WHITE PAPER ON GEOMETRY AND MEASUREMENT

(for the Principles and Standards for School Mathematics Writing Group)

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General Issues

Geometry and measurement have always played a relevant role in the advance of mathematics as a science. They were the core of mathematics for centuries, and geometry and measurement remain helpful tools and starting points for developments in other areas of mathematics and sciences. However, in recent decades we have seen a decadence of both topics in the primary and secondary school curricula all over the world (with only a few exceptions, as shown by the recent assessment results). In some countries geometry and measurement are taught only in primary schools, having been reduced to the recognition of the shape of several basic figures, the rote learning of their names, a few properties, and some formulas to calculate perimeters, areas and volumes. In other countries they are reserved for secondary school, in the very formalized contexts of deductive synthetic geometry and analytic geometry. Fortunately, the tendency is changing in the 90s, and new curricula in many countries all over the world recognize the importance of geometry and measurement. Geometry and measurement are vehicles to promote a deeper understanding of all areas of mathematics at any level, to improve students' abstract reasoning, and to highlight relationships between mathematics and sciences.

The aim of this paper is to present some reflections, in light of recent research on mathematics education, about the teaching and learning of geometry and measurement, and the reasoning processes generated by students when working on these topics, in the hope that they will be useful for the new NCTM's Principles and Standards for School Mathematics.

After this introductory section, the paper is divided in two parts, devoted to measurement and to geometry. Each part contains three sections, where we propose standards for the different grade ranges, we detail sets of benchmarks matching the standards, and we discuss related teaching and learning issues, including specific examples.

Perspective plays a critical role in seeing the world and no less substantial a role in "seeing" the function of research on mathematics education in developing educational standards. Hence, we briefly outline several theories of learning and development that shall be used as frameworks to structure and justify the contents of this paper.

Mediated Action

Recent research in mathematics education have "rediscovered" the theory of L. Vygotskii, and many successful experiments based on the consideration of the social interactions in the classroom have been carried out. Wertsch (1998) proposes that learning is a form of mediated action, suggesting that it is useful to consider performance as assisted and even enabled by mediational means (see Ellice Forman white paper on socio-cultural perspectives). When considering a geometry or measurement education, it is important then to consider a range of mediational means. Carpenter & Lehrer (in press) suggest that teaching and learning mathematics for understanding should pay careful attention to:

(a) the potential affordances of the tasks posed to students (e.g., the problems they are asked to solve, such as developing a map of their neighborhood, or making some measurements at the school);

(b) the nature of the tools available (e.g., ruler and compass, Polydron™, computer);

(c) the symbol systems and notations students invent or are asked to appropriate (e.g., drawings of objects and their relations, two-dimensional representations of three-dimensional objects, etc.);

(d) the modes and means of argumentation developed in the classroom (e.g., single cases, multiple cases, formal proof); and

(e) the activity structures prevalent in the classroom, i.e., the forms of ritualized activity that set the stage for learning (e.g., individual work, group work, whole-class discussion).

This perspective suggests that conceptual change occurs as students solve problems, invent and refine symbol systems, and participate in meaningful mathematical arguments.

Piaget

Traditionally, curricular development has thought of the growth and development of student thinking about geometry and measurement within traditions that do not easily accommodate Vygotskii's social view of learning. In Piaget's theory, conceptual change results from a learner's accommodations to new tasks and the opportunities for learning about space and measure that these tasks provide. Learning is, however, not simply task-driven, but rather, constrained by general characteristics of reasoning abilities, mental logic broadly associated with age, and personal beliefs. Although Piaget's theory serves as a wellspring of important insights and ideas, it is relatively silent about conceptual change in light of symbol systems and arguments used by students.

Van Hiele

Van Hiele offers a powerful alternative to Piaget, one focused more on characterization of different kinds of reasoning, and conceptual change in light of assisted performance (e.g., cycles of inquiry in classrooms). The van Hiele framework is based on the identification of four levels of reasoning:

Level 1: Children recognize shapes as a whole. Descriptions of figures are based on their appearance and similitude to known objects (doors, wheels, etc.). At this level, the mathematical properties of figures play no relevant role in identifications or descriptions.

Level 2: Children identify, define and classify figures based on their mathematical properties. Properties are discovered and generalized from observation of a few examples. To prove a property, students just check it in some examples.

Level 3: Students perceive the relationships among properties of figures, and they base classifications on such relationships. Examples continue playing a relevant role in the elaboration of proofs, although they are no longer the owners of truth, but they are used by students as a help to organize abstract deductive arguments. However, students cannot understand formal proofs.

Level 4: Students understand the characteristics and role of the elements of an axiomatic system, and they can develop formal proofs by different techniques.

Van Hiele suggests that progress in the refinement of reasoning is gained after acquiring experience in using the new way of reasoning, and, in the absence of suitable instruction, such progress may be arrested (Fuys, Geddes & Tischler, 1984, p. 246). To help teachers in the organization of teaching to promote their pupils' acquisition of a new level of reasoning, the van Hiele model proposes five phases of learning:

Phase 1: Students learn about the nature of a new topic to be studied, and teachers become aware of their pupils' previous knowledge and thinking levels in this topic.

Phase 2: Students begin to learn the basics of the new topic by completing simple directed tasks. Teacher's role is to direct students' work to the object of study.

Phase 3: Students learn the new terms while they explain their solutions to the activities and compare different solutions. During the early part of the process, students tend to use their own terms for the new concepts but, over time, they refine their language with the help of the teacher.

Phase 4: Students are proposed a variety of activities aimed to improve and deepen their knowledge of the topic, the students needing to combine their knowledge in new ways to solve them. Some activities should allow multiple solution paths, and others should be open-ended problems.

Phase 5: The teacher helps the students to make an overview of the topic they have been studying, and show connections to other related topics.

A number of references can be found containing more detailed descriptions and different analysis of van Hiele levels (e.g., Burger & Shaughnessy, 1986; Clements & Battista, 1992; Crowley, 1987; Fuys, Geddes & Tischler, 1984, 1988; Gutiérrez & Jaime, 1998; Gutiérrez, Jaime & Fortuny, 1991; Van Hiele, 1986). Yet studies conducted in this tradition often assess student's knowledge without strong coupling to the design of the learning environment. There are, however, some exceptions, such as the design of geometry courses for elementary school based on Logo environment (Clements & Battista, 1990, 1992) and the design of units for teaching plane isometries in grades 3 to 12 and teacher training courses (Jaime & Gutiérrez, 1989, 1995, 1996). Hence, research should be promoted to develop the new NCTM's Principles and Standards and gain knowledge about what students might learn if they participate in well-designed learning environments for prolonged periods of time. Accordingly, we highlight classroom-based research about students' conceptual development in the domain of geometry and space. This paper is intended to suggest what might be possible *if* students have the opportunity to learn about space and geometry. Because there is little work that couples teachers' professional development with student learning (theories of mediated action assume teacher assistance as a vital form of mediation), some of our suggestions are speculations, albeit based on research evidence.

In some of the research we have used to produce our proposal for the elementary grades, teachers develop (or appropriate) tasks, encourage students to invent symbol systems and notations, and promote mathematical arguments. Teachers collaborate to increase their collective knowledge of the growth and development of student thinking, and it is this shared interest that provides the "glue" for continued innovation over time. Although there is significant teacher-to-teacher and student-to-student variation, nevertheless there are replications of student learning outcomes across schools and teachers. Although the work does not follow students into the secondary grades, perhaps it sheds some light on what we could reasonably expect, if geometry became a more significant contributor to students' mathematics education through primary and secondary grades.

MEASURE

This section of the paper describes the growth and development of children's understanding of measure, and how fundamental ideas about measure can scaffold fundamental ideas about space. Our assumption is a classical one, derived from the Greek derivation of geometry as "earth measure." By learning about spatial measure of length, area, volume and angle, students mentally structure and revise their constructions of both large-scale and small-scale spaces. Hence, although measure theory can be treated as distinct from geometry, and ironically often is due to Greek emphasis on straight-edge and compass construction, there is much pedagogical value in returning geometry to its roots in spatial measure.

Benchmarks of Understanding and Learning Measure in Grades Pre-K to 12

The benchmarks of understanding -the conceptual landmarks- that we aim for are those suggested by Piaget and his collaborators as important conceptual attainments. These include various conceptions involving the very idea of a unit of measure and also, the notion that measure involves the organized accumulation of standard units. Besides conceptual understanding, the work done by students has to include the learning of definitions, relationships among units, formulas, etc. The work to be done in classrooms should include developing students' understanding and learning of the following:

• **Unit**

a. The relationship between an attribute and the unit measuring that attribute.

b. Iteration (accumulation of units).

c. Tiling (units tile lines, planes, volumes).

d. Identity (identical units allow one quantity to represent the measure).

e. Proportionality (measurements with different sized units allow different quantities to represent the measure, these being inversely proportional to the size of the units).

• Origin (Zero-point)

a. Measures involve construction of intervals, so that, for example, the distance between 0 and 10 is the same as that 30 and 40.

• Conservation

a. Some transformations of an object modify its form but do not modify its measure (for instance, decomposition and recomposition of lengths, areas and volumes). This property is a logical consequence of those above.

• Multilineality

a. Relationship among the dimension of a magnitude and the linear dimensions used to calculate its measurement.

• Formulas

a. Discovery and understanding of usual formulas for perimeters, areas, and volumes.

Recommendations: Measure Standards

Grades Pre-K - 2

(1) Measure lengths and areas.

(2) Develop ideas of unit-attribute (e.g., units designed for length won't generally serve as indicators for area).

(3) Develop notions of importance of attributes of measure like identity, iteration, and tiling.

(4) Develop conservation of length.

Grades 3 - 5

(1) Measure areas, volumes, and angles.

(2) Revisit ideas of unit-attribute (e.g., angle measure as arcs or turns)

(3) Revisit attributes of measure like identity, iteration, and tiling for area, volume, and angle measure.

(4) Develop notions of zero-point for length measure, as a general means of introducing the notion of origin of measure.

(5) Develop conservation of area, and volume.

(6) Extension to other qualities, like mass.

(7) Develop formulas for rectangles and triangles.

Grades 6 - 8

(1) Revisit area, volume, and angle measures but now emphasize relationships, like surface area to volume, or planar vs. dihedral angles.

(2) Emphasize precision of measure.

(3) Connect measure to data, most especially the notion of a distribution resulting from repeated measurement.

(4) Explore measure of a circle.

(5) Develop formulas for other polygons, for some simple three-dimensional forms, like rectangular prisms and, perhaps, cylinders.

Grades 9 - 12

(1) Expand notion of measure to include levels of measure and related issues of scale (e.g., nominal, ordinal, interval, ratio).

(2) Connect measure to data, especially constructs of contingency and covariation.

Measure Standards for grades Pre-K - 12

The standards stated for the different grade ranges can be summarized in the following ones:

(1) Learn to measure length, area, volume, and angles.

(2) Acquire the main concepts related to measurement, specially the unit of measure and conservation of magnitudes.

(3) Learn formulas and algorithms to calculate different measures of some plane and space figures.

(4) Develop strategies for approximate measurement, and use them to connect measures to other contexts like calculus and statistics.

Children's Acquisition of Measure Concepts

As we noted previously, much of the work in the field draws upon the seminal contributions of Piaget (Piaget & Inhelder, 1948, Piaget, Inhelder & Szeminska, 1960), which continue as a wellspring for contemporary research. Piaget's analysis suggested that conceptions of spatial measure were not unitary but instead consisted of a web of related constructs leading to the eventual construction and coordination of standard units. Understanding of measure entailed a successive mental restructuring of space. Piaget and his colleagues further suggested that conceptual change was tightly coupled to the overall development of reasoning. Accordingly, conservation (recognition of invariance under transformation) of length, area, volume, and angle was a hallmark of, and constraint on, development in each domain of spatial measure studied by Piaget.

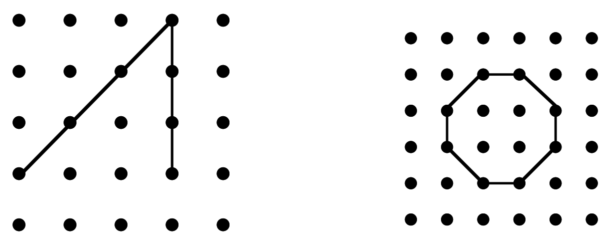
However, studies conducted in the last two decades generally fail to support a tight coupling of the development of understanding of spatial measure and general capacities for mental logic (i.e., conservation). For example, Hiebert (1981 a, b) conducted an instructional study in which first-grade children, some of whom conserved length and some of whom did not, were taught important underpinning of length measure, like iterating units to measure lengths. Hiebert found that acquisition of ideas like iteration was generally unrelated to a child's status as a conserver. The sole exception was a child's recognition of the inverse relationship between the length of a unit of measure and the resulting count (e.g., smaller units produce larger counts). The lack of any tight coupling between conservation and understanding of measure is also characteristic of other domains of spatial measure.

In the sections that follow, we review developmental progressions of children's understandings of length, area, volume, and angle measure. These progressions are based largely on clinical interviews conducted with children at various ages or grades, or on longitudinal study of transitions in children's reasoning during the course of traditional schooling. More recent work examines the acquisition of measure concepts in classrooms designed to promote understanding of measure, rather than simple procedural competence. This contemporary work often suggests the need to revise previous accounts of development to include consideration of the mediational means employed. (The work reviewed is intended as representative, not exhaustive.)

Length Measure

Understanding length measure requires restructuring of space so that one "sees" counts of units as representing an iteration of successive distances. Thus, a count of n units represents a distance of n units. Developmental studies suggest that acquisition of this understanding involves coordination of multiple constructs, especially those of unit and zero-point (**RICH, ADD SOME REFERENCE/S HERE**). Construction of unit involves a web of foundational ideas including procedures of iteration, recognition of the need for identical units, understanding of the inverse relationship between magnitude of each unit and the resulting length measure, and understanding of partitions of unit. Understanding zero-point involves the mental coordination of the origin and end-point of the scale used to measure length, so that the length from the 10 cm mark on the scale to the 20 cm mark is equivalent to that between 2 and 12 cm. Studies of children's development indicate that these constructs are not acquired in an all or none manner, nor are they necessarily tightly linked. Most studies suggest that these understandings of units of length are acquired over the course of the elementary grades, although there are significant variations in developmental trajectories when different mediational means are employed.

Children's first understandings of length measure involve direct comparison of objects (Lindquist, 1989). Congruent objects have equal length, and young children (first grade) typically understand that the length of two objects can be compared by representing them with a string or paper strip (Hiebert, 1981 a, b). First graders can also use given units to find the length of different objects, and they associate higher counts with longer objects (Hiebert, 1981 a, b) However, this apparent ease of counting need not imply understanding of measure as a distance. For example, only first-grade children in Hiebert's study who conserved length understood the inverse relationship between counts and magnitude of units (e.g., smaller units yield higher counts). Young children (first- and second-grade) often do not understand the purposes of identical units of length measure, so they freely mix, for example, inches and centimeters, counting all to "measure" a length. **I'VE ADDED FROM HERE Another situation reflecting the same misunderstanding is often found when children are asked to calculate length of segments on a geoboard or square lattice. Many children do not recognize the different distances between two consecutive points in vertical/horizontal and diagonal directions, so they say that the two segments in Figure 1a are the same length, 3 units (Saunders, 1977). In another experiment, more than 40% of students aged 12, 13, and 14 years wrote that the perimeter of the octagon in Figure 1b is 8 cm (Hart, 1981). TO HERE** A significant minority of young children do not spontaneously iterate units of measure when they "run out" of units (Lehrer, Jenkins & Osana, 1998).



- a - - b -

Figure 1.

Children's understanding of zero-point of length measure is somewhat tenuous. Only a minority of young children understand that any point on the scale can serve as the starting point, and even a significant minority of older children (e.g., grade 5) respond to non-zero origins by simply reading off whatever number on a ruler aligns with the end of the object (Lehrer, Jenkins & Osana, 1998). Children often report the length of a 2 1/2 cm object as 3 1/2 because they don't coordinate the interval between the origin and endpoint, relying instead on counting. For example, Lehrer et al. (1999) noted that some second-grade children (7-8 years of age) measured a 2 1/2 unit strip of paper by counting 1, 2, [pause], 3 [pause], 3 1/2. They explained that the 3 referred to the third unit counted, but “there’s only a 1/2,” so in effect the last half unit was represented twice, first as a count of unit and then as a partition of a unit. **I’VE ADDED FROM HERE This problem does not happen only on early grades, but it is still important in upper grades. Hart (1981) asked students aged 12, 13, and 14 years for the length of a 6 cm segment drawn over a ruler from the mark 1 to the mark 7; she found that 46.2% of 12 year olders, 30.6% of 11 year olders, and 22.8% of 14 year olders said that the segment was 7 cm long. According to these results, it seems that it is at the junior high school where most students complete the acquisition of concepts related to length. TO HERE**

Other recent work has focused on establishing developmental trajectories of understanding in classrooms that promote understanding. These studies suggest that there are important gains in understanding when children’s learning is mediated by systems of inscription and notation. For example, Clements, Battista & Samara (1998) report that using computer tools which mediate children’s experience of unit and iteration help children mentally restructure lengths into units. Other recent studies place a premium on making transitions from embodied forms of length measure, like pacing, to inscribing and symbolizing these forms as “foot strips” and other kinds of measurement tools (Lehrer et al., 1999; McClainet al., 1999). By constructing tools, children have the opportunity to discover the measure principles that guide their design. Although this has a long tradition in teaching practice, studies like these provide important details about how these practices contribute to conceptual change.

<u>Area Measure</u>

In many ways, studies of children’s conceptions of area measure parallel those discussed for length. We focus here on a longitudinal investigation of 37 first-, second-, and third-grade children who were followed for three years (e.g., through the fifth grade) because the results of the study are representative of much of the literature (Lehrer, Jenkins & Osana, 1998). Young children (first- and second-grades) often treat length measure as a surrogate for area measure. For example, Lehrer, Jenkins & Osana (1998) found that some young children measured the area of a square by measuring the length of a side of a square, then moving the ruler over a bit and measuring the length between the sides again, and so on, treating length as a space-filling attribute. When provided manipulatives (squares, right triangles, circles, and rectangles) for use in finding the area measure of a variety of forms, most children in the primary grades (1-3) freely mixed units and then reported the total count of such units. The two most commonly observed strategies with use of manipulatives were boundedness and resemblance. Children deployed units in ways that would not violate the boundaries of irregular forms (non-polygonal), and they often used units that resembled the form being measured (e.g., triangles for triangles). Young children were also likely to ignore the space-filling properties of units, preferring instead to honor the boundaries of the forms, so that when presented a choice with "leaving cracks" or overlapping a boundary, they invariably chose the former.

During the course of development over the elementary grades, area measure became differentiated from length measure, and the space-filling (tiling) requirement of the unit became more apparent to most children. However, other aspects of area measure remained problematic, even though students could recall standard formulas for finding the areas of squares and rectangles. Less than 20% of the students believed that area measure required identical units, and less than half could rearrange a series of figures so that known area measures could be used to find the measures of the areas of unknown figures. In other words, students found it very difficult to decompose and then recompose the areas of forms, to see one form as a composition of others. This points to a more general phenomena of spatial structuring, as noted by Piaget. Battista (1999), for example, reports that students in the primary grades often cannot structure a rectangle as an array of units.

In sum, conceptual development in area measure lagged that of length measure. Understanding of core conceptual notions like identical units and tiling were typical of students by the end of the elementary grades for length measure but not for area measure. Younger children are prone to use resemblance as the prime criterion for selecting a unit of area measure, suggesting the need for attention to the qualities of unit that make it suitable for area measure. Current practices of declaring squares as units may lead to procedural competence but violate students' preconceptions about what makes a unit suitable.

Like length measure, studies of developmental trajectories of area measure in classrooms that promote understanding reveal significant departures from patterns typically described in the literature. For example, Lehrer et al. (1998) found that second-grade students developed comprehensive understandings of area measure when they began with solving problems involving partitioning and reallotment of areas without measuring and only later considered area problems involving units of measure that they first invented. By the end of the school year, these children had little difficulty creating two-dimensional arrays of units for rectangles and even for irregular (non-polygonal) forms. Thus, the developmental patterns noted so often in the literature probably reflect the nature of instruction, with the possible exception of the need to be able to conserve area in order to understand reallotment of forms.

Volume Measure

Measure of volume presents some additional complexities for reasoning about the structure of space, primarily because units of measure must be defined and coordinated in three dimensions. Although the evolution of student conceptions of units of volume measure is not well understood, there is an emerging body of work that addresses strategies students employ to structure a volume, given a unit. For example, Battista & Clements (1998) noted a range of strategies deployed by students in the third and fifth grades to mentally structure a three-dimensional array of cubes. Many students, especially the younger ones, could only count the faces of the cubes, resulting in frequent instances of multiple counts of a single cube and a failure to count any cube in the interior of the array. The majority of fifth-grade students, but only about 20% of third-grade students, structured the array as a series of layers.

Layering enabled students to count the number of cubes in one layer and then multiply or skip-count to obtain the total number of cubes in the array.

I'VE ADDED FROM HERE When students calculating volumes are asked to move from counting units in arrays to use formulas, they face a new aspect of geometric measurement since they have to use length units to calculate volumes. This is usually labeled as the problem of multi-linearity. Tipically, students add the dimensions of solids instead of multiplying them. TO HERE

Angle Measure

Freudenthal (1973) suggested that multiple mathematical conceptions of angle should be entertained during the course of schooling. Henderson (1996) suggests three: (a) angle as movement, as in rotation or sweep; (b) angle as a geometric shape, a delineation of space by two intersecting lines; and (c) angle as a measure, a perspective that encompasses the other two. Mitchelmore (1998) and Lehrer, Jenkins & Osana (1998) suggest that students in the elementary grades develop separate mental models of angle as movement and angle as shape. In Mitchelmore's (1998) study, students in grades 2, 4, and 6 increasingly perceived how different types of turning situations might be alike (e.g., those involving unlimited turning, like a fan, and those involving limited turning, like a hinge), but they rarely related these to situation involving "bends" or other aspects of intersecting lines. Lehrer et al. asked children to find ways of measuring the "bending" in a hinge (with a sweep demonstrated from one position to another) and the bending in a bent pipe cleaner. Like the students in Mitchelmore's studies, students in grades 1-5 rarely saw a relationship between these situations, but their measurement actions were very similar. Children most often chose to measure the distance between the jaws of the hinge and the ends of the pipe cleaners.

Studies of student learning with Logo generally confirm the existence of separate models of angle. Logo's turtle geometry affords the notion of angle as a rotation, although students often confuse the interior and exterior (turtle) angles of figures traced by the turtle. Nevertheless, with well crafted instruction, tools like Logo mediate the development of measures of rotations (Lehrer, Randle & Sancilio, 1989). However, students rarely bridge these rotations to models of the space in the interior of figures traced by the turtle (e.g., Clements et al., 1996). Simple modifications to Logo help students perceive the relationship between turns and traces, and in these conditions, students can use turns to measure static intersections of lines (Lehrer, Randle & Sancilio, 1989).

In static contexts ("bends"), students typically believe that angle measures are influenced by the lengths of the intersecting lines or by their orientation in space. The latter conception decreases with age, but the former is robust at every age (Lehrer, Jenkins & Osana, 1998). Hence, it remains a major challenge to design pedagogy to help students develop understanding of angle and its measure. Unlike the spatial structuring of linear dimensions (length, area, and volume), developing understanding of angle requires novel forms of mental structuring (e.g. developing notions of turn, tracing a locus of a turning movement, relating turning movements to traces in environments like Logo). In addition, understanding angle involves coordination of several potential models and integration of these models in a theory of their measure. Common admonitions to teach angles as turns run the risk of monoculture because students will rarely spontaneously relate situations involving rotations to those involving shape and form. As we have stated often in our tour of measure, the form of mediation matters as much as the problems posed.

GEOMETRY

This section of the paper describes the progress of children's understanding of geometric concepts and properties (the term "geometry" includes both two- and three-dimensional spaces). Although there is another white paper specifically devoted to reasoning and proof, we will also consider the growth of these children's abilities in the context of geometry, since geometry is a privileged environment for teachers to promote the improvement of the above mentioned abilities.

Benchmarks of Understanding and Learning Geometry in Grades Pre-K to 12

The benchmarks defined below are related to different topics that should integrate the school geometry contents, students' proficiency, and teaching methods. Most of them follow from the Van Hiele model of reasoning, although influences from other researchers can also be noted. A part of the benchmarks are aimed to gain understanding of the concepts, while the others are aimed to memorize definitions, main properties, formulas, etc. and get proficiency in using them on problem solving. During their schooling, students should reach the following benchmarks:

• Plane figures

a. Identify, draw, describe, classify, and define the main kinds of plane figures: The classes of triangles and quadrilaterals; concave, convex, and regular polygons; circle and the other conic curves; straight and curved lines, segments, and arcs; the relative positions of straight lines in the plane; the different types of angles.

b. Discover, generalize and prove the main properties of plane figures: Parallelism, perpendicularity, angle bisector, etc. Master techniques of drawing (both in paper-and-pencil or computer environments).

• Space figures

a. Identify, draw, build, describe, classify, and define the main kinds of space figures: The classes of prisms and pyramids; concave, convex, and regular polyhedrons; sphere and the other solids of revolution; the relative positions of planes and/or straight lines in space; straight/plane and curved lines and surfaces; the different types of angles in space.

b. Discover, generalize and prove the main properties of space figures: Parallelism, perpendicularity, collinearity, regularity, etc.

c. Understand and practice "reading" and "writing" the most usual kinds of plane representations of solids (perspective; levels; parallel, isometric, and orthogonal projections).

• Isometries and congruence

a. Understand the different plane isometries (translation, reflection, rotation, and glide reflection).

The focus in lower grades is on the dynamic aspects, and only in upper grades the formal view of plane transformations and the algebraic language are introduced (Jaime & Gutiérrez, 1995).

b. Prevent and/or correct the usual misconceptions present in the students' answers (horizontal reflection around a slanted axis, etc. Hart, 1981).

The role of teachers is to pay attention to identify pupils' errors.

c. Formalize the concept of congruence, discover its main properties, and prove them (criteria of congruence of triangles, etc.). Work on applications of isometries (tessellations, and others).

d. Understand the different space isometries, apply them to describe the main space figures, and understand the differences among these and plane isometries.

• Similarities. Geometric proportionality

a. Understand the concept of geometric similarity, including both graphical and numerical aspects, and their relationships. Work on applications of similarity and dilatation (maps, pictures, the golden ratio, introduction to trigonometric ratios, etc.).

The first approach should be just manipulative (by using magnifying glasses, sun shadow, pictures, pantograph, etc.) and graphic (paying attention to visual and basic geometric characteristics of shapes). Then, numeric relationships are introduced (the type of numbers depending on students' arithmetical ability) with the help of drawings.

b. Generalize the concepts of ratio and proportionality in geometrical contexts.

First, students operate with ratios like double, triple, half, third, etc. Then, "easy" rational ratios and, finally, any rational and real ratio is used.

c. Prevent and/or correct the usual misconceptions present in the students' answers (additive "ratios", etc. Hart, 1981, 1984, 1989).

The role of teachers is to pay attention to identify pupils' errors.

d. Formalize the concept of similarity, discover its main properties, and prove them (dilatation, criteria of similarity of triangles, Thales theorem, etc.).

• Non-Euclidean geometry

a. In upper grades, interact with some non-Euclidean geometries, understanding the differences among the axiomatic systems and the behavior of figures and properties in each geometry.

Several possibilities, having different formal requirements, are available: The taxi-cab geometry is very basic, but challenging; spherical geometry can be easily visualized by drawing on a ball; other geometries, like the Poincaré's hyperbolic geometry, can be experimented with the help of dynamic geometry software. Given the counter-intuitive characteristics of non-Euclidean geometries, teachers should carefully avoid an excessive formalization, inadequate for their pupils' level of reasoning.

• Space and visualization

a. Develop strategies of visualization (i.e., mental analysis and transformations of plane and space objects, like decomposition, comparison, displacement, etc.).

b. Use both visual and analytic ways of reasoning and work (Krutetskii, 1976).

The role of teachers is to help pupils to increase their ability of using both ways of representation and thinking.

• Geometry problem solving

a. In the context of general problem solving, master the methods specific to geometry problems, mainly the method of two loci (Polya, 1981), and different heuristics (Schoenfeld, 1985).

• Rigor and proofs

a. Explain their ways of solving problems or activities, and give reasons to convince teacher and other pupils of their correctness.

From early grades, teachers should ask students for the reasons of their outcomes, the sophistication of the reasons depending on pupils' level of reasoning. Teachers should promote the improvement of their pupils' understanding of proof by preparing adequate activities or problems (for instance, to make students understand that one or a few examples are not enough to prove the general validity of a statement).

b. In upper grades, begin the learning and use of formal language and ways of proof (kinds of implications, types of proofs, etc.).

Recommendations: Geometry Standards

The following recommended standards summarize the main aspects of the benchmarks stated above, that should be mastered by every student when finishing each grade range. Other parts of the benchmarks are not explicitly mentioned in the standards, but they appear to be necessary when developing the standards. As mentioned in a previous section, a correct learning needs understanding, practice, and rote learning, the three parts being considered in the following standards.

Grades Pre-K - 2

(1) Identify different plane and space figures, and learn their basic facts.

(2) Observe and identify objects sliding, rotating, and reflecting on mirrors. Develop dynamic notions of slide, rotation and reflection.

(3) Observe and identify similar plane or space figures.

Grades 3 - 5

(1) Learn mathematical properties of plane and space figures by discovering and generalizing them after manipulation and induction.

(2) Learn the basic characteristics of slides, rotations and reflections by discovering and generalizing them after manipulation and induction.

(3) Use instruments to show, produce, and measure similar figures. Approach the concept of ratio. Verbalize the concepts of dilatation and similarity.

(4) When solving a problem or activity, draw sketches, diagrams or figures to help to solve it and explain the solution. Practice the drawing of appropriate kinds of plane representations of space figures.

Grades 6 - 8

(1) Discover and prove inductively mathematical properties of plane and space figures. Define and classify families of figures.

(2) Discover and prove inductively the results of products of isometries. Use those results to justify the criteria of congruence of triangles.

(3) Learn the concepts of ratio and proportion. Learn the main properties of similarity, and use them to prove inductively the criteria of similarity of triangles.

(4) Master the usual kinds of plane representations of space figures (perspective; parallel, isometric, and orthogonal projections). Solve activities requiring mental visualization of plane or space figures (both static and dynamic).

Grades 9 - 12

(1) Prove properties of plane and space figures. Proofs are inductive in the lower grades and deductive in the upper grades.

(2) Complete the algebraic study of the group of plane isometries, and prove the main theorems.

(3) Complete the study of similarity by proving the Thales theorem and others. Use those theorems to solve different kinds of applied problems.

(4) Take contact with non-Euclidean geometries (like taxi-cab, spherical or Poincaré's hyperbolic geometries). Understand the axiomatic structure of such geometries, check or prove some theorems, and explain differences with Euclidean geometry.

(5) Combine visual and analytic reasoning to solve geometry problems and tasks. Apply strategies of visualization (mental analysis and transformation of visual information) to devise ways of solving them.

Geometry Standards for grades Pre-K - 12

The standards stated for the different grade ranges can be summarized in the following ones:

(1) Acquire a detailed knowledge of families of plane and space figures, including definition, classification, discovering and proving properties, and using them in context.

(2) Learn the characteristics of Euclidean transformations (isometries, dilatation and similarity) and use them as tools to solve problems and prove theorems.

(3) Master abilities of mental visualization and representation of geometric two- and three-dimensional figures, and use them to solve problems in mathematics and other contexts.

(4) Understand the components and structure of an axiomatic system. Work with Euclidean and non-Euclidean geometries.

Children's Acquisition of Geometric and Spatial Concepts

In this section we elaborate the benchmarks and standards listed above and detail the psychological and research background for our suggestions. Some examples of students' activity and classroom interactions are also included to show implementations of our proposals. This section has been organized linked to the grade ranges, to make it more operative and helpful to readers.

A characteristic of school geometry is that most topics can be studied with quite diverse degrees of sophistication, ranging from a manipulative and visual orientation, appropriate for grades up to lower primary, to a formal and abstract one, appropriate for students in upper high school or, even, university. Therefore, teachers and textbook writers should consider as a central component of their curricular elaborations the idea of spiral curriculum. In a spiral curriculum, some topics are studied in several grades (no necessarily consecutive), with new concepts, properties, and techniques arising each time, basing new knowledge on the previously acquired, and, when possible, analyzing previous knowledge from a higher viewpoint (for instance, a property is, first, discovered by manipulation, later it is inductively justified and, finally, it is deductively proved). The concept of spiral curriculum may be traced also in the proposal we present in this paper.

Another characteristic of a geometry course in any grade should be the attention to problem solving, looking at the particular methods and heuristics for geometric problems. In the broader context of teaching problem solving, each area of mathematics (geometry, measurement, algebra, calculus, probability, etc.) has its specificity, and teachers should pay attention to them.

• Grades Pre-K - 2.

The first contact of children with geometry is the exploration of shapes. They can recognize, differentiate and describe plane shapes having special characteristics, like squares, circles, triangles, rectangles, or rhombuses (diamonds), granted that the shapes have differentiated appearances (e.g., do not try to teach that a square is also a rectangle and a rhombus). Even in the earliest grades, children can differentiate among squares, equilateral triangles, "long" rectangles, and circles. In the same way, three-dimensional shapes like spheres, cubes, pyramids, cylinders, or cones can be studied. The number of shapes that can be identified increases with the grades. Some typical activities for this range of grades are the following:

1. *Provide your pupils with a set of 15 shapes, equilateral triangles, squares, rectangles, and circles. Ask them to classify the shapes, and to name each group. After finished the classifications, ask the students to tell characteristics differentiating each group* (Figure 2).

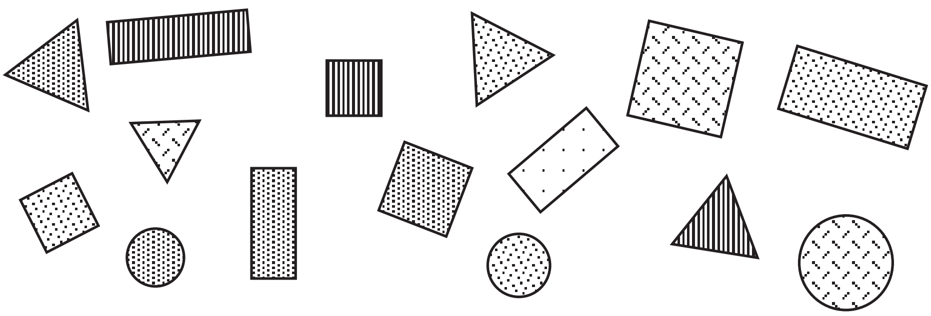


Figure 2.

The shapes can be made of cardboard, plastic or wood, and should be, at least, 10 centimeters (4 inches) wide. Do not paint in the same colour nor make the same size all the shapes of a same kind.

You can increase the difficulty of this activity by including more kinds of figures (for instance, non-equilateral triangles), different sizes and/or shapes, etc.

Similar activities can be prepared with solids.

2. *Provide your pupils with a black and white picture based on geometric shapes, like the one in Figure 3. Ask them to color triangles in yellow, circles in red, etc.*

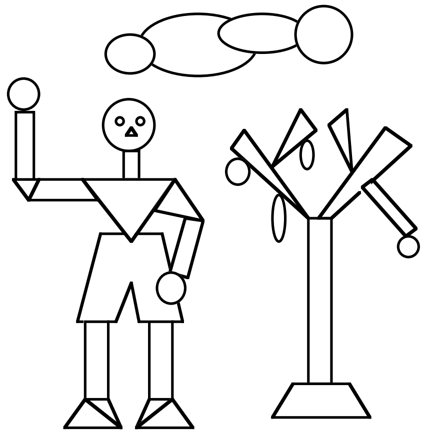


Figure 3. Look for triangles, circles, and rectangles.

Even when children can only recognize a few shapes, teachers should try to avoid the detrimental influence of prototypical examples (Hershkowitz, Bruckheimer & Vinner, 1987; Vinner, 1991) by drawing and showing as many shapes, sizes, and positions as possible. For instance, triangles are not only equilateral, and they do not need to rest on an horizontal side. Handling real shapes helps children very much although, coherently to their level of reasoning, they may have reasons for using only prototypical positions. For instance, when a kindergarten teacher drew a set of equilateral triangles in different positions (Figure 4), her pupils recognized all the shapes as triangles, although they said that some triangles could not be in that position (standing on a vertex) “because they would fall down”.

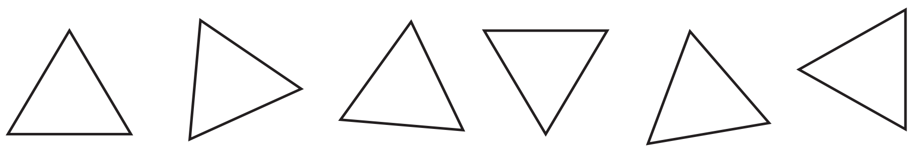


Figure 4. Equilateral triangles in different positions.

Isometries can be introduced through games or design activity. For instance, Logo-like activities where a child has to follow a peer’s directions offer opportunities to practice translations (sliding) and rotations (turns) (Clements & Battista, 1992; Clements et al., 1996).

A mirror may be the introduction to a game where two children simulate to be on both sides of the mirror and one of them has to act as the image of the other one. Folding sheets of paper and making cut-outs is a way of introduction of the axis of symmetry. A typical activity is to fold the paper along the dashed line (Figure 5), cut along the border of the half butterfly, and unfold the resulting piece. This activity may be extended to produce nice sets of translated, rotated, and symmetric shapes. Now the sheet of paper is folded along several dashed lines, depending the result on the position of the folding lines, parallel or intersecting at a point (Figure 6).

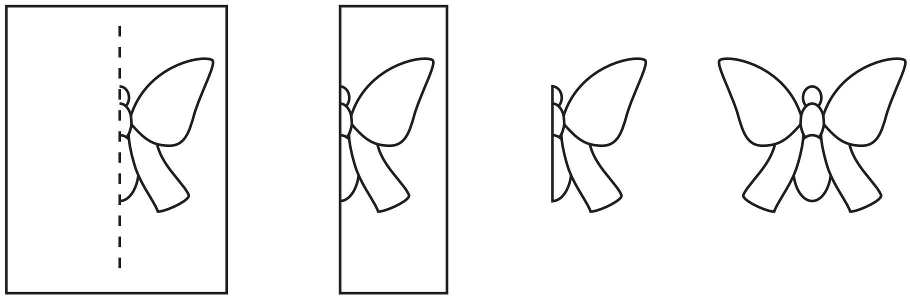


Figure 5. Fold, cut the butterfly, and unfold it.

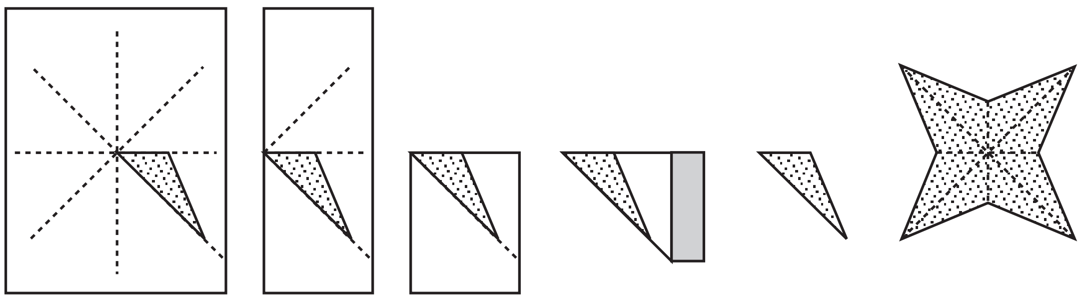


Figure 6. Fold, cut the shape, and unfold it.

When deciding the teaching strategies, teachers have to be aware that children are reasoning in the first van Hiele level, so they should be provided with opportunities to manipulate the figures in different ways. For instance, cardboard or wood polygons and circles, or solid models of polyhedra, cylinders, etc. can be used to make classifications. Plasticine and sticks can be used to build polygons and polyhedra. Computers with specific software are a source of different kinds of activities. A computer is specially useful for activities requiring a certain degree of accuracy that children cannot reach. For instance, if we want accurate results, it is much better to ask children to draw polygons on a computer with a "draw" package than on a sheet of paper, since it is quite difficult to talk about equal sides while looking at drawings made by children in second grade or younger. However, in these early grades children usually perceive better the characteristics of real objects than those of virtual objects on the screen, since handling provides them a very important feeling. Therefore, computers should be used only when they provide a real advantage over manipulatives.

Teachers should induce students to name the geometric objects they are using and to describe their relevant characteristics (count the sides, vertices or faces, compare lengths, etc.). The descriptions will likely contain a mixture of mathematical and "ordinary life" terms and properties of figures, proper of the first van Hiele level of reasoning, and these activities are a necessary training for children, helping them to gain experience and progress to a higher level of reasoning. Quite often, children in these grades are unable to use the standard names. In this case, the teacher should agree with her pupils an alternative name. Teachers should accept this kind of vocabulary, although trying to make pupils to increase slowly the number of mathematical terms used.

• Grades 3 - 5.

Students in grades 3-5 should advance from reasoning at Van Hiele level 1 to level 2, so they base their arguments on, and are convinced by, observation in examples of mathematical elements and regularities of figures. Only when ending grade 5, some students might begin the progress to level 3, accept the insufficiency of examples to guarantee the truth of a statement, and try to arrange an abstract argumentation, although based on experimentation. Therefore, at these grades, students' learning must be based on their activity, by exploring, conjecturing, checking, and talking about their conclusions.

In the topic of plane and space figures, activity has to focus on the discovery of the different elements of the figures (sides, angles, faces, diagonals, etc.) and their main properties (number of elements, congruence, parallelism, perpendicularity, etc.). This knowledge is very important since it is the base for definition and classification of families of figures.

Traditionally there has been a tendency to over-simplify the kinds of classification students in van Hiele level 2 can do, stating that they do exclusive classifications, i.e. sets of disjoint families of figures. Such statement is derived from research observing students' activity when they only classify squares, rectangles and rhombuses, or equilateral and isosceles triangles. However, it is usual to find students arguing, for instance, that a square cannot be a rectangle "because squares have four congruent sides and rectangles must have sides with different length" and, at the same time, accepting that both squares and rectangles are parallelogram "because their opposite sides are parallel". The origin of this apparently contradictory behaviour is the kind of properties used by students to do each classification and the kind of logic particles implied: On the one hand, students characterize rectangles and squares on the base of the lengths of their sides, but students in van Hiele level 2 are unable to understand the logic term "some ..." (for instance, *some* rectangles have four equal sides), and consider "none" as the negation of "all". Then, students believe that *all* rectangles must have sides with different length. On the other hand, the property used to identify parallelograms does not have such kind of logical difficulty, so students accept that squares, rectangles, and rhombuses are specific cases of parallelograms. The following case shows clearly such behaviour:

As part of a research project (Gutiérrez et al., 1994) designed a set of activities for teaching quadrilaterals, including classification of the main families. A question in the post-test was about classification of rectangles and squares. A student, reasoning in level 2, wrote: "You [the teacher] has taught us that squares are rectangles, but I do not believe it. I believe that squares are not rectangles".

In grades 3 to 5, students may begin to use dynamic geometry software to draw figures, measure them, observe properties, and check their truth. This may be the first approach of students to proof, certainty, and mathematical truth. Dynamic geometry software, like Cabri or Sketchpad, provide an environment where, in a few seconds, students can observe a variety of examples much richer than using any static tool or manipulative. Results as important as Pythagoras and Thales theorems may be discovered, experienced, and understood by students if they are provided with appropriate computer plus paper-and-pencil environments. Furthermore, they can attempt a variety of problems that, usually, are too difficult for them when stated to be solved in paper-and-pencil context, like the problem of the isoperimetric rectangles: “Given all the rectangles with a certain perimeter, find the rectangle with maximum area”. If students are provided with a file containing the construction (Figure 7), by dragging point *v* they can find that the rectangle with maximum area has all the sides the same lenght, i.e. it is the square having the given perimeter.

Insert here movie “Isoperimetric.mov”

Figure 7. Which rectangle has maximum area?

Students can continue learning about plane isometries (slides, turns, and reflections), focusing on the discovery of their main characteristics. Manipulatives are essential to help students to work on the activities. A difficulty of teaching this topic in grades 3 to 5 is the unability students have to make accurate drawings. Using cutouts helps them to solve the problems, since they do not need to spend too much time drawing. Students can also begin to use coordinates to characterize translations. At this stage, the meaning students give to the coordinate system is the record of the vertical and horizontal movements of objects equivalent to a given translation. Then, instead of negative numbers, children usually invent their own codes to differentiate opposite movements. For instance (Figure 8), the coordinates (3d, 5r) mean that the slide moves the objects 3 steps down and then 5 steps to the right.

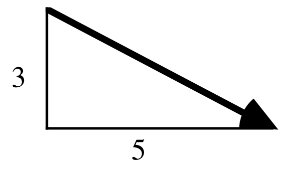


Figure 8. Coordinatizing a vector of translation.

The following are some activities adequate to students in grades 3 to 5 (Jaime & Gutiérrez, 1995, 1996):

1. *Shape A has been translated to shape B* (Figure 9). *Mark some points in shape A and the equivalent points in shape B. Draw the vector from each point in A to the equivalent point in B. What do you note? Write your conclusion down.*

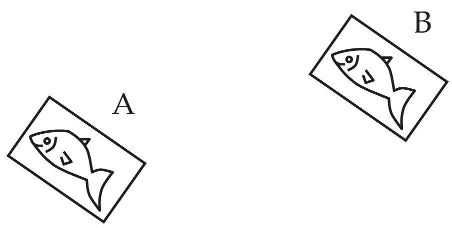


Figure 9.

After having learned to use coordinates, repeat the activity by asking students to calculate the coordinates of the vectors drawn.

2. *Rotate shape A -100° around each of the four points marked* (Figure 10). *Which relationship do you note among the four images?*

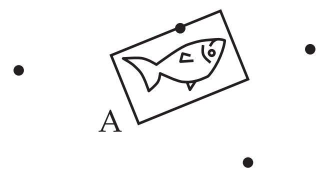


Figure 10.

3. *Move the shape A* (Figure 11) *with the product $S_2 \bullet S_1$ (first reflect it on axis e_1 and then reflect the image on axis e_2). By comparing first and last positions of shape A, identify which simple movement would have moved figure A to the same position.*

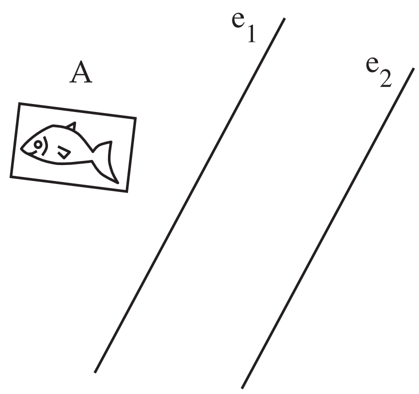


Figure 11. Product of symmetries with parallel axes.

After moving a few shapes from different positions, and using several pairs of axes, students in these grades discover that the movement equivalent to the product $S_2 \bullet S_1$ of symetries having parallel axes is a slide. However, students are not able to understand the mathematical relationship among the vector and the axes until they are reasoning in level 3.

Children in grades 4 and 5 can make their first approach to use some plane representations of solids, by drawing representations of given solids, and building solids from given representations. This learning should continue, at least, up to grade 8, since only then most students can master different usual types of plane representations (levels, isometric projection, and orthogonal projection).

When preparing activities to teach plane representations, teachers should be aware that even the simplest representation has some specific difficulty for children in grades lower than

6. According to results from research (Gutiérrez, 1996), representations by levels are the easiest, but children in grades 3 to 5 have difficulty in understanding the role and necessity of the rectangles framing each representation of a level when they are provided with “buildings” made of linking cubes (for instance, Multilink™ or Centicube™) and are asked to draw its “floors”. Figure 12 shows such behaviour: The shapes of the layer are correct, but the frameworks are wrong because they do not show the relationship between the three layers, so it would not be possible to rebuild the solid.

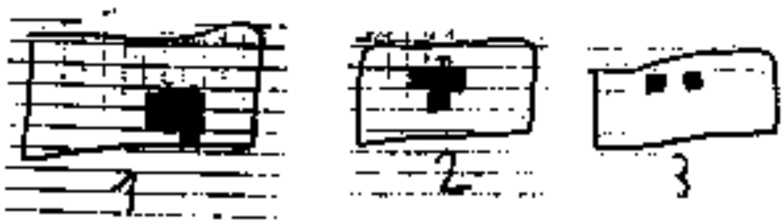


Figure 12. The three levels of a building.

• Grades 6 - 8.

Students entering grade 6 reason in van Hiele level 2. Now teachers should propose their pupils activities to reinforce their level 2 thinking and to introduce them to the ways of reasoning of level 3. The learning of level 3 thinking will continue for several year, ending in grade 10 or later. The main objectives for teachers in grades 7 and 8 should be to help students to discover that checking properties with examples do not garantee their truth, and to discover that the different properties of a figure are not independent but some of them are a consequence of the others. However, even when student reason in level 3, they still need the help of manipulatives or dynamic software to experiment, generate conjectures, and use examples to devise inductive reasoning.

As students accept the new role of examples (examples are no longer the holders of truth), they will be able to produce inductive proofs. To practice and deepen this new way of reasoning, students should work both on new properties and relevant theorems already studied before, like Pythagoras and Thales theorems. As in lower grades student just checked those theorems with some examples, now they should try to produce inductive abstract proofs. For instance, after students have learned the basics of graphical representation of functions, and parabolas, they can work again on the problem of the isoperimetric rectangles. Now the teacher provides them with an elaborated version of the construction they used before (compare Figures 7 and 13), where they may discover that the graphical representation of the area as function of the base of the rectangle is a parabola. Students should be able to produce an inductive proof of the answer to this problem based on the symmetry of the parabola.

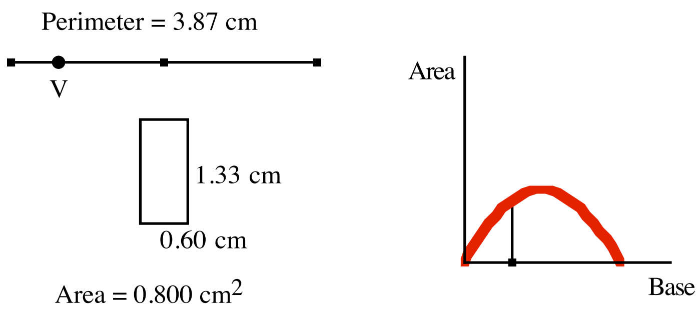


Figure 13. Prove that the square is the rectangle having maximum area.

In grades 6 to 8 students should complete the study of families of plane and space figures, focusing on definition, classification, metric properties, and relationships between properties of a family. An evidence students' level 3 reasoning is that they are able to understand the different classifications of quadrilaterals induced by different definitions of rectangle[1], rhombus[2] or trapezoid[3]. This study should also include an analysis of the relationships among different families of polygons and polyhedra, paying attention, for instance, to the influence of classifications of polygons on classifications of polyhedra. For

[1]) Rectangle defined as “quadrilateral having four right angles” vs “quadrilateral having four right angles and sides of different length”.

[2]) Rombhus defined as “quadrilateral having four equal sides” vs “quadrilateral having four equal sides and angles of different amplitude”.

[3]) Trapezoid defined as “quadrilateral having at least two parallel sides” vs “quadrilateral having exactly two parallel sides”.

instance, if squares are not rectangles, then cubes are not right parallelepipeds (prisms with all the faces rectangles).

In these grades, students should continue learning about isometries and similarity. Respect to the isometries, the main objective should be to broaden the knowledge of the algebraic structure of this set. Therefore, the study of different products an decompositions of isometries is essential, since they provide the relationships among the different isometries and they are the base for the technique used in the formal proofs of many theorems (Jaime & Gutiérrez, 1995). Some cases may have been studied before (for instance, the product of two slides), but the most important cases, like product of rotations having different centers, or decompositions of slides and rotation in products of symmetries, can only be understood by students reasoning in level 3 or higher.

An appropriate teaching strategy is to begin this topic stating some activities where students complete the learning of the different cases of products and decompositions, and continue asking the students for some level 3 justification of such results. Proving the commutativity (or the lack of it) of some products is a good opportunity for students to practice techniques of proof, since this property is true in some cases, but it is false in others. This forces students to look, in some cases, for a proof or, in other cases, for a counter-example. The following activity is designed to induce students to see that a rotation can be decomposed as product of two symmetries, with axes intersecting in the center of rotation, in infinitely many ways (Jaime & Gutiérrez, 1996):

There is a rotation, with center C (Figure 14), *moving shape A to shape B, but now we have to use a product of two symmetries to make such movement. Could you draw the two axes? Is it possible to find other solution (compare your result with those from your peers)?*

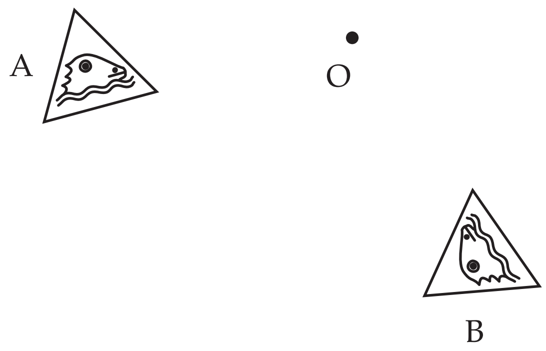


Figure 14.

After working in the activity, the teacher collects the decompositions produced by different students and show them to the whole class. The variety of results should raise a discussion about their correctness and, finally, help students to understand that there are infinitely many correct answers.

It was mentioned elsewhere in this paper that students in grades lower than 6 can generate and use a coordinate system for slides. In grades 6 to 8, students can continue learning about the Cartesian coordinate system by coordinatizing other elements of the isometries, like centers of rotation or axies of symmetry.

Respect to the similarity, students also progress in learning the basic concepts, main properties, and procedures to identify or draw similar figures. Like for the isometries, students should try to prove some properties of dilatation and similarity. They can learn, for instance, to calculate coordinates of images of points under dilatations. Golden ratio is an interesting topic combining history, art, and mathematic where students can give meaning to the concept of similar figures.

The study of congruence and similarity is completed with the criteria of congruence and similarity of triangles. A quite usual way of teaching these topics is to provide the students with the different cases, then ask them to memorize the information and, finally, propose some exercises where students have to check if two triangles are congruent (or similar). Instead, it would be much more interesting if teachers give their puils the oportunity to discover (in a guided discovering environment) and prove the different cases by applying their knowledge of isometric and similarity transformations. This can be made both in paper-and-pencil or computer environments. As an example, Figure 15 shows a way to prove the SAS criterium of congruence (the product S•R•T moves ΔABC to triangle ΔMNP):

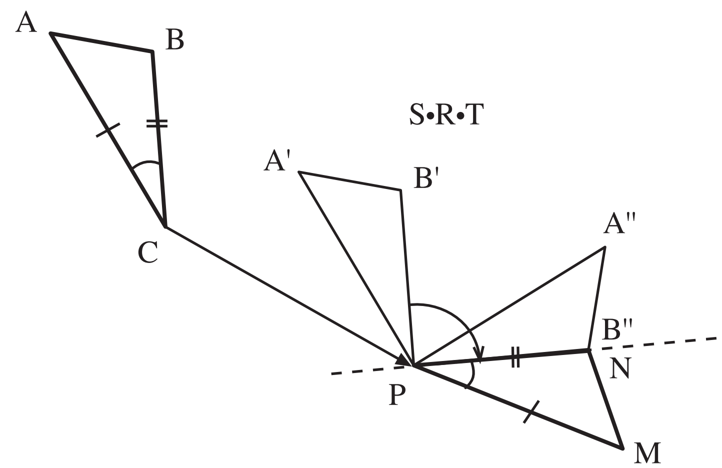


Figure 15. SAS criterium of congruence of triangles.

Given triangles ΔABC and ΔMNP, with AC = MP, BC = NP, and ∠C = ∠P,

1. If C ≠ P, let T be the slide with vector CP. Let ΔA’B’P be the result of such translation of ΔABC.

2. If B’ ≠ N, let R be the rotation with centre P and angle ∠B’PN. Let ΔA’’B’’P be the result of such rotation of ΔA’B’P. As BC = NP, then B’’ = N.

3. If A’’ ≠ M, let S be the reflection with axis NP. As AC = MP, the image of side A’’P is side MP, so the image of ΔA’’BNP is ΔMNP.

In grades 6 to 8 students have completed the acquisition of capabilities of visualization and spatial reasoning allowing them to acquire a good mastering of plane representations of three-dimensional figures in both directions, drawing representations and building solids (Gutiérrez, 1996). Students should practice the different kinds of representations (nets and projections) by a set of diverse activities, like to following ones:

1. The teacher provides students with some polyhedra made of Polydron-like pieces, and asks them to unfold the polyhedra so all the pieces (faces) rest on the table and remain connected.

2. Given some hexaminoes (Figure 16), students have to identify those that, when folded up, produce a cube.

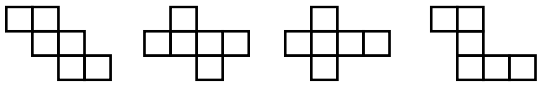


Figure 16. Which hexaminoes are nets of a cube?

3. Students are provided with linking cubes and some plane representations of modules (Figure 17). They have to build the solids represented.

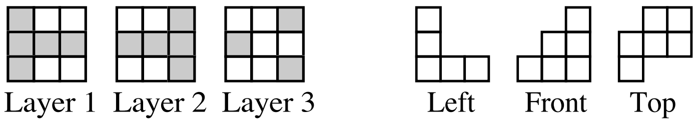


Figure 17. Build the solids represented.

4. Students are provided with a blank sheet of isometric paper and an isometric representation of a solid (Figure 18). They have to draw the isometric representation of the solid resulting after adding a cube on each shaded face. Depending on the expertise of students, they will be allowed to buid the solid or not.

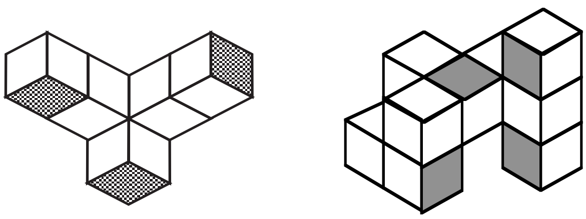
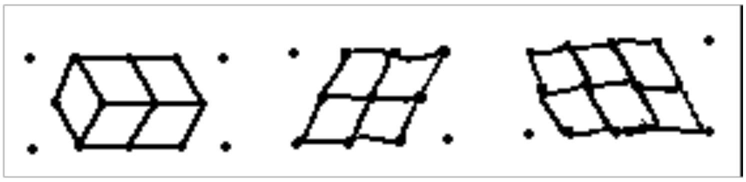


Figure 18. Add a cube on each shaded face, and draw the isometric projection of the resulting solid.

It is important to note that drawing isometric projections is quite more difficult than it could be expected. Figures 19b and 19c show two attempts of a student who was asked to copy on isometric paper the drawing provided on the same sheet (Figure 19a). As can be observed, such difficulty comes from the necessity of coordination between the different planes of the faces (Gutiérrez, 1996).



- a - - b - - c -

Figure 19.

Specific dynamic three-dimensional geometry software provides an excellent environment for students to improve their abilities of visualization. In particular, are useful those pieces of software representing solids on the screen, and allowing to rotate them in any direction (Gutiérrez, 1995).

• Grades 9 - 12.

An objective of upper secondary school mathematics courses is to introduce students in formal mathematical thinking. Those students who succeeded in beginning to reason at level 3 during grades 7 or 8, should continue acquiring expertise in this type of reasoning during grades 9 and 10. Then, in grades 11 and 12 the students are ready to learn the basics of formal mathematical reasoning (van Hiele level 4), formal proofs being only one of the elements characterizing such kind of reasoning. Although a benchmark for courses in grades 11 and 12 may be the students to be able to do formal proofs, teachers have to be aware that many students in these grades will not be able to reach such level of reasoning.

A way to introduce students to formal proofs, their meaning and necessity is to go back to results previously proved by inductive methods, analyze the flaws of these inductive proofs, and try to produce better, deductive, proofs. For instance, the problem of the isoperimetric rectangles (Figures 7 and 13) may be analyzed again after students have learned to calculate maximums and minimums of functions. Now, students can use dynamic software, but the focus of the activity is to let students understand that the evidence derived from the screen (the symmetry of the curve) has to be verified by a more formal way guaranteeing that such curve is really a parabola. Then, they should work to obtain an analytical expression showing the area of a rectangle as function of its base, and then to calculate the maximum of such function:

For a certain perimeter *p,* the relationship among base and altitude of a rectangle is h = $\frac{p - 2b}{2}$, hence the area of the rectangle is A = f(b) = b $\frac{p - 2b}{2}$. This function has its maximum in b = $\frac{p}{4}$, that is for the square.

Although dynamic software can be a very interesting tool helping students in lower grades to progress in their understanding of proofs (from examples to inductive reasoning), the immediateness and endlessness of examples is a potential obstacle for students to feel the necessity of abstract formal arguments. In this situation, the format of proofs is not important, but students' awareness of its necessity. After the emphasis put during 80s on "two columns" proofs, research shows that there is no evidence of advantage in teaching such format of writing proofs instead of other (Martin, 1990). Furthermore, current research pays much more attention to the process of understanding the concept of formal proof (see the proceedings of recent PME international conferences).

The study of isometries and similarity is completed in grades 9 to 12 by presenting to the students the whole algebraic structure. They should have acquired in lower grades all knowledge necessary to make a formal study of the group of plane isometries, including, results like:

- Translation, rotation, and reflection are isometries (i.e. the distance between the images of any two points is equal to the distance between the points).
- Any isometry can be decomposed as product of, at most, three reflections.
- Any similarity can be decomposed as product of an isometry and a dilatation.

- The classification theorem for plane isometries: There are only four isometries, translation, rotation, reflection, and glide reflection.

Apart from the geometric (graphic) and algebraic studies of isometries, in grades 9 to 12 the analytic approach to those transformations can be studied. From the coordinates of each transformation, their matricial representations can be obtained, and also their equations.

it is also interesting to study in these grades the space isometries. Students can compare both sets of transformations and analyze the similitudes and differences among them. Specific three-dimensional dynamic software is of great help in this study, since it is quite difficult to provide students with physical models of all the solids they could need to use.

Those students in upper high secondary school beginning to acquire the level 4 reasoning should be put in contact with several components of a formal geometry system. Techniques for formal proofs (direct, converse, ad absurdum, etc.) are the main component, but it is also important to learn the structure of an axiomatic system. The first pages of Book I of Euclid's Elements (**RICH, INCLUDE HERE A REFERENCE OF EUCLID'S ELEMENTS AVAILABLE IN USA**) contain a clear example of such structure: Definitions, postulates, axioms, and propositions (theorems).

After some refection on the Elements, students can be induced to experiment with other geometric axiomatic systems. Each geometry presents certain differences with Euclidean geometry that can be analyzed. For instance:

1. The taxicab geometry is defined on a square lattice (Krause, 1986). It is a particularization of Euclidean geometry, since the same concepts and metric are used, but the space is only the set of points in the lines of the lattice. Then, distances can be measured only moving in horizontal and vertical (Figure 20).

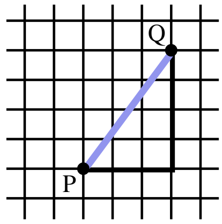


Figure 20. Euclidean distance from P to Q = 5, and Taxicab distance from P to Q = 7.

Some problems can be stated to highlight the differences between both geometries: Calculate the expression for the taxicab distance between two arbitrary points P (p_1, p_2) and Q (q_1, q_2). Draw a taxicab circle. Draw the perpendicular bisector between two points A and B.

2. Any high school student knows that straight lines are not real objects but mental constructs and, when we draw a straight line on the floor, we “ignore” the imperfections of the drawing. A way of introduction of spherical geometry is to ask students to look at the earth from the distance, like from a space shuttle orbit, and to decide how straight lines are seen from there. Then, some other questions can be asked: What is a “straight” line on a sphere? How can we measure distances on the surface of a sphere? How angles? Draw the usual plane Euclidean shapes (triangles, squares, circles, etc.) on a sphere. What characteristics of Euclidean geometry are lost in spherical geometry? (Henderson, 1996).

3. Historically, the negation of the Fifth Euclidean Axiom (axiom of the parallel lines) was the origin of Gauss, Bolyai, and Lobachevsky’s hyperbolic geometry. Secondary school students with highest level of reasoning can approach this geometry, and can experiment another example of non-Euclidean geometry, more abstract and formal than the previous ones (Coxeter, 1969). A particular representation of hyperbolic geometry is the Poincaré’s hyperbolic plane, where the horizon is a circle and “straight” lines are the parts contained in the horizon of circles orthogonal to the horizon (Figure 21).

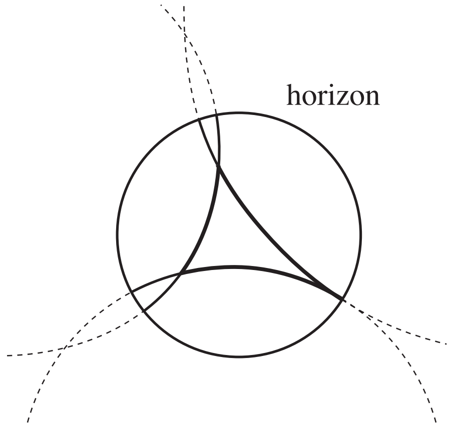


Figure 21. An hyperbolic triangle.

Dynamic geometry software can be a great help for students since, even in highly formalized contexts like this one, students need to visualize the concepts to better understand them. For instance, they can measure the angles of a hyperbolic triangle and transform it to observe the change in the angles and make conjectures about the value of their sum (Figure 22).

Insert here movie “Hypergeom.mov”

Figure 22. How much add the angles of a hyperbolic triangle?

In any of the previous examples of non-Euclidean geometries, more sophisticated problems can be stated, like those dealing with perpendicularity, parallelism, coordinate systems and equations, trigonometry, etc.

Grades 9 to 12 are the appropriate place to begin the study of analytic geometry. First, students should work on simple cases, like equations for straight lines, circles or parabolas. Then, they can apply such knowledge to solve other problems. For instance, they can look at the zeros of a parabola to obtain the usual expression of the roots of a quadratic equation (Gutiérrez, 1981). Then, they can continue working on more complex problems.

The higher students’ level of reasoning, the higher their capability to integrate knowledge from different areas of mathematics or sciences. Then, in last secondary school grades, students in level 4 of thinking should be given the opportunity to explore the uses of geometry in other contexts. These uses are, some times, from mathematics, like in calculus and the geometric interpretation of the concepts of derivative and integral, or probability and the geometric representation of probability (NCTM, 1988). Other times geometry is used in sciences, as in mechanics, where some geometric machines, like pantographs, are used (Figure 23). Then, geometry students can analyze the structure of such machines to discover, and then prove, their geometric characteristics (Bartolini, 1993). Again, dynamic geometry software can be used, even more easily than real models since software allows for more measurements and explorations.

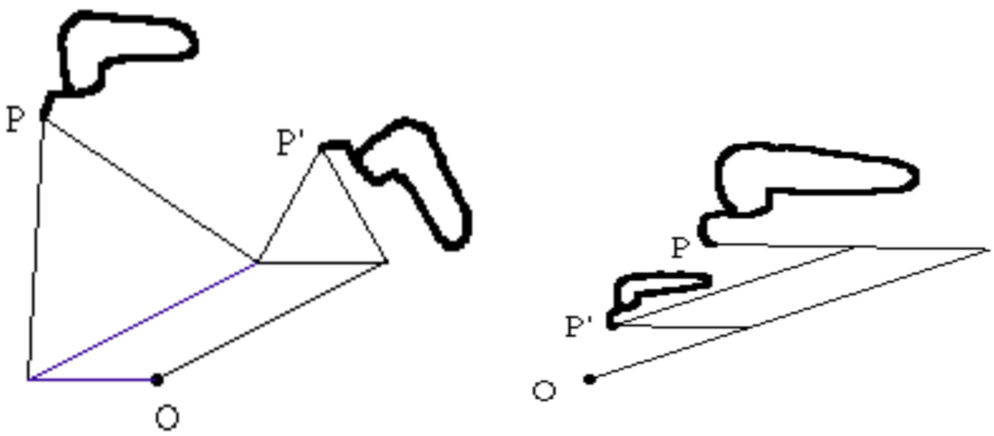


Figure 23. Pantograph of Sylvester (left), and usual pantograph (right).

Research shows that some people feel more comfortable when reasoning in a geometric (visual) way, while others like better to do analytic (non-visual) or harmonic (mixed) thinking (Krutetskii, 1976). On the other side, recent research on learning secondary school and university mathematics shows clearly that better results are obtained when students receive visual information to help them understand abstract and formal contents. In this way, geometry becomes a tool that can be used in learning different topics of mathematics and also other subjects.

As students grow in knowledge and thinking capability, they are able to deal with more complex concepts and problems. Then, parallel to the increment of students' quality of reasoning, it is necessary an increment in their ability of visualization and mastering in using visual strategies for representing information and solving problems. There are several ways to do it. We sketch below some examples.

The work on plane representations of three-dimensional figures can be continued by exploring more deeply relationships between plane and spatial shapes, experimenting, conjecturing, and proving the conjectures. Both static and dynamic problems can be stated. For instance, students can be asked to rotate a solid given on a computer screen from its current position to math exactly the position shown in a picture. Depending on characteristics of the represented solid (its complexity), and the software, (the way it receive inputs from the user or the way it shows the rotation), several degrees of difficulty can be obtained, appropriate for students with different levels of expertise in using visualization abilities (Gutiérrez, 1996).

Students in grades 9 to 12 can work on cross sections of familiar solids like prisms, pyramids and solids of revolution, and combinations of visual (i.e. based on mental images), deductive, and manipulative activities can be organized. A well known activity is to ask which shape has the cross section of a cube when the plane cuts specific points (for instance the midpoints) of three edges?

Usually students begin by sketching cubes on paper, drawing different cross sections, and conjecturing about the number of sides, regularity, shape, etc. of the sections. Then, they should try to arrange a deductive argument to prove that the conjectures are correct. Finally, the activity can be completed by making the cross sections if students are provided with cubes made of plasticine or styropor, or some specific dynamic three-dimensional software, like Doorzier (Doorman & Schumann, 1996). Figure 24 shows the consecutive steps to cut a cube by selecting three midpoints of edges. An advantage of this software is that sections can be moved in the solid and, therefore, it allows to analyze the problem much more in detail than using real solids.

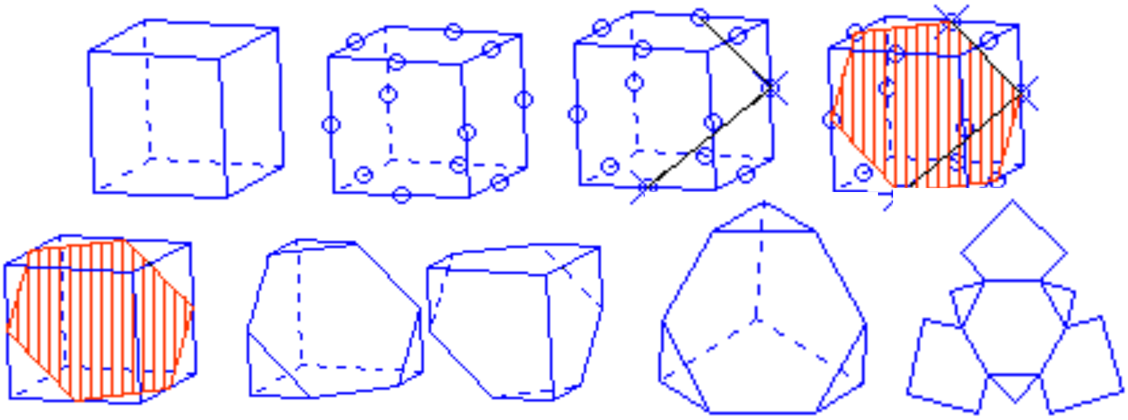
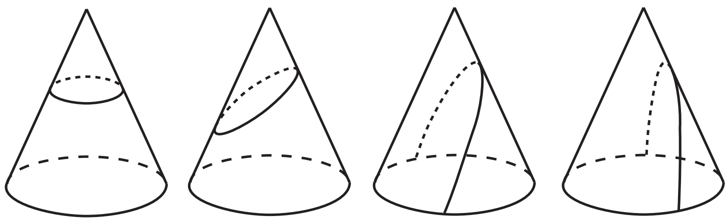


Figure 24. Process of cutting a cube and creating the net of a part.

Students can work on conics by integrating geometry, algebra, and calculus approaches. On the one side, sectioning solids of revolution in several ways produces the different conic curves (Figure 25). A complementary conception of these curves is induced by their locus characterizations. Finally, the algebraic expressions can be obtained, and an analytical study can be performed.



Circle Ellipse Parabola Hyperbola

Figure 25. Cross sections of a cone.

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