Adela Jaime and Angel Gutiérrez

Guidelines for Teaching Plane Isometries in Secondary School

Two key concepts strongly supported by the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) are (1) that geometry should be taught from multiple perspectives and (2) that it is necessary to make mathematical connections. Both concepts can be implemented by appropriately selecting topics that can be presented to students from several points of view in different environments and that can also link different branches of mathematics or mathematics and other sciences.

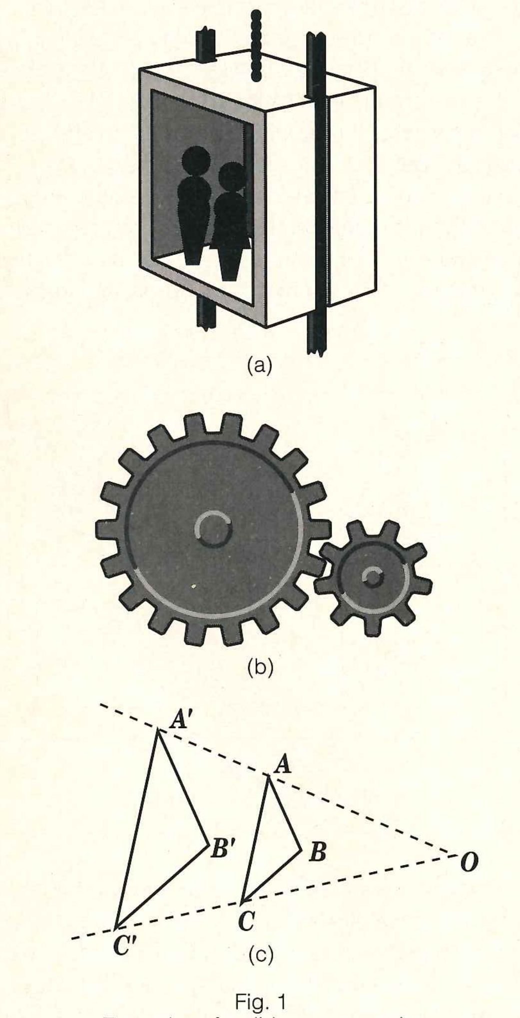

Fig. 1
Examples of a slide, a turn, and a similarity transformation

Isometries is such a topic. An isometry in the plane is a transformation from the plane to itself, preserving the distance between points. Translations (slides), reflections (flips), and rotations (turns) are types of isometries. Apart from their mathematical definitions, isometries can be represented as physical movements. Slides (translations) are movements along straight lines, like the movement of an elevator **(fig. 1a).** Turns (rotations) are movements along circles, like the movement of wheels in a machine or the hands in a clock **(fig. 1b).** Flips (reflections) can be represented by images in a mirror. A more detailed description of the relationship between the mathematical and the physical aspects of isometries can be found in O'Daffer and Clemens (1977). Not all transformations of the plane are isometries, although some transformations have properties in common with isometries. For instance, although similarity transformations maintain the shape of figures, they do not preserve the size, as do isometries. In **figure 1c,** triangle $A'B'C'$ is an enlargement of triangle ABC from the point O and is an example of a similarity transformation.

Isometries furnish a rich diversity of activities

On the one hand, the study of isometries can be organized from multiple approaches based on (1) different kinds of manipulatives, such as mirrors and Miras, paper folding, geoboards and dot paper, tessellations, and squared paper and coordinates; (2) computer software, such as Logo, Cabri, The Geometer's Sketchpad, and The Geometric Supposer; and (3) real-life problems involving the structure

Edited by ***J. Michael Shaughnessy***
Portland State University
Portland, OR 97207

Adela Jaime and Angel Gutiérrez teach in the mathematics education department at the University of Valencia, 46071 Valencia, Spain; angel.gutierrez@uv.es. As researchers, they are interested in the process of learning and teaching geometry and, more specifically, in the application of the van Hiele model and visualization in spatial geometry.

of objects, buildings, companies' logos, industrial design, art, and so on. This wide range of possibilities presents a rich diversity of activities useful for students from the early primary grades through secondary school.

On the other hand, isometries can serve as a conceptual and technical tool useful in many other areas of geometry. For instance, isometries can be used to classify polygons and solids by type of symmetry, to construct plane representations of three-dimensional objects, and to analyze motions in trigonometry or analytic-geometry settings. The common tools, techniques, and vocabulary of isometries can enable students to make connections and to transfer knowledge across branches of mathematics as well as to physics, architecture, and other subject areas. The increased capability of new computers has resulted in the development of computer-assisted-design (CAD) and other software dealing with three-dimensional objects. In this field, translations, rotations, and reflections are needed to create objects, to move them, and to build virtual-reality environments. More detailed arguments in support of the relevant place of isometries in the curriculum of secondary schools, including many examples of specific activities, can be found in the NCTM's *Addenda* Series books on geometry (Coxford 1991; Geddes 1992).

Insight is possible only after direct experience

THE VAN HIELE MODEL

The van Hiele model of mathematical reasoning has become a proved descriptor of the progress of students' reasoning in geometry and is a valid framework for the design of teaching sequences in school geometry as acknowledged by the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) and by the *Addenda* Series books devoted to geometry. The van Hiele model establishes four levels of reasoning, from a holistic reasoning based on physical attributes of figures (level 0) to a formal abstract reasoning (level 3). According to the van Hiele model, an important characteristic of mathematical reasoning is that growth in age does not necessarily imply growth in a student's level of reasoning. Instruction plays a central role in a student's progression through the levels.

The van Hiele model also specifies five phases of teaching to be considered when teachers organize their students' activity: (1) inquiry and information, (2) directed orientation, (3) explication, (4) free orientation, and (5) integration. A more detailed description of the van Hiele model can be found in Burger and Shaughnessy (1986), Crowley (1987), Fuys et al. (1988), Geddes (1992), or Hoffer (1983). According to Geddes (1992), mathematics teaching should be based on van Hiele levels 0 and 1 for grades K–4 and on levels 0, 1, and 2 for grades 5–8. We would add that instruction for grades 9–12 should be based on levels 1, 2, and 3.

A TEACHING EXPERIMENT

In this article we present results from research aimed to design and experiment with a set of units for teaching plane isometries in grades 3–12 (Jaime 1993). The activities in the units are organized according to the levels of reasoning and the phases of instruction defined by the van Hiele model. We have carried out this research project in Spain, funded by the Valencian government and the University of Valencia, during a five-year period and have used the teaching units with primary and secondary students and with preservice teachers at the university level. In some situations, whole classes taught by their teachers participated in the experiments. In others, the experiment took place under laboratory conditions, with groups of two to six students, directed by the researchers themselves.

The teaching sequences that we have designed can be implemented without expensive or sophisticated material. The basic manipulatives needed in the activities are sheets of paper, glue, and cutouts as shown in **figure 2.** The only restriction is that the cutouts cannot have figures on them that are invariant under rotations. For example, a circle will not work, since it will look the same after it has been rotated. The size of the cutouts may vary depending on the students' age and ability. In addition to the cutouts, students can use such materials as rulers, compasses, drawing triangles, protractors, transparent circles, Miras, and mirrors.

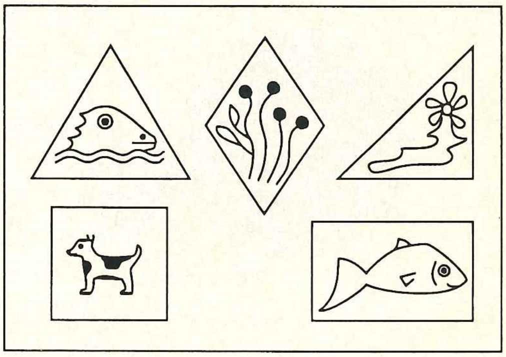

Fig. 2
Cutouts used in the activities

Students are asked (*a*) to move the shapes physically and (*b*) to glue them on a sheet of paper after completing a movement so that they can recall and observe the results. For students at levels 0 and 1, the manipulation of objects is absolutely necessary to grasp the meaning of each isometry. Insight into the mathematical characteristics of each type of isometry is possible only after some direct experience. For students at level 2, physical manipulation is still necessary to solve many activities, both as a source of generating ideas and as a way of verifying conjectures. Although students at level 3 may be able to use abstract reasoning, manipulation may help them to understand better the statements of propositions or proofs.

THE OBJECTIVE OF THE TEACHING UNITS ON ISOMETRIES

We will not attempt to show here the entire sequence of activities for each isometry across all the van Hiele levels and phases, which can be found in Jaime (1993). Examples of activities will be limited to levels 1, 2, and 3, relevant for secondary school, along with a brief summary of the main characteristics of isometries for each van Hiele level and the corresponding objectives of the teaching units **(table 1).** The teaching units in our sequence are numerous and sufficiently diverse so that teachers can give their students enough experiences to help them achieve the objectives for their current level.

TABLE 1

Characteristics and Teaching Objectives of the van Hiele Levels in Isometries

Levels	Characteristics of Thinking	Teaching Objectives
0	Students have a global nonmathematical view of isometries.	—To recognize each isometry as a movement and to perform isometries using appropriate manipulatives —To recognize the invariance of shape and size under an isometry
1	The elements and properties that define each isometry are considered by students.	—To explicitly use the elements that characterize each isometry —To begin to use the definition of each isometry —To discover new properties of isometries from experimentation, to state and use them —To make products of isometries by moving cutouts —To discover that compositions of translations commute, as do compositions of rotations around the same center —To begin to use mathematical notation and vocabulary for the different isometries
2	Students can discover and use properties of, and relations among, isometries; can understand general mathematical reasoning; and can make informal arguments as proofs.	—To understand and use the intersection of perpendicular bisectors to determine the center of a rotation —To complete the analysis of the product of two isometries started in level 1 —To understand and use the infinite number of ways of decomposing rotations or translations into a product of two symmetries —To justify and use properties of isometries —To understand the formal definition of each isometry —To understand simple formal proofs that are shown and explained by the teacher
3	Students can reason without any concrete support.	—To complete the knowledge in previous levels by deducing and proving more complex properties of isometries —To acquire a global view of the set of plane isometries and its algebraic group structure —To do formal proofs, in particular the theorem of classification of isometries: *Every isometry is one of these four: a translation, rotation, symmetry, or glide reflection.*

EXAMPLES OF STUDENTS' REASONING ON ISOMETRY ACTIVITIES

We present several activities from our teaching units that correspond to different van Hiele levels, along with some students' answers and an analysis of the answers in terms of the van Hiele levels. The answers are representative of secondary school students' responses using different levels of reasoning.

Activity 1. To discover that if a figure is rotated through the same angle about several different centers, translations occur among all the images of the rotated figure. (See **fig. 3.**)

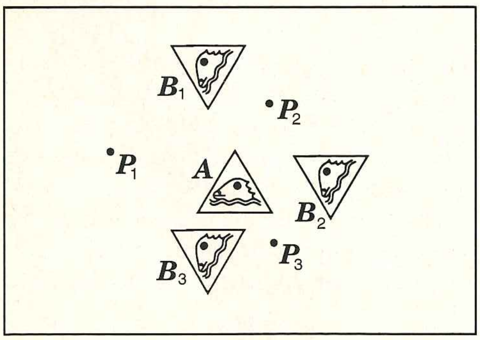

Fig. 3
The figures B_i are rotations $R(P_i, 60°)$ of cutout A.

This activity corresponds to van Hiele level 1, since it asks students to obtain a conclusion from experimentation. It also corresponds to the van Hiele instructional phase 4 of free orientation because it can be solved in several ways and uses properties of rotations and translations that have been studied previously.

The property suggested in activity 1, recognizing that images are all translations of one another, is quite visual, which the students soon realized. Nevertheless, recognizing and generalizing this property do not necessarily imply the ability to relate it to other properties. This thinking is typical of students who are reasoning at van Hiele level 1. The following are comments by students after they had obtained the image of a figure under several rotations of 90 degrees, each with a different center:

Josefa: All [the images] are looking at Mónica.
Mónica: All are translations.
Josefa: I realized that, too.

After a few similar exercises, these students were asked to rotate the cutout 90 degrees three times, so that the center of the turn was at a different point each time. (See **fig. 4.**)

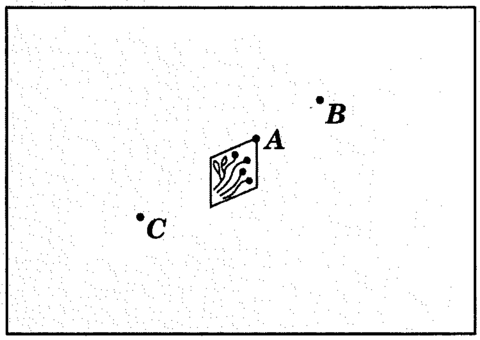

Fig. 4
Students rotated the cutout 90 degrees three times.

Rosa: That's right. All [the images] are translations of each other. [She says to Josefa] It [Josefa's answer] is wrong because it is not a translation.

Next the teacher asked for a rotation of the same figure, with vertex A as the center of the rotation. This time the students were not allowed to use a protractor or a compass. Mónica realized that the result should be *parallel* to the other figures but said, "I can't do it." However, at a later point, she did manage to complete the exercise.

Rosa had taken a cutout, placed it over the figure, and rotated it with her hand around vertex A so that it was parallel to the other images that had been obtained after rotations of 90 degrees. In contrast, Josefa had just placed the image parallel to the initial figure without using the rotation or the center, likely because she heard from Mónica that the solution should be "parallel." These students used the word *parallel* to describe the three images of the initial figure after the rotations $R(A, 90°)$, $R(B, 90°)$, and $R(C, 90°)$.

Although these students did justify their results from the regularities they observed on their sheets, they did not try to explain their results, which is typical of students using level-1 reasoning.

Activity 2. To deduce and justify that the product of rotations is equivalent to a translation when the sum of the rotation angles is a multiple of 360 degrees

If the sum of the angles is not a multiple of 360 degrees, then the product of two rotations is equivalent to a rotation whose angle is the sum of the component angles. For example,

$$R(C, 130°) \cdot R(B, 85°) \cdot R(A, 145°) = T_v$$

and

$$R(B, -30°) \cdot R(A, 90°) = R(C, 60°).$$

(See **figs. 5** and **6.**)

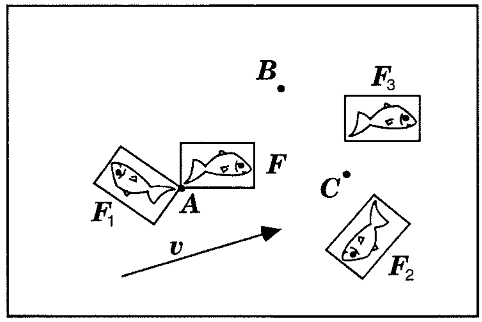

Fig. 5
F_3 is the image of F by $R(C, 130°) \bullet R(B, 85°)$ by vector v. Thus, $F_3 = T_v(F)$.

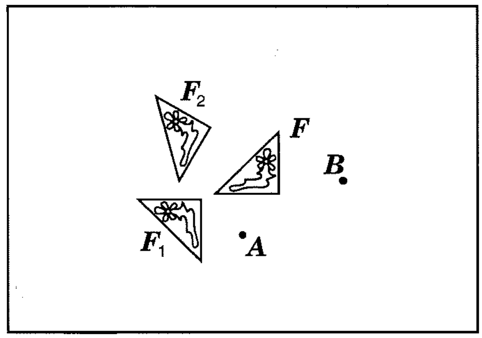

Fig. 6
F_2 is the image of F by $R(B, -30°) \bullet R(A, 90°)$.

Different notations are used for the isometries in publications or textbooks. For rotations, R_ϕ is used frequently, where R stands for rotation and ϕ is the angle of rotation. However, this notation is confusing when several rotations with different centers have to be used. We use the notation $R(C, \phi)$, where C is the center and ϕ is the angle of rotation, to avoid confusion. Another equivalent notation would be

$$R_\phi^C.$$

We also use the notations T_v, where v is the vector showing magnitude and direction for translations, and S_e, where e is the axis of a reflection for a symmetry. The word *symmetry* is often used in texts as a synonym for the word *reflection*. Lastly, a "•" is used to represent the product operation on isometries, also referred to as the composition of isometries. In a product, the motions are performed from right to left.

This activity corresponds to van Hiele level 2, since students have to connect several properties of rotations and translations and to use some abstract reasoning to generalize the results. Activity 2 is also typical of instructional phase 2 because it presents a basic property of rotations and introduces students to a new kind of problem.

To solve activity 2, the students were given two exercises. First, they were asked to rotate a figure 90 degrees and then to rotate its image 180 degrees, that is, $R(B, 180°) \cdot R(A, 90°)$. Once the students had obtained the image under this product, the following dialogue occurred:

Teacher: Is there any rotation that moves the first figure onto the last one?

Rosa: 90. No, there is not. Yes, 90.

Josefa: [Takes a cutout; places it over the initial figure; rotates it with her hand, following approximately the path of the required rotation; and places it in the final position.] Of course! And it is alike. [She means that the cutout fits the image exactly.]

Teacher: Why 90 degrees?

Rosa: If we apply 90 degrees, then there is a translation. [Between it and the initial figure. To explain, she takes a cutout, places it on the initial figure, rotates it 90 degrees, and then translates it onto the image.]

Since Rosa's reasoning level was moving ahead of the other students' reasoning, more instruction was necessary so that the other students could understand Rosa's explanations. The van Hiele levels of thinking used by this group, and therefore the kind of instruction they needed, varied from Josefa, who was beginning level 1, to Rosa, who had completely acquired level 1 and was beginning to reason at level 2.

Josefa's vocabulary was simple and ambiguous, as were the techniques she used to justify properties. She understood the properties' having strong visual components, which is typical of students beginning to learn at level 1. Meanwhile, Mónica was progressing through level 1. She understood and used mathematical terms and applied recently learned properties when a direct relation was found or after some hints were given by the teacher. Rosa, in contrast, already understood the properties that she could apply in specific cases to justify new properties. This understanding is typical of level-2 reasoning.

When activity 2 was given to Alicia and María, more advanced students, using

$$R(B, -30°) \cdot R(A, 90°)$$

for the product, they used mathematical notation and tried to give a valid reason for the value of the angle resulting from the product. However, neither was able to complete it all correctly. Alicia referred to activity 1's property, but she could not establish all the implications required for a good proof, either at a formal level 3 or an informal level 2:

Alicia: [immediately] $R(A, 60°)$. They [the final images obtained by both of the students] have the same inclination. [By "inclination" of a figure, the student is referring to the angle that the figure makes with the horizontal.] They are parallel because of the angle measurement. It is the same, and we both have the figure in the same position.

While helping students, the teacher asked for the center and angle of the rotation. Students made several mistakes while trying to obtain the center. Finally, they measured the angle and obtained 60 degrees:

Alicia: [It is] because 90 – 30 = 60.

Teacher: Why?

María: Because it has to . . . .

Alicia: Might it be for the same reason as before? When the center is different, it results in a . . . .

Teacher: Inclination?

Alicia: Yes. Well, all [the images] that were parallel. All are parallel even though the centers are different; since it is a translation of this figure, it will always end up in the same [position].

These students were given other similar exercises. They eventually realized that they had to add the angles, but they were not able to present a proof or justification other than to measure the resulting angle. The teacher had to give a proof that used the property of translations discovered in activity 1. However, the students did understand this proof, since they correctly applied it later in several different situations. For example, the students were asked to resolve several products of two rotations that were equivalent to a translation, such as

$$R(P, -70°) \cdot R(S, 70°):$$

Alicia: [Quickly]: The image will be as it [the initial figure] is now. Parallel to this one. [Pointing to parts of the two figures] This side will be parallel, also this one, and this one too.

Teacher: Will it be in the same place [as the initial figure]?

Students: It depends on the centers of the turns.

Teacher: Where should the centers be located so that the figure does not move?

María: All [the centers] should be the same point.

Teacher: Are you sure about this?

Alicia: Well, not completely sure! I do not know. It may happen that some of these specific cases happen from time to time. [That is, they might not follow a general rule.]

It is necessary to build carefully on previous knowledge

Alicia and María were progressing in level 2. They understood the necessity of general proofs as the way to validate a statement or conjecture, and they tried to make some informal proofs with a good deal of support from specific cases and figures. Also, they were able to understand a proof explained by the teacher and to replicate the proof in a similar case, which is typical of level-2 reasoning.

Activity 3. To prove that the product of two rotations is either a rotation or a translation, a more precise mathematical statement of the property that was being investigated in activity 2

Before work begins on this activity, students need to know how to decompose a rotation or a translation into the product of two reflections. In particular the students need to decompose

$$R(B, \beta) \cdot R(A, \alpha) = S_4 \cdot S_3 \cdot S_2 \cdot S_1$$

and choose the axes such that $S_2 = S_3$ (see **fig. 7** for an example).

In our teaching unit, activity 3 was one of the last activities in phase 4 (free orientation) of van Hiele level 2. It introduces the process of making an abstract proof and links informal and formal reasoning. We accepted informal explanations from students and guided them through the steps of the proof. Activity 3 could also be considered at van Hiele level 3 for those students who are themselves able to make a formal proof.

To solve activity 3, students were asked to work on several particular cases before trying the general case. The teacher first suggested that they work on the decomposition

$$R(P, 60°) \cdot R(O, 90°) = S_4 \cdot S_3 \cdot S_2 \cdot S_1.$$

Here is a student's response:

Teacher: The rotation $R(P, 60°)$ has to be equivalent to $S_4 \cdot S_3$, and the rotation $R(O, 90°)$ has to be equivalent to $S_2 \cdot S_1$.

María: Are they independent of each other? Because I draw here [the center O] two axes with 45 degrees and here [the center P] two axes with 30 degrees.

Teacher: Okay. Does any line exist that can be used as [axis of] symmetry S_2 and also S_3 [at the same time]?

María: An axis going through both centers [**fig. 7** shows her correct solution] because the axis has to go through the center [of rotation] and it has to be used for both rotations.

Teacher: Is it possible to simplify this product $S_4 \cdot S_3 \cdot S_2 \cdot S_1$?

María: $S_4 \cdot S_1$ because S_2 and S_3 are the same.

Teacher: This example is a specific case, but after a few more examples I will ask you for the general proof that the product of two rotations with a different center is a rotation.

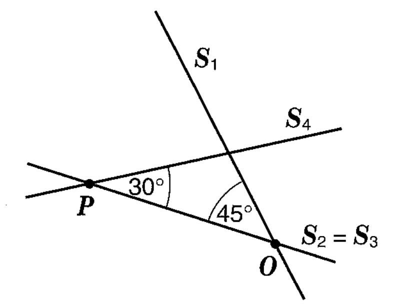

Fig. 7
Reflections through axes $S_4 \bullet S_3 \bullet S_2 \bullet S_1$ to obtain $R(P, 60°) \bullet R(O, 90°)$

María: But is it possible always to simplify two axes?

Teacher: What do you think?

María: It is [always possible] if neither of the axes has been fixed previously.

María was asked to solve several exercises like the previous one. Then the teacher asked this question:

Teacher: How would you do a general proof?

María: [After making the drawing in **fig. 8**] Where the axes cut each other there would be a point, and the angle of the rotation would be $\alpha + \beta$.

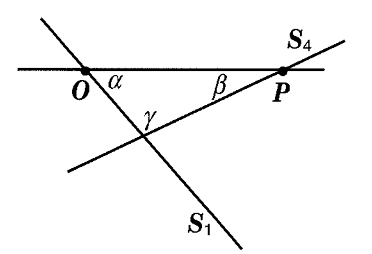

Fig. 8
María's drawing for preparing a general proof

María made a drawing with α, β, and γ as angles (**fig. 8**) and reasoned about the general case of a product $R(P, \beta) \cdot R(O, \alpha) = R(Q, \gamma)$, where $\gamma = \alpha + \beta$, but she made some mistakes when marking the angles in her figure and in the algebraic manipulations of the relation

$$\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = 180°.$$

She also made some other drawings, and in one of them, by chance, the axes S_1 and S_4 happened to be parallel. Although María was able to solve each step, the teacher had to guide her through the proof.

The excerpts from these three activities indicate the progress of a few students as they reason at level 2. It also appears that María was ready to begin work at level 3. At the beginning of these activities, the students were not able to justify all their steps, but once the teacher explained it for the first time, they were able to use similar reasoning in other situations. María was able to do some general proofs with the teacher's guidance.

We have not yet been able to experiment with level-3 students, so we cannot show examples of their reasoning on isometries at this level. We would expect that students at level 3 could construct a formal proof of their own for activity 3, which is based on the decomposition of rotations and translation into pairs of reflections. The main mathematical conceptual work to be done at level 3 on isometries is to complete the investigation of the algebraic structure of the group of plane isometries and its subgroups on the basis of the properties studied throughout our teaching units. In our teaching unit, we also introduce several types of formal proof that are typical in the world of mathematics and we do prove the main properties of isometries.

FINAL REMARKS

Knowing the characteristics of the van Hiele levels of reasoning on isometries is useful for secondary school teachers who wish to implement appropriate activities in transformational geometry for their students' levels of reasoning. We have shown how a single activity was solved in different ways by students using different levels of reasoning. The van Hiele model allows teachers to characterize their students' answers by identifying student's reasoning levels. In this way, the diverse ways of thinking present in every classroom can be better understood and managed by teachers, as they work with each student according to a specific level of thinking.

One consequence for teaching mathematics from this research is that to help our students progress through the van Hiele levels of reasoning, it is necessary to build carefully on previous knowledge. When faced with new situations, students should be given enough activities so that they can acquire an in-depth understanding of new concepts and properties. It takes time to build a conceptual network of understanding in our students, as pointed out by these activities on isometries.

REFERENCES

Burger, William F., and J. Michael Shaughnessy. "Characterizing the van Hiele Levels of Development in Geometry." *Journal for Research in Mathematics Education* 17 (1986):31–48.

Coxford, Arthur F. *Geometry from Multiple Perspectives. Addenda* Series, Grades 9–12. Reston, Va.: National Council of Teachers of Mathematics, 1991.

Crowley, Mary L. "The van Hiele Model of the Development of Geometric Thought." In *Learning and Teaching Geometry, K–12,* 1987 Yearbook of the National Council of Teachers of Mathematics, 1–16. Reston, Va.: The Council, 1987.

Fuys, David, Dorothy Geddes, and Rosamond Tischler. *The van Hiele Model of Thinking in Geometry among Adolescents. Journal for Research in Mathematics Education* Monograph Series, no. 3. Reston, Va.: National Council of Teachers of Mathematics, 1988.

Geddes, Dorothy. *Geometry in the Middle Grades. Addenda* Series, grades 5–8. Reston, Va.: National Council of Teachers of Mathematics, 1992.

Hoffer, Alan. "van Hiele Based Research." In *Acquisition of Mathematics Concepts and Processes,* edited by Richard A. Lesh and Marsha Landau, 205–27. New York: Academic Press, 1983.

Jaime, Adela. "Aportaciones a la Interpretación y Aplicación del Modelo de van Hiele: La Enseñanza de las Isometrías del Plano. La Evaluación del Nivel de Razonamiento." Ph.D. diss., Universidad de Valencia, Spain, 1993.

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: The Council, 1989.

O'Daffer, Phares G., and Stanley R. Clemens. *Geometry: An Investigative Approach.* Menlo Park, Calif.: Addison-Wesley Publishing Co., 1977.