Identification of pre-algebra problems of generalization that discriminate mathematical giftedness

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Adequate attention to mathematically gifted students starts with identification. Problem-solving has proved to be an excellent tool to identify them. However, not every problem is adequate to discriminate mathematically gifted students. We present a part of a research project aimed to create a set of problems for primary school adequate to evaluate the acquisition and use of different mathematical capacities and identify candidates to be mathematically gifted students. We focus on pre-algebra and the evaluation of the capacity of generalization by means of geometric pattern problems. The aim of this paper is to present a research methodology based on the definition of a set of descriptors for this kind of problems and a mixed (qualitative and quantitative) methodology to analyse students' answers in order to differentiate the problems which are or are not good discriminators of mathematical giftedness in the capacity of generalization.

Keywords: Mathematical giftedness, pre-algebra, primary education, problem solving.

Introduction

Mathematics educators often suggest that students should solve tasks and use tools in a way similar to that of mathematicians (NCTM, 2000). This teaching methodology is particularly interesting for mathematically gifted students (MG students hereafter), since it allows them to develop their mathematical capacities, learn new tools, and improve their mathematical abilities (Chamberlin & Moon, 2005). This can be accomplished in different ways, mainly by practicing their mathematical capacities, like proving, generalizing, etc. or solving the same types of problems mathematicians face, like mathematical modelling (Stillman et al., 2020). Generalization is a necessary capacity for mathematicians. For students, generalization is necessary to understand algebra, learn algebraic thinking, and improve their algebraic language. More and more countries begin the learning of algebra in primary school, e.g., Costa Rica includes pre-algebra since 2012 (MEP, 2012), proposing to base its teaching on numerical and graphical patterns.

Like Cai and Knuth (2011), we consider that *pre-algebra* (or early algebra) refers to the beginning of teaching algebra in primary school, organized to promote a soft transition from the arithmetical meanings, operations, and language to the algebraic ones, trying to avoid the factors that make such transition difficult for students. This way of teaching includes focusing on noticing structures, producing generalizations, doing operations and their inverses, and interpreting letters and the equal sign, among other components of *algebraic thinking* (Fritzlar & Karpinski-Siebold, 2012).

Researchers recognize the early development of the capacity of generalization, when learning pre-algebra, as a distinctive trait of primary school MG students (Fritzlar & Karpinski-Siebold, 2012); these students can generalize rapidly and easily (Krutetskii, 1976) and are more successful than regular students in finding and verbalizing generalizations (Sriraman, 2003). Teachers can identify candidates to be MG students by observing their pupils' solutions to problems of generalization.

Solving geometric pattern problems (GP problems hereafter) has proved to be a good context to introduce students into algebraic thinking and to identify MG students, since they can be solved in several ways located in different levels of algebraization (Nolte, 2012). But not every GP problem is adequate to identify MG students of a given school grade. Then, there is a need for research results informing on ways to differentiate GP problems which can or cannot discriminate students having high development of their capacity of generalization and use of algebraic language.

We are working on a research project aimed to create sets of problems, for different grades of primary school, adequate to evaluate the acquisition and use of different mathematical capacities and identify candidates to be MG students. One of such capacities is generalization. The experimental part of the research is based on the answers to problems posed in the three phases of the Costa Rican Olympiad of Mathematics for Primary School (OLCOMEP) by olympic and regular students.

The research objective of this paper is to present and discuss the research methodology that we are using to analyse and evaluate GP problems, in order to identify the problems which can better discriminate traits of MG related to generalization and be useful to identify potential MG students.

Theoretical framework

Pattern problems (in particular, GP problems) are often used to introduce generalization and algebraic verbalization to primary school students. In this context, a *pattern* is an increasing sequence of whole numbers having a regular increment from a term to the next one. In our experiments, we have used patterns having constant increments. A students' *algebraic generalization* of a pattern is "the capability of grasping a commonality noticed on some elements of a sequence S, being aware that this commonality applies to all the terms of S and being able to use it to provide a direct expression of whatever term of S" (Radford, 2006, p. 5). Based on previous literature on GP problems (Gutiérrez et al., 2018; Radford, 2006; Rivera, 2013), we have identified the different actions students have to do to solve a GP problem and, for each action, have defined a set of specific *descriptors of types of answers* to any GP problem, summarized as follows:

Action 1. *Handling of the data*: Students can interpret and use in different ways the visual data provided by a GP problem:

- DG1.1. Figural handling of the data.
- DG1.2. Arithmetic handling of the data.

Action 2. *Calculating the value of the term in a given position (direct questions)*: Students can calculate in different ways the value of the term of the GP in a given immediate, near, or far position:

- DG2.1. Counting the visual parts of the term.
- DG2.2. Recursive calculation.
- DG2.3. Functional calculation.
- DG2.4. Proportional calculation.

Action 3. *Verbalization of the general relationship ruling the pattern*: Students can identify and express verbally (orally or written) in different ways a relationship defining the GP (Radford, 2006):

DG3.1. Naive induction (by guessing)

DG3.2. Arithmetic generalization (recursive)

DG3.4. Algebraic contextual generalization (expressed based on natural language).

DG3.3. Algebraic factual generalization (expressed based on particular numbers)

DG3.5. Algebraic symbolic generalization (expressed based on algebraic language).

Action 4. *Calculation of the position of the term having a specific value (inverse questions)*: Students can calculate in different ways the positions of the terms of the GP, given their values:

- DG4.1. Recursive calculation.
- DG4.2. Trial and error.
- DG4.3. Wrong inversion of operations.
- DG4.4. Incomplete inversion.
- DG4.5. Inversion with wrong order of operations.
- DG4.6. Correct inversion of operations.
- DG4.7. Solving an equation.

Methods

The olympiads are considered a context adequate to identify candidates to be MG students. We have designed a group of GP problems to observe the descriptors presented in the theoretical framework in students' answers. We posed the problems in the OLCOMEP and regular schools. To fulfil the research objective of this paper, we present in detail our research methodology to analyse students' answers and decide which problems are good to discriminate candidates to be MG students.

Based on the above-mentioned descriptors, we first analyse and classify students' answers and, then, we analyse and compare the results of different types of students (regular and olympic, olympic reaching the different phases of the olympiad, etc.), to get diverse conclusions, in particular information about the usefulness of the GP problems posed to discriminate MG students. To do it, we use a mixed analysis (qualitative and quantitative) of students' answers. We will base the description of the methodology and the examples showed on one of the problems (Figure 1).

Adela is building a crib mobile for a baby. She has the structure of the mobile and adds some pieces every day, following a rule she has decided, which can be seen in the figures.

A) How many pieces will the mobile have on day 7?
B) Explain the rule that Adela has used to decorate the mobile.
C) How many days will Adela need to decorate the mobile if she has exactly 181 pieces?

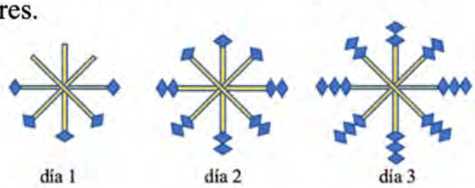


Figure 1: Problem of generalization of the OLCOMEP Phase II, 6th grade, 2019

We first make the qualitative analysis. For each problem, we create the solutions' space of the problem, made up of the solutions that we consider within the reach of the students, according to the contents available to them, their grade, and the official curriculum. We also create the collective solution space of the problem, which includes the descriptors of the capacity of generalization evidenced in each solution, presented in a table (Figure 2) showing, for each descriptor: the ways in which it can be evidenced; examples of answers; comments on the way the indicator is evidenced in each answer; and the codes of the students whose answers evidenced that descriptor in the problem.

All the information about each student' answers to the problems of a phase of the olympiad is organized in a spreadsheet, including (Figure 4): the score (0-5) obtained in each of the three problems we are analysing, with the partial scores obtained in each section of the problem, the problem score; the total score (0-15); and, for each capacity, the descriptors evidenced in the answers.

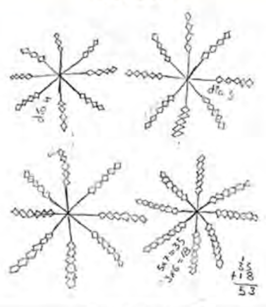
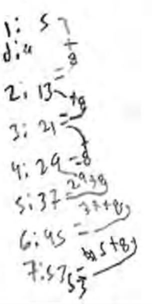
Code	How to show it	Example	Comments	Students
DG2.1	*Performs a calculation process by counting.* Represents the requested term to count the elements that make it up.	E62008 	To know the number of pieces that the mobile uses on day 7, make the drawings of the following days, until reaching day 7 and then count the number of pieces that make it up.	E62002, E62008, E62015, E62018.
DG2.2	*Performs a calculation process recursively.* Obtain the requested term by successively adding the difference between consecutive terms.	E62013 	To calculate terms from the previous term, adds the difference between terms to find the next one, so to determine a term you must have calculated all the previous ones.	E62001, E62003, E62004, E62007, E62012, E62013, E62016, E62020, E62021, E62022.

Figure 2: Example of data collection sheet for descriptors DG2.1 y DG2.2 in the problem

Next, we carry out the quantitative analysis based on the data obtained from the qualitative analysis of students' answers, by comparing the frequencies with which a certain capacity, or one of its descriptors, is present in the answers of the students with upper and lower total scores in the set of problems analysed. To do it, we use the *discrimination index (d)*: "An item has power of discrimination if it distinguishes, discriminates, between those subjects who score high on the test and those who score low, that is, if it discriminates between the successful and the unsuccessful in the test" (Muñiz, 2002, p. 219). In our context, the discrimination index of a problem evaluates the ability of the problem to differentiate subjects with higher scores. To calculate d, we use the technique of "discrimination by thirds[, which] is the difference in the proportion of correct answers to an item, between the group made up of the examinees with the higher marks in the test and the examinees with the lowest marks. Each group is made up of a third" of the sample (Rojas, 2014, p. 3; emphasis added). As the sizes of the subgroups to analyse are not so large, we divide the samples into *modified thirds* when necessary: we enlarge or reduce the strict thirds to place in the same third all students having the same total score.

If $\sum P_1$ ($\sum P_3$) is the sum of the scores in a problem of the students in the lower (upper) third of the sample, then the mean score in the problem of the students in the lower (upper) third is $\overline{P_1} = \frac{\sum P_1}{N_1}$ ($\overline{P_3} = \frac{\sum P_3}{N_3}$), where N_1 (N_3) is the number of students in the lower (upper) third. The discrimination index is $d = \frac{(\overline{P_3} - \overline{P_1})}{S}$, where S is the maximum score of the problem (S=5 in this case). The value of d is between -1 and 1; $d = 0$ means that the problem does not discriminate (the students in the upper and lower thirds had the same mean scores). The discriminative capacity of a problem increases as *d* moves away from 0. A problem discriminates: very well if $0.40 \leq |d| \leq 1$, well if $0.30 \leq |d| < 0.40$, little if $0.20 \leq |d| < 0.30$ and very little if $0 \leq |d| < 0.20$.

To calculate *d* for the problem of Figure 1, we order the data of the 22 olympic students who solved it according to their total score, from the lowest to the highest. Then, we identify the 7 students in the upper and lower strict thirds. As the 7th student in the lower third had the same score (7) as the next students, we move that student out of the lower third. Furthermore, several students in the upper strict third had the same score (10) as the last student in the middle third, so we include this student into the upper third. Thus, we have in the upper third the 8 students with the highest scores (10-12), in the lower third the 6 subjects with the lowest scores (2-5) and in the central third the 8 students with intermediate scores (7-9), which we do not use. Table 1 shows the means of the scores obtained in this problem by the students in the lower and upper thirds and the value of the discrimination index *d* for the problem, suggesting that this problem discriminates little MG students.

Table 1: Calculation of the discrimination index of the problem

Mean of the lower third	Mean of the upper third	Index *d*
$\overline{P_1} = \frac{\sum P_1}{N_1} = \frac{20}{6} = \mathbf{3,33}$	$\overline{P_3} = \frac{\sum P_3}{N_3} = \frac{37}{8} = \mathbf{4,62}$	$\frac{(\overline{P_3} - \overline{P_1})}{5} = \mathbf{0,26}$

The discrimination index *d* of the descriptors of generalization gives us information about which descriptors discriminate better the candidates to be MG students and, among them, those who have the best results in the olympiad. We apply the same methodology presented above, with some differences in the way of calculating the means of the extended thirds: to calculate the index *d* for a problem, the students are ordered by their total score, but, to calculate the index *d* for a descriptor they are ordered by their problem score of a particular problem and, as a second criterion (if necessary), by their total score. We order the students according to their scores in the generalization problem we are analysing, identify the upper and lower thirds, and organize the data in a table, including the descriptors of generalization evidenced in that problem by each student. To calculate the index *d* for each descriptor of generalization present in this group of students, we determinate the frequency of each descriptor in the upper and lower thirds and perform the same calculations shown in Table 1. The scoring for the descriptors is to be evidenced (1) or not (0), so S=1. Table 2 presents the values of *d* obtained for each descriptor of generalization evidenced by the olympic and regular students who solved the problem of Figure 1. The descriptors DG2.4, DG3.1, DG3.5, DG4.3, DG4.5, and DG4.7 were not evidenced by the students, so *d* cannot be calculated for them.

Table 2: Index *d* for the descriptors of generalization in the problem

	DG 1.1	DG 1.2	DG 2.1	DG 2.2	DG 2.3	DG 3.2	DG 3.3	DG 3.4	DG 4.1	DG 4.2	DG 4.4	DG 4.6
Olympics	-0,02	0,02	-0,05	0,25	-0,07	0,20	0,10	0,07	0,25	0,30	-0,27	-0,12
Regular	-0,02	**0,50**	-0,22	**0,56**	0,22	**0,67**	0,11	No evid.	**0,50**	0,00	No evid.	0,06

When working with regular students, it is necessary to have a parameter that would allow us to differentiate which students stand out from the rest in the use of their problem-solving skills. According to the normal distribution of intelligence (Figure 3), it is expected that “the common” is to have an average intelligence, with IQ values close to the mean. For students to be considered “out of the ordinary”, their IQ must be at least one standard deviation away from the mean. Subjects with an IQ ≥ 115 are considered more able, and those with an IQ ≥ 130 are considered gifted.

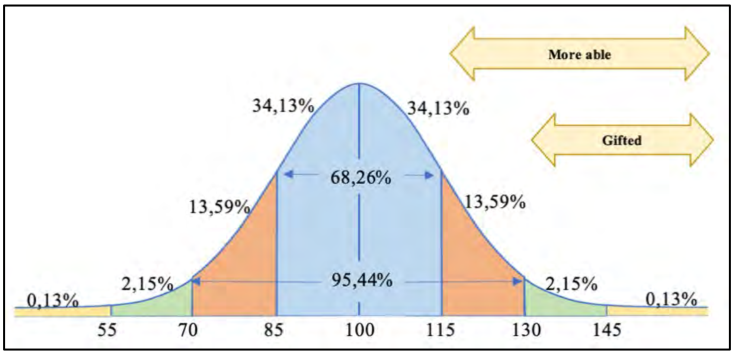


Figure 3: Normal curve of intelligence

If we transfer this approach to our sample and the scores in the generalization problems, we can obtain a score landmark allowing us to say that students can be considered candidates to be MG students when their score is equal to or greater than such value. To get this, we created a subgroup of the sample whose distribution adjusts as closely as possible to the normal curve. Such subgroup is made up of the 52 students from three full sixth-grade regular classrooms (where 2 students were olympic) plus 7 randomly selected olympic students, for a total of 59 students, including 9 olympic students. We have added such number of olympics considering that they are more able and, in a normal population, approximately 15.87% of the students are more able. The total scores in the three problems of generalization range from 0 to 15 points, the mean of the total scores of the 59 students in the sample subgroup is 4.20 points, and the standard deviation is 3.73. Then, when posing these problems to groups of regular students, applying the criterion of normality, we consider that students obtaining a total score in these problems of 7.93 or higher are candidates to be MG students. Applying this criterion to the 50 sixth-grade regular students in our sample, we get (Figure 4) that there were 8 regular students with scores higher than 7.93.

Students		Descriptors of generalization																					
		Scores				DG	DG	DG	DG	DG	DG	DG	DG	DG	DG	DG	DG	DG	DG	DG	DG	DG	DG
Total score	Code	A	B	C	problem	1.1	1.2	2.1	2.2	2.3	2.4	3.1	3.2	3.3	3.4	3.5	4.1	4.2	4.3	4.4	4.5	4.6	4.7
8	EC62001	0	2	0	2		X			X			X										
8	EC62009	1	1	0	2		X		X				X										
8	EC62029	1	1	1	3	X				X			X				X						
8	EC62041	0	1	2	3	X			X				X				X						
9	EC62014	1	1	2	4		X		X				X				X						
9	EC62032	1	2	2	5		X		X				X				X						
11	EC62030	1	2	2	5		X		X				X				X						
11	EC62028	1	2	2	5		X		X				X				X						

Figure 4: Descriptors showed by the candidates to be MG students from the regular group

Conclusions

We have presented a methodological technique to identify generalization problems that are good discriminators of potential MG students in regular classrooms. With the methodology applied to the showed case (the problem in Figure 1 and the sample of olympic and regular students), we can raise several conclusions. The discrimination index *d*=0.26 of this problem for the olympic students indicates that the problem discriminates little between the olympic students with upper and lower total scores, since both kinds of students got nearly the same problem scores; the students with higher total scores made correctly most calculations in the direct and inverse questions, but they lost points because they did not know how to verbalize the generalization of the pattern. On the other hand, a few students with lower scores failed the calculations in the direct questions, other students did not answer or lost points in the inverse question, but all of them were fully or partially successful in verbalizing the generalization of the pattern. Asking to explain the pattern allowed everyone to achieve points in this section, but those students who did not know how to express the pattern clearly lose some point. For this reason, this problem provided little discrimination.

In contrast, the discrimination index of this problem with regular students is *d*=0.50, meaning that the problem differentiates very well the students in the regular classrooms who get higher total scores from those who get lower total scores in the test. While the olympic students seemed to understand the problem without difficulty, the regular students with the lower total scores could not even solve the easiest direct questions. Most of the regular students with higher total scores solved the direct questions and they made at least a partially correct verbalization of the generalization; almost half of these students could solve the inverse question. Therefore, in regular classrooms, the discrimination index *d* of this problem is very good.

The data in Table 2 shows that some descriptors, DG1.1, DG2.1, DG2.3, DG3.3, DG3.4, DG4.2, DG4.4, and DG4.6 were used almost to the same extent by the students with the higher scores and the lower problem scores, so they do not help us to discriminate students. DG1.2 was used to a greater extent by students with higher scores in the regular group (*d*=0.50), while olympic students with lower and upper scores used it almost equally. DG2.2 helps us to discriminate between the students with higher scores, because in the olympic group it discriminates a little (*d*=0.25), due to the particularities previously analysed, but in the regular group it discriminates very well (*d*=0.56). DG3.2 is the descriptor that best discriminates the students in the regular group with higher scores (*d*=0.67). It happens because almost all the regular students who made some kind of generalization and had higher scores, made arithmetic generalization. The use of DG4.1 is present in a greater extent in the answers by the students with higher scores, both olympic (*d*=0.25) and regular (*d*=0.50).

The descriptors that best discriminate students with higher scores are those presented by the eight students in the regular group who, according to our reference value, stand out as candidates to be MG students. These descriptors correspond to the arithmetic handling of the data, recursive calculation, arithmetic generalization, and the recursive calculation strategy in the inverse question. All descriptors were exhibited mainly by students with higher scores in both olympic and regular students. Generalization is only one of several capacities MG students need to show, so these students should be followed up to get more information about their mathematical capacities.

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