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THE LEARNING OF PLANE ISOMETRIES FROM THE VIEWPOINT OF THE VAN HIELE MODEL

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Abstract

The aim of this communication is to suggest a new application of the Van Hiele model of reasoning. We present a description of the Van Hiele levels for the learning of plane isometries, and some examples of activities for each level. We have obtained this theoretical description of the Van Hiele levels from experiments in school settings carried out with primary school students and with pre-service primary school teacher.

Introduction

The Van Hiele model of reasoning claims that there exist several levels of reasoning for the students; one of its main claims is that, for successful teaching, it is necessary to take into account the students' current level. Therefore, one of the main aims of the Van Hiele model, is to analyze each area of geometry (or of mathematics in general) and to characterize each level of reasoning using elements belonging to a given area, in order to develop teaching units for the classroom. In the existing literature there are general descriptors (see Usiskin (1982), Burger & Shaughnessy (1986), Hoffer (1983), and Fuys, Geddes & Tischler (1985)) and also specific descriptors and teaching units focused on several areas of plane geometry, such as polygons, angles or surfaces (see Fuys, Geddes & Tischler (1985) and Scally (1987)). But there are other important topics which have not yet been investigated; one such topic is geometric transformations and, in particular, plane isometries; although Hoffer (1983) and Alsina, Burgues & Fortuny (1987) do present descriptions of the levels in terms of plane isometries, they are simply theoretical statements lacking any further practical application. In our current research we have continued the study of the Van Hiele model that we began some time ago, with relation to measurement and spatial geometry (see Gutiérrez & Jaime (1987a) and Gutiérrez, Fortuny & Jaime (1988)) by working on plane isometries.

The results that we show here have been obtained from our work over several years on teaching plane isometries to primary school pupils and to future teachers in Valencia (see Gutiérrez & Jaime (1987b and 1988)). Our wish to provide pupils with activities according to their reasoning abilities has led us to use the Van Hiele model. Therefore, we have first determined the characteristics of translations, rotations and symmetries for each Van Hiele level and, within each level, those corresponding to the learning phases which allow access to the higher level. Secondly, we have designed teaching units for each isometry, taking into account these characteristics.

Now we shall present the general characteristics of each level, related to plane isometries. As we think that it will be clearer if we give examples of just one isometry instead of using all three symmetries for different examples, we will confine ourselves to the translations in the examples.

Of the various opinions on the number of levels of the Van Hiele model, we assume (see Gutiérrez & Jaime (1987a)) the existence of four levels of reasoning, namely, (1) recognition, (2) analysis, (3) classification and (4) deduction. Table 1 shows a summary of the Van Hiele levels of reasoning for plane isometries.

Now we shall make a detailed description of the characteristics of the four levels and the most significant results of our experiments. The activities on plane isometries that we propose to the students include, in general, the use of cut-outs, to promote active learning and to avoid difficulties caused by the children's lack of drawing ability.

Table 1: The Van Hiele reasoning levels in plane isometries

Level 1	Visual identification of translations, rotations and symmetries. Static recognition: Identification of isometric figures. Dynamic recognition: The movements are carried out automatically.
Level 2	Experimental discovery of the elements and basic properties of the isometries. The isometries are made and identified by means of their elements and basic properties.
Level 3	Experimental deduction of relations and properties of the isometries. Justification of properties and relations already known. Formal definition of translation, rotation and symmetry. Products and decompositions of isometries are determined.
Level 4	Global insight of plane isometries: Properties are proved formally; the structure of group is taken into account; the relations existing between the isometries are generalized; ...

<u>LEVEL 1</u> There are two ways to beginning to discover the plane isometries: static and dynamic. The <u>static</u> approach consists of the visual recognition of figures which correspond to each other under an isometry; this recognition includes the use of figures arranged in non-standard positions. In the <u>dynamic</u> approach, the students move the figures physically; in the early phases of this level they use some devices (rulers, discs, mirrors, computers, folding, ...), and in the later phases the students can begin to perform the movements without those tools, by remembering what they have done before.

Some *types* of activities for translations on level 1 are:

- Giving examples and non-examples of translations.
- Moving figures or objets along a ruler or a straight line.
- Asking pupils to talk about the differences between translated and non-translated figures; to do so, they can use a ruler and make the movements physically or tell by looking at the figures.

- Asking the students for some examples of translations from his environment.
- Translating a figure so that one of its segments maps onto another given segment.

It is evident that when students use visual recognition (a behaviour characteristic of the first level), they use the elements of isometries (directed segment, center, reflection line) and some of their basic properties, but they will only become conscious of them when they have reached the level 2.

<u>LEVEL 2</u> The work with the students at this level begins with the discovery of the basic elements and characteristics of each isometry: Directed segment and parallelism (translations), center, directed angle and movement along circumferences (rotations), reflection line, equidistance, perpendicularity and inversion (symmetries). When identifying which figures correspond to each other under an isometry, at this level the students do not base their reasoning only on visual recognition, but they also verify the presence of the basic properties of the identified isometry; this allows the students to use ruler, compass and protractor to move points of the figures.

However, the students do not relate the properties to each other, that is, they have not yet built up the network of relations; consequently, they are not able to determine minimal sets of properties that characterize an isometry and, theren, they cannot properly define isometries.

One typical piece of behaviour observed in the early phases of level 2 is to expect different images after moving a figure under the same translation when the origin of the arrow has been placed on different points of the figure (see figure 1).

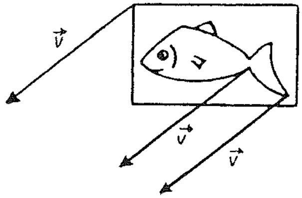

Figure 1

After experimentation they realize that the result will be the same, but they do not understand why.

The absence of the network of relations can also be seen in the way students manipulate two figures to check if they correspond under a specific isometry. Moreover, the students do not realize that they can locate the whole figure image under a given isometry when they know the image of two points of the figure (see figure 2).

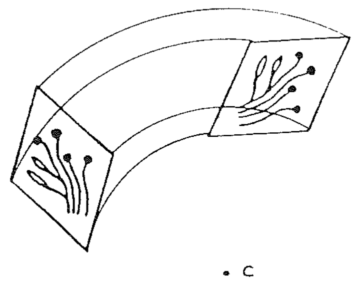

Figure 2

At level 2, the students learn to distinguish and to use the characteristics of translations (length, slope and direction); they also discover by experimentation other basic properties, such as parallelism between the corresponding figures. When working with squared paper, students can also discover the coordinates of the arrow defining a translation, and they can describe them by means of whole numbers qualified by words such as right/left up/down (if students already know the negative numbers, they can use them). They can find products of translations and deduce from experimentation some properties, such as commutativity.

With respect to rotations, some of the facts that the students will discover at level 2 are equidistance from the center, variation of slope (according to the rotation angle) of the rotated figure, the importance of angle direction and the existence of equivalent rotations. The students can also handle products of rotations with the same center and discover some algebraic properties.

As for symmetries, the students will discover equidistance and perpendicularity with respect to the reflection line of two symmetric points. They also recognize other properties such as the parallelism between the segments that join several points and their respective images under a symmetry, the inversion of figures, the fact that the position of the image of a line varies according to its position relative to the reflection line, etc.

There are some examples of different *types* of level 2 activities for translations:

- Performing a translation given its directed segment.
- Performing a translation on squared paper, given the coordinates of its directed segment (for instance, 3 squares to the right, 5 squares down).
- Completing several frieze patterns from the same figure, by means of translations whose arrows differ only in slope, length or direction. Comparing the results and discussing the differences.
- Checking whether two figures correspond to each other after a translation and, if they do, finding its arrow.
- Obtaining products of translations and observing the results.

<u>LEVEL 3</u> At this level the students have already acquired the ability to relate the properties they already know and to discover new properties by experimentation and informal deductive reasoning. They give definitions for each isometry, that is, they identify minimal sets of sufficient conditions to characterize an isometry. They can give informal proofs for properties discovered at level 2.

The students now know the minimal number of point-images of a figure needed to locate the whole image, and can justify this.

With respect to the product of isometries, students can deduce the result of products of two symmetries or two rotations. This will allow them to begin to build up a network of relations between various isometries in the later phases of level 3 (because they can find products which include different kinds of isometries) and to acquire a global understanding of isometries when they reach level 4. It is also possible at this level to work with glide reflections and to discover or deduce some of their properties from the knowledge they already have of translations and symmetries.

In the later phases of level 3 the students can handle the general decomposition of isometries and, in several cases, obtain all the possible solutions (infinite sometimes).

As for rotations, the students can deduce that the perpendicular bisector of a segment is the set of the centers of all the possible rotations which map one endpoint of the segment onto the other; in this way, they will be able to discover the centers of rotation and will understand the meaning of the usual algorithm to discover the center of a circle.

There are some *types* of level 3 activities about translations:

- Finding products of several translations and discovering, from the coordinates of their directed segments, the coordinates of the resulting directed segment. Generalizing and justifying the result.
- Decomposing a translation into several products of translations (and justifying that there are infinite possibilities).
- Decomposing a translation into two symmetries, a) when one reflection line has been fixed, b) when no reflection line has been fixed. Discovering and comparing the number of possible solutions in each case.
- Predicting and justifying the result of the product of a translation and a rotation.

<u>LEVEL 4</u> The main activity which students develop at this level is formal and consists in deducing and proving complex properties and theorems which in the previous levels were out of the students' reach.

These are some of the facts which must be used in the activities belonging to level 4, because they help to acquire a global insight of isometries:

- The group structure of the plane isometries as a basic tool.

- The Classification Theorem of the Plane Isometries (every isometry is equivalent to a product of at most three symmetries).
- Equivalent movements, decompositions and products.
- Given the characteristics of several isometries, identify the movement which results from their product

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