follows:

1. No answer.
2. The correct altitude on the specified base. A small error in the perpendicularity of the segment drawn was accepted.
3. The median to the specified base.
4. The perpendicular bisector of a side.
5. The correct altitude to a side different from the specified base.
6. A segment perpendicular to the specified base, but with the wrong length.
7. A segment internal to the triangle, from the opposite vertex to the specified base, but not perpendicular to it nor the median to the base. The difference between Codes 2 and 7 is that the sloping of the segment drawn is big enough to discard imprecision in the drawing.
8. Other incorrect responses.

TABLE IV

Incorrect Responses (%) With Respect to Error Type

Item	Code 3	Code 4	Code 5	Code 6	Code 7	Code 8
1	0	0.6	0	6.3	0	0
2	3.1	1.3	0	4.4	0	3.8
3	0	0.6	0	5.0	0	1.3
4	4.4	1.9	0	5.0	0.6	2.5
5	6.3	1.3	4.4	7.5	0.6	1.9
6	0	0.6	3.1	6.9	0	1.3
7	18.9	1.9	5.0	4.4	3.1	5.0
8	28.3	1.9	11.3	2.5	9.4	2.5
9	10.1	1.3	7.5	6.3	1.3	7.5
10	9.4	1.3	1.3	2.5	4.4	4.4
11	5.0	1.9	0	6.3	1.3	1.9
12	13.8	1.3	5.0	8.8	4.4	0
13	11.9	1.9	10.7	5.0	5.0	3.1
14	17.0	1.9	6.9	7.5	3.8	0

Table IV shows the frequency (in percent of the set of 159 responses) of each kind of Codes 3–8 observed. We attempt to analyze prototypes of each solution and to infer the concept image associated with it.

In the analysis of the errors for each item, it is necessary to realize that some errors cannot be detected in some triangles. For instance, items involving isosceles triangles (Items 1, 3, and 6) did not allow us to detect whether students confused altitude, median, and the perpendicular bisector (Code 3 and 4 errors) because these three segments coincide, except when the student considers the altitude on a side different from the specified one.

Next, we present and analyze examples that represent the most common types of responses. The answers allow us to recognize the consistency of the students' concept image with respect to the altitude of a triangle. Some students' responses are not included in the analysis because those students made only a few mistakes, likely due to negligence or oversight, or they gave mostly wrong answers but without any apparent pattern.

In each case below, although only a few student responses are presented, those responses allow us to identify the characteristics of the student's concept image. The student's other responses were consistent with the ones provided.

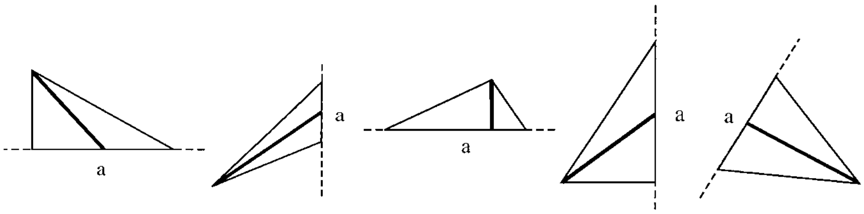


Figure 6. Responses reflecting confusion between the concepts of altitude and median (Ana, Group D).

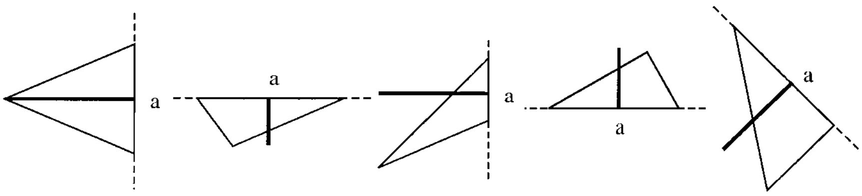


Figure 7. Responses reflecting a partial concept image influenced by the concept of perpendicular bisector (Student 22, Group A).

Altitude vs. median. Figure 6 shows responses of a student whose concept image of altitude is a mixture of altitude and median (Code 3 error). As can be seen in Table IV, this is the most frequent error. The confusion of the two concepts may be due, in part, to both concepts usually being studied at the same time in the primary school. It also appears that the students, rather than using the definition of altitude when drawing, used only their concept image, a result that confirms Vinner's (1991, p. 73) findings. Code 3 was recorded in 26.9% of the tests without the definition of altitude and in 22.4% of the tests with the definition. It is interesting to observe that, in reality, most of these students, as the one featured in Figure 6, have a partial concept image; they correctly drew the altitude in triangles with internal altitudes, but they were unable to draw an external altitude or an altitude that coincides with a side.

Altitude vs. perpendicular bisector. The examples in Figure 7 represent responses of a student who mixed the concepts of altitude and perpendicular bisector, an error that rarely occurred in our sample (Code 4). The student does not have a wrong concept image of altitude in that the drawn segments are, in fact, altitudes on the required sides. The student has, however, only a partial image, which makes him/her draw the altitude with an endpoint on the midpoint of the required base of the triangle.

Limitation to internal altitudes. Figure 8 shows the answers of a student who, in all cases, has drawn an internal altitude to a side of the triangle.

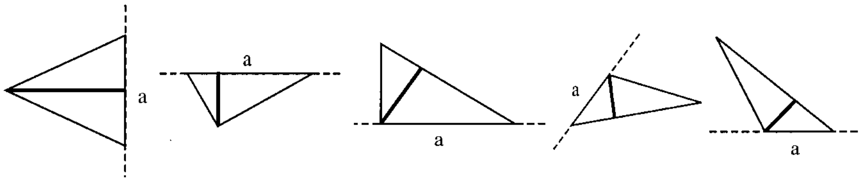


Figure 8. Responses reflecting a partial concept image that excludes external altitudes (Marta, Group D).

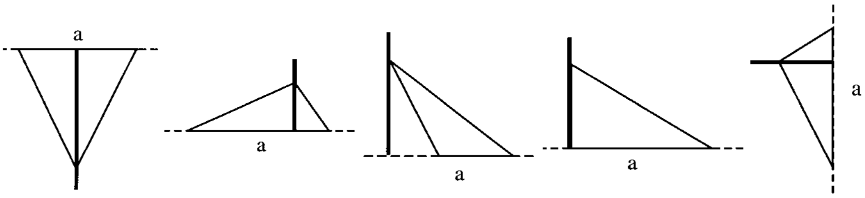


Figure 9. Responses reflecting a partial image which does not take into account the length of the altitude (Student 25, Group A).

The student considered a side different than the specified base when the correct solution would have not been internal to the triangle (Code 5 error). This student has a partial concept image, which does not include triangles with non-internal altitudes. That is, the prototypical case is that of an acute triangle, in which the altitude is internal to the triangle.

Disregard of length. Figure 9 presents the case of a student who did not consider the length of the segment as relevant, but who took into account all the other characteristics of the concept of altitude (Code 6 error). It appears that this student associated the altitude with a ray or a segment of an undetermined length, but having all the other characteristics of the altitude. Again, one way of changing this partial concept image into a complete one is to present problems that require students to use the characteristic that they had not considered, in this case, the length. A disregard of the length of the altitude was the second most frequent error in our students.

Fixation on side. Figure 10 presents the work of a student (the only one in our sample who answered in this way) whose concept image of altitude appears to include only isosceles triangles, because only Items 1, 3, and 6 were answered correctly. The student had no image for other cases in which he/she highlighted some element of the triangle, in particular, one side. More specifically, this student drew a side of the triangle or the dotted segment (provided in the test to indicate the side on which the altitude should be drawn).

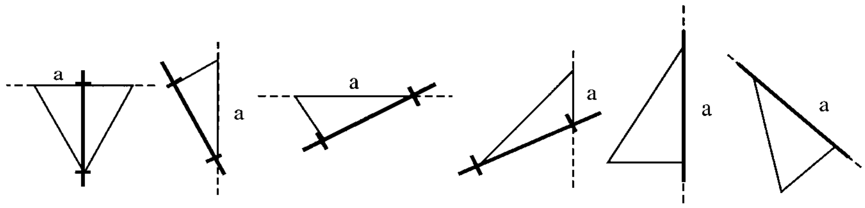


Figure 10. Responses representing fixation on a side of the triangle (Student 23, Group A).

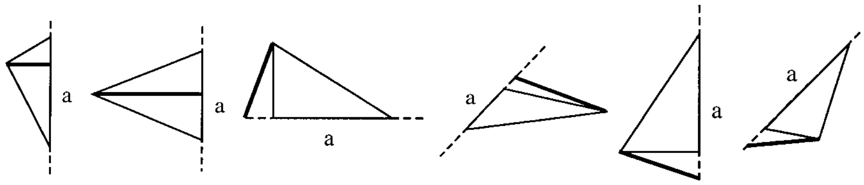


Figure 11. Responses in which the highlighted base served as distracter (Student 32, Group A).

Despite the apparent regularity of the answers, a closer look shows that the student had no criterion for choosing a particular side because the side marked was sometimes side a, and the other times a different side.

Marked base as distracter. Finally, the student whose answers are included in Figure 11 presents a concept image that is quite confusing. The use of incorrect visual images prevailed over the use of the mathematical properties of the concept. For this student, the dotted segments that were supposed to help identify the base might have served as distracters that induced the student to draw wrong segments. Once again, this concept image includes the prototype of an internal altitude, although the answers to the first and last triangles in Figure 11 (the same triangle in two different positions) are contradictory. Furthermore, for this student, the altitude must be a segment different from any of the sides (in the case of right triangles). This student did not consider perpendicularity when other visual characteristics made the drawings more like the figures that seemed to exist in his/her memory.

We have shown several examples of preservice primary teachers' concept images that represent various incorrect and/or partial conceptions of altitude of a triangle. Some errors occurred quite frequently, whereas others seem very unusual. The reason for certain errors is evident, although, as the two last examples show, interpretation of answers is sometimes difficult. In light of the previous results, several consequences can be raised for pre- or inservice teacher training courses.

IMPLICATIONS FOR TEACHER EDUCATION

Certainly the concept of altitude of a triangle is not an easily grasped concept by either pupils or preservice teachers. In light of our research and other similar studies (Hershkowitz, 1989; Hershkowitz & Vinner, 1984), we can begin to provide a basis for teachers of teachers to make informed decisions when designing programs for preservice and inservice teachers. We found that many preservice teachers had the same poor concept images as primary or secondary students. This situation can provide a context for the teachers to examine their and their classmates concept images and concomitantly learn what kinds of concept images pupils are likely to have. They could examine, for example, their understanding of the altitude and the median, and subsequently investigate how they differentiate these two concepts and how their students might as well. This kind of an analysis could provide them with insights when they become teachers with the responsibility to teach these concepts to their students. The question then becomes: What kind of instruction could best enable teachers and students to develop a complete concept image?

We have based our research on the Vinner model. We found that the core constructs of this model, the student's *concept image* and *concept definition*, and the description of the relationship among them provide valuable tools for interpreting the learning of basic geometric concepts. Our research has contributed to the particular context in which this model could be used with pre- or inservice teachers to enhance their understanding and for them to develop a basis for understanding their students' concept images. Further, we recommend that future research focuses on topics such as space, measurement, or isometries to provide an even richer research base for teacher education programs.

We see the following specific suggestions for teacher education as emanating from our research. We offer them in the hope they are of value to other teacher educators.

- Any instructional program intended to enhance preservice teachers' concept images should take into account their previous concept images and possible misconceptions and should consider specific learning situations useful for addressing those misconceptions.
- Teachers should be given opportunities to present, explain, and defend their particular conceptions of altitude and other basic geometrical concepts. These discussions could provide the opportunity to consider and resolve any cognitive conflicts that arise from different concept images. These concept images should be compared with the formal definition.

- Teachers could conduct class discussions with their pupils to assess their concept images of the altitude of a triangle. Analysis of students' outcomes would also provide a context for teachers to reflect on their own concept images.
- Teachers could be given pupils' responses to questions like the ones presented in this study and asked to describe these pupils' concept images. As they reflect on and discuss pupils' concept images they might at the same time reflect on their own concept images.
- Teachers could be presented with research similar to the one reported in this article and use that research as a basis for reflecting on their own concept images. They could, for example, study the nature of students' understanding of subconcepts when difficulty occurs regarding the primary concept. They could engage in designing activities aimed at examining possible conflicts related to the different subconcepts, their relationships, and the logical nature of the definition.

A final remark addresses the use of research such as ours and the education of teachers. There are many mathematical concepts that depend on lower order concepts (Skemp, 1971) or subconcepts. It often happens that high school or university teachers assume that their pupils understand these lower order concepts and subconcepts. Using the Vinner model with its distinction of concept image and concept definition, teachers and teacher educators can examine the nature of students' poorly formed concept images and determine what geometric concepts, properties, and facts are lacking. This, then, could become a focal point for teacher education programs.

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Departamento de Didáctica de la Matemática Angel Gutiérrez
Universidad de Valencia
Apartado 22045
46071 – Valencia
Spain
E-mail: angel.gutierrez@uv.es

Departamento de Didáctica de la Matemática Adela Jaime
Universidad de Valencia
Apartado 22045
46071 – Valencia
Spain
E-mail: adela.jaime@uv.es