PROBLEMS OF VISUALIZATION AS A TOOL TO IDENTIFY MATHEMATICALLY GIFTED STUDENTS

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Visualization has emerged in various investigations as a characteristic associated to mathematically gifted students' reasoning, highlighting that these students tend to use spatial aspects and mental images to a great extent. Furthermore, problem-solving has proved to be an effective context to identify mathematically gifted students. We present the way in which we use visualization problem-solving as a useful way to observe certain characteristics that students show in their solutions and identify those who stand out as potential mathematically gifted students. In the experiments that we have carried out, we have considered different abilities of visualization and posed three types of problems. We present here one problem of each type and analyses of students' answers. Our results show that each type of problems is useful to evidence the use of some abilities of visualization and all together help to identify potential mathematically gifted students who have a well-developed capacity of visualization.

INTRODUCTION

Visualization is recognized as a key component of reasoning and problem-solving, which may help students to understand the problems and solve them. Various authors point it out as a fundamental component of mathematical reasoning (Clements & Batista, 1992), recognizing in this way its contribution to the development of problem-solving strategies by the students.

Several researchers indicate that mathematically gifted (MG) students tend to use spatial features and mental images to a greater extent than their peers with average mathematical ability, recognizing MG students as good visualizers (Van Garderen, 2006). Therefore, an increasing number of researchers focus on analyzing the use of visualization by MG students (Ramírez & Flores, 2017; Ryu et al., 2007), aiming to characterize traits of visualization observed in these students when they solve problems. Some authors suggest that the capacity of visualization should be a part of the processes of identification of MG students, by posing problems whose solution involves this capacity (Rojas et al., 2009; Webb et al., 2007).

To bring new information to this research issue, we designed a set of novel and challenging problems related to visualization and posed them to average and potentially MG students participating in the Costa Rican Mathematical Olympiad for Primary School (OLCOMEP). The aims of this text are i) to present the types of problems we designed and analyze the abilities of visualization evidenced in students' solutions and ii) to highlight the abilities of visualization, among those observed for each type of problems, that can be helpful to identify potential MG students.

THEORETICAL FRAMEWORK

Based on Gutiérrez (1996), we consider visualization as the *capacity of reasoning based on the use of spatial or visual elements (photographs, diagrams, drawings, graphs, solid objects, etc.), both physical and mental, used to solve problems, prove conjectures, or understand concepts and properties.* To use their capacity of visualization to solve problems, students have different abilities of visualization at their disposal, which may help them create or manipulate mental images. To ground the analysis and interpretation of students' solutions to our visualization problems, we have used as theoretical framework the abilities of visualization defined by Del Grande (1990), Gutiérrez (1996), and Van Garderen (2006). We integrated various theoretical elements to establish a set of descriptors of visualization observable in students' outcomes. To characterize the abilities of visualization, we have defined a set of *operational descriptors* of each ability, aimed to make the analysis of students' outcomes more feasible. The descriptors are:

- *Visual identification:* identify a figure that is part of a complex context by isolating it from the context.
- *Mental rotation:* produce or transform a mental image by rotating it in space. It can be based on a dynamic image or on a pair of pictorial images.
- *Conservation of perception of partially hidden figures:* identify a regularity of a partially hidden or deformed by perspective figure that allows to imagine how the figure would look like if it could be viewed completely.
- *Conservation of perception of hidden figures:* identify a regularity of a structure that allows to assume that it includes hidden figures, and what the figures would look like if could be fully seen.
- *Recognition of positions-in-space with respect to oneself as observer:* recognize the positions of objects located in space in relation to oneself as observer, acting as the reference center.
- *Recognition of positions-in-space with respect to a reference object:* recognize the positions of objects located in space in relation to another fixed object, acting as a reference center.
- *Recognition of spatial relationships between objects:* recognize the positions of objects located in space (and/or their mental images) in relation to each other.
- *Recognition of spatial relationships between elements of objects:* recognize the positions of elements of one or more objects located in space (and/or their mental images) in relation to each other.
- *Visual discrimination:* compare objects, identifying their visual similarities and *differences*.

Choosing the most convenient abilities according to the requirements of a problem is a skill which, from the literature, seems to be more developed in more mathematically able students.

METHODOLOGY

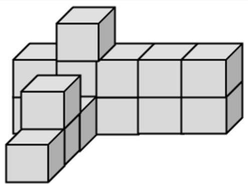
We designed 16 problems of visualization for students in grades 2 to 6 (7 to 12 years old) to be posed in the successive rounds of OLCOMEP. There are 3 types of problems: A) identify the quantity and/or type of objects that form a complex structure; B) mentally disassemble or rearrange the elements that make up a given structure, to determine or verify the instructions stated in the problem; and C) analyze different side-views of a given complex structure made of simple elements.

We collected solutions from 300 students and classified their ways of using their capacity of visualization by analyzing how they put into play the descriptors of visualization in their solutions. For each problem, we ordered the students who solved it according to their score in the problem and divided the students into three subgroups of approximately equal size, with the students having the lowest, intermediate, and highest scores. These subgroups were established to analyze the prevalence of descriptors within various score subgroups and differentiate those descriptors that are observed to a greater (or lesser) extent by the potential mathematically gifted students. Furthermore, we selected some students from each subgroup to interview them. The objectives of the interviews were to complete some shortcomings in the answers that impeded us to interpret them or to ask the students about their reasoning to get the results. The interviews were focused to complement the information collected in the written answers. In the following section we present examples of the different types of problems and synthesize some students' solutions where different descriptors are evidenced.

RESULTS

Type A problems: identify the quantity and/or type of objects that form a complex structure

The example in Figure 1 is a problem posed to grade 4 students (aged 9-10). Some students used *visual identification*, which allowed them to identify each small cube, with its faces and edges, as part of the complex structure. On the other hand, to solve the problem it was necessary to count the number of cubes in the structure, so students, based on observable regularities, had to identify cubes that are totally or partially hidden, or identify the rhomboids in the drawing as real squares which are faces of cubes deformed by perspective. To do it, it was necessary for students to use the *conservation of perception of totally* or *partially hidden figures,* respectively.



With cubes that measure 1cm on the edge, Mariela began to build a large cube whose edge measures 5cm, as shown in the figure. From the previous information: if each small cube weighs 5dag, how many grams do all the cubes that Mariela needs to complete the large cube weigh in total? justify your answer.

Figure 1: Problem of type A

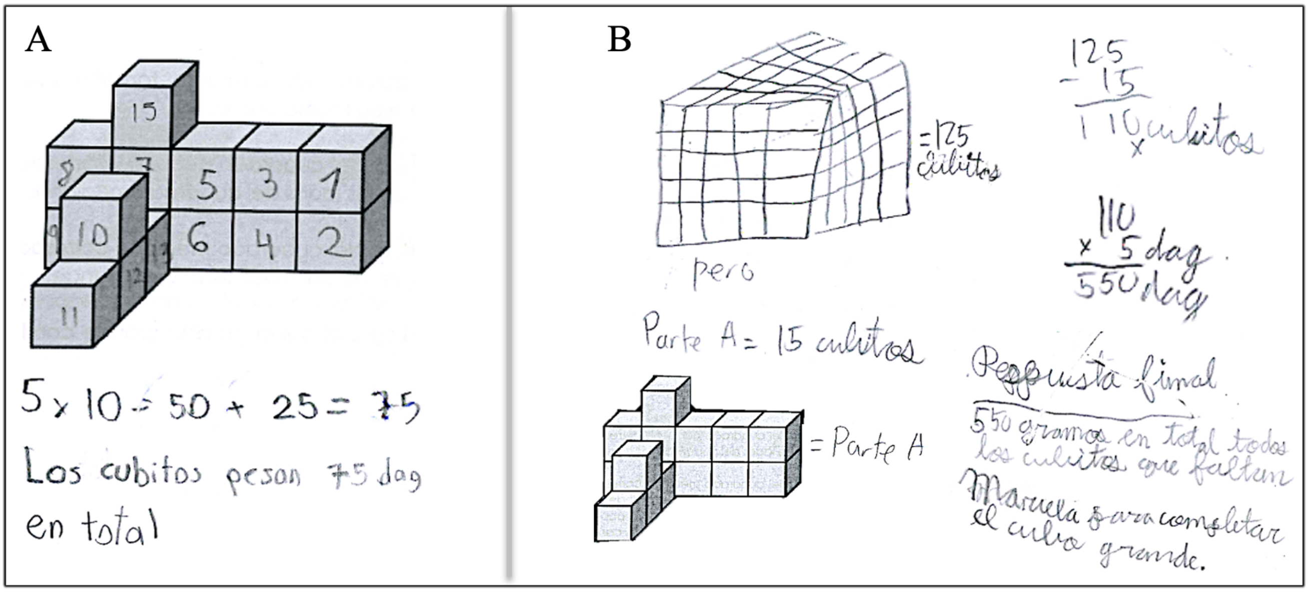


Figure 2: Answers to the problem of type A

These descriptors were evidenced in students' answers, for example the student in Figure 2.A evidenced *visual identification* because he counted each cube that makes up the structure and wrote the numbers corresponding to the count on one of its faces. The student also evidenced *conservation of perception of partially hidden figures*, by counting cubes 7, 8 and 9, of which only a part of their face is visible, and cubes 12 and 13, having one of their faces visible but deformed by perspective. Furthermore, the *conservation of perception of hidden figures* was evidenced in the answer, since the student counted cube 14, located below cube 7.

The student in Figure 2.B made a drawing of how the large cube that Mariela wants to build would look like, to determine the number of cubes needed, the missing ones and then their weight. He evidenced *visual identification* by identifying all the small cubes, their faces, and edges. He also evidenced: *conservation of perception of partially hidden figures*, by including in the count the cubes of which only a part of their faces is visible, or a face deformed by perspective; and *conservation of perception of hidden figures,* by including in the count the cube that is completely hidden.

The mentioned descriptors were evidenced in the solutions to this problem of almost all students with the highest scores, while none of them was evidenced in the solutions of the students with the lowest scores. So, these descriptors differentiated very well in this problem the best visualizers, candidates to be MG students.

Type B problems: mentally disassemble or rearrange the elements that make up a structure

The example in Figure 3 is a problem posed to grade 5 students (aged 10-11). We argue that those students had to use the *mental rotation* when they solved it by producing or transforming mental images of the cube or one of its nets and mentally rotated them, as well as the figures on its faces, with respect to their initial position. This descriptor was observed in almost all students with highest scores and only them, so it differentiated very well the best visualizers.

Students must build a cube that has figures in specific positions on its faces. To do this, they build models to assemble the cube, drawing the figures in the indicated spaces and positions. Everyone must get an identical cube. Observe the work that some students have done so far:

Complete the following views of the cube that students will obtain at the end of the work. Must respect the positions of the figures.

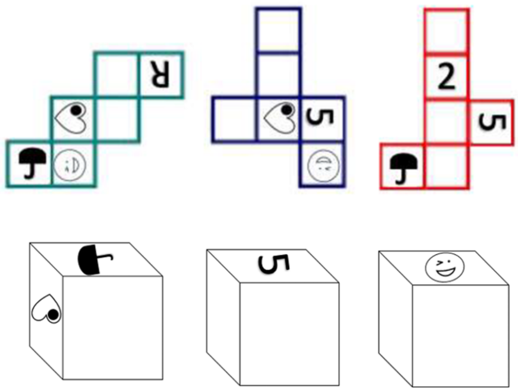
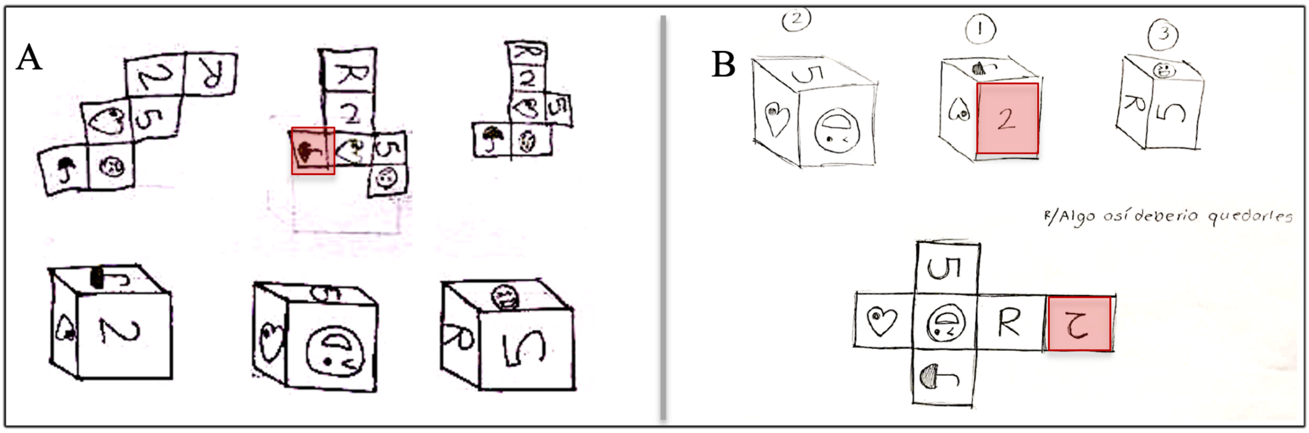


Figure 3: Problem of type B

Some written answers evidenced the use of *mental rotation.* For example, in Figure 4.A, the student was able to make the mental rotations of the faces of each net necessary to close the cube, because the student completed correctly the three cubes based on the partial information given in each net. Furthermore, she completed the figures on each face of the cube according to the given position (in this case there is a mistake in the orientation of the umbrella in the second net. Nevertheless, this error did not affect her answer).

These types of problems (A, B and C) involve a certain degree of abstraction that makes it difficult for students to solve them and communicate their reasoning in a written form. So, we expect that, in addition to those students who were unable to establish relationships between the data, there were many others who mentally progressed through some steps of the solution process, but there is no evidence of the descriptors related in their written outcomes. Therefore, the interviews we made after the written solutions were helpful to let us complement the evidence of these descriptors.

Some students used *recognition of positions-in-space with respect to a reference object* to correctly identify relationships between the positions of several figures on the faces using one of them as the reference. We confirmed that student of Figure 4.B used it when he explained how he identified the correct place of the "2", by using the "5" as a reference center: he observed the three nets and said "in this image [third net], there is the 5. It is repeated in all the images [nets], in fact, they are in line [pointing to the line of faces of the three nets having the heart next to the "5"] and the heart should go below. As the heart goes here, [pointing to the face next to the "5" in the third net] it helps me imagine that the "2" could go here, [he imagined adding a face in the first net, above the heart, in the same place as the "2" in the third net]". In this case there is a mistake in the orientation of the 2, however, the place is correct. The mistake occurred when he transferred the 2 to the new net (made by him) because we verified in the interview that he was clear (in the net given) about the correct orientation since he said that "as the 2 it is facing away from the 5, I would have to rotate this 2 to the right and move it to this face [the one next to the R], so that it would form like a 25." This descriptor was exhibited by one student with lowest scores and 50% of students with highest scores, so we can argue that it helped well to differentiate the best visualizers.



*Red squares show figures in correct places but wrong orientation.

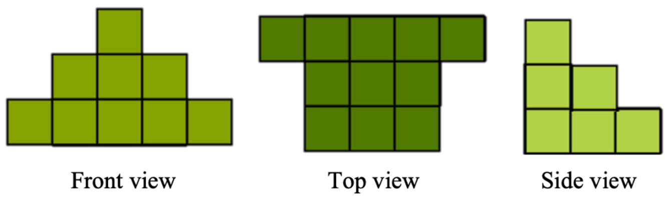
Figure 4: Answers to the problem of type B

On the other hand, when students identified characteristics of the figures on two faces that allowed them to relate those figures directly to each other, they needed to use *recognition of spatial relationships between objects*. In the case of the student in Figure 4.B, we confirmed that he identified characteristics of the figures on the faces to related to other figures; in the interview, he explained that in the second net "it seen that the "5" is above the heart and that allows to transfer that information to the first net and place the "5" in that same position on the face that is next to [above] the heart". We also evidenced this descriptor in the explanation given previously about the orientation of the 2, when the student mentioned that it was placed next to the 5 oriented so that it is like a 25. This descriptor was evidenced by 50% of the students with highest scores and only by them, so it differentiated very well the best visualizers.

Type C problems: analyze side-views of a complex structure made of simple elements

The problem in Figure 5 was solved by grade 5 students (aged 10-11). It can be solved with the help of *mental rotation,* although no student evidenced in the written solution having used it. It can also be solved by using *visual identification*, if students observe the given side-views, they could recognize their squares as faces of the boxes, and extract information from groups of squares to deduce the location that the boxes should have in the stage, and they try to create a mental image of the complete stage. In Figure 6.A we can note that one student evidenced *visual identification* to interpret the given data and represent it in his own way (a representation of the stage by layers but using only the base layer) to establish relationships between groups of squares (rectangles with thicker border, highlighted) that represents cube-shaped boxes.

Luis and Ana must build a stage for the celebration of independence at their school, joining several cube-shaped boxes that are in the hall. They had a plan of the stage, but they have lost it, now they only have these images:



How many boxes should Ana and Luis use to build the stage? How would you explain to Ana and Luis the way in which they should build the stage, based on the images they have? Is it the only possible way to build the stage?

Figure 5: Problem of type C

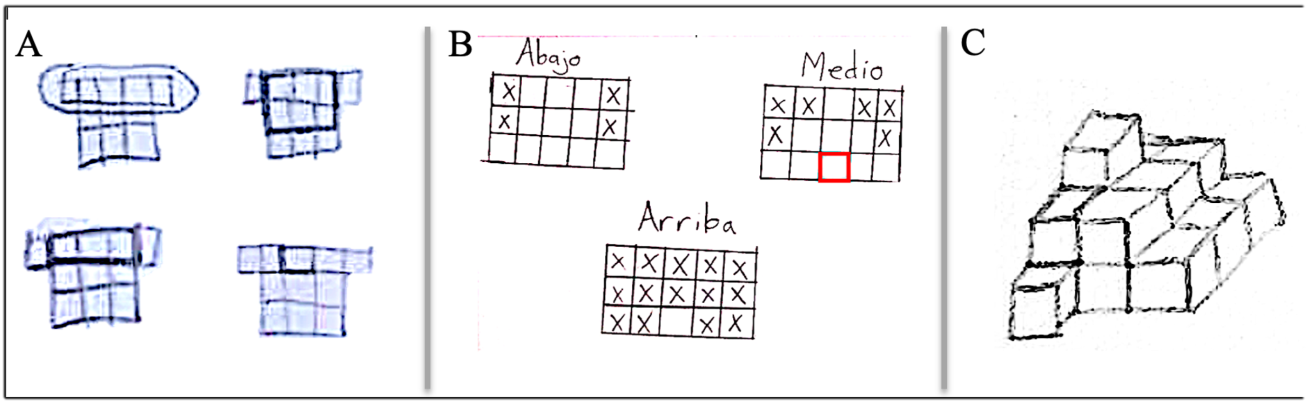


Figure 6: Answers to the problem of type C

When relating the information given to determine the shape of the stage, some students evidenced the descriptor of *recognition of spatial relationships between objects* when they related the positions of different boxes to each other. Figure 6.B presents the answer by another student who also drew a representation by layers of the stage in which we highlight the central square in the down row (squared in red). The student knew that this box was in that position because he noted the relationships between this box and the other boxes below it. We confirmed in the interview the reason to affirm that this box was there. He explained that "it must be there [in the second layer] to support the box on top [third layer]", even though none of the views showed it.

Other students related the dimensions of the side-views to calculate the height and depth of the stage and generate a mental image of it; in this case, they evidenced the *recognition of positions-in-space with respect to oneself as observer*. If one student takes the information provided by the different views and do a mental image of the stage, he could produce a mental image that helped him draw a stage like the one showed in Figure 6.C. This way of solution seemed to be more complex than the previous ones, because it was evidenced only by all the students with the highest scores.

None of the three descriptors evidenced in the solutions to this problem was observed in the students with lowest scores, so all of them differentiated very well the best visualizers, potential MG students.

CONCLUSIONS

We aim to highlight problem-solving visualization as a useful approach to delve into the characteristics exhibited by students in their solutions and identifying key aspects that are more commonly observed in potentially mathematically gifted students. To observe aspects related to students' visualization abilities and their proficiency in utilizing them, we have chosen to operationalize them through descriptors to do their observation more accessible.

After analyzing students' performances on the problems previously indicated (A, B, and C), we have consistently observed the presence of abilities related to identifying visual information provided in the problem. Descriptors such as visual identification are evident across all three types of problem. Additionally, other descriptors facilitating the location of visual data within the problem, including identifying totally or partially hidden elements, are observed in various problem types.

Some type A problems are limited to the use of abilities related to the identification of visual information. While type B or C problems utilize them to get information about the problem and perform another more complex action with the data. Descriptors related to hidden images generally pose greater complexity compared to those related to partially hidden images. However, variations in difficulty arise due to personal factors (such as age and level) or specific task factors (problem context).

Therefore, these descriptors aid in graduating the abstraction of interpreting visual information, from an elementary identification of explicitly provided elements to identifying partially or totally hidden elements at a more advanced level of abstraction.

Some descriptors are highly specific to types of problems and are only apparent in problems where rotations or rearrangements of elements are required, such as the case of mental rotation. Others are exclusively applied when students are tasked with analyzing a structure from different perspectives, whether from their own viewpoint or a reference point provided within the problem.

Therefore, it is suggested to incorporate problems of types A, B and C to obtain a more complete profile of the student's visualization abilities. These three types of problems offer a broad range of possibilities and address most visualization abilities, including all the descriptors presented.

According to the results, descriptors were exhibited by few or no students with lower scores, while half or more students with the highest scores demonstrated them. It indicates that these descriptors effectively differentiated the best visualizers in these problems. They are potential candidates to be MG students.

We provided examples of students' responses and how they demonstrate the descriptors to illustrate how this approach can be used to identify MG students. Our results show that each type of problem highlight the use of specific visualization abilities, and collectively, they aid in identifying potential MG students with well-developed visualization skills.

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